Asymptotics of eigenvalues and eigenvectors of Toeplitz matrices

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Abstract

Analysis of the asymptotic behaviour of the spectral characteristics of Toeplitz matrices as the dimension of the matrix tends to infinity has a history of over 100 years. For instance, quite a number of versions of Szegö 's theorem on the asymptotic behaviour of eigenvalues and of the so-called strong Szegö theorem on the asymptotic behaviour of the determinants of Toeplitz matrices are known. Starting in the 1950s, the asymptotics of the maximum and minimum eigenvalues were actively investigated. However, investigation of the individual asymptotics of all the eigenvalues and eigenvectors of Toeplitz matrices started only quite recently: the first papers on this subject were published in 2009-2010. A survey of this new field is presented here.

Main object.

Spectral properties of larger finite Toeplitz matrices

$$A_n = (a_{j-k})_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ldots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \ldots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \end{pmatrix}.$$

$a(t) = \sum\limits_{j=-\infty}^{\infty} a_j t^j, \ t \in \mathbb{T}$-symbol of $\{A_n\}_{n=1}^{\infty}$

Eigenvalues, eigenvectors singular values, condition numbers, invertibility and norms of inverses, e.t.c.

$n \sim 1000$ is a business of numerical linear algebra.

Statistical physics - $n = 10^7 - 10^{12}$ - is a business of asymptotic theory.

Applications:

- the numerical solution of differential and integral equations;
- Ising model (in statistical mechanics);
- stochastic processes and time series analysis;
- signal processing;
- image processing;
- quantum mechanics.

Qualitative significance

Structure of location of eigenvalues

Distance between eigenvalues

Points of concentration

Fast numerical calculations

I. Two parameters:

n- dimensions of matrices;

j- number of eigenvalue

$$1 \leq j \leq n$$

Asymptotics by n uniformly in j.

II. Distance between λ_j and λ_{j+1} is small:

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n}\right) - \text{normal case}$$

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n^\gamma}\right) - \text{special case}$$

$$\lambda_j = \lambda_{j+1} \qquad - \text{exceptional case}$$

Publications about asymptotics of individual eigenvalues. Simple Loop case.

1. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Inside the Eigenvalues of Certian of Hermitian Toeplitz Band Matrices. Computational and Applied Mathematics, 233 (2010), 2245-2264 pp.

2. Deift P, Its A, and Krasovsky I. Eigenvalues of Toeplitz matrices in the bulk of the spectrum. Bulletin of the Institute of Mathematics Academia Sinica (New Series) 7 (2012), 437-461 pp.

3. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols. Journal of Mathematical Analysis and Applications Volume 422, Issue 2, 15 February 2015, 1308-1334 pp.

4. J.M. Bogoya, S.M. Grudsky and E.A. Maksimenko.. Eigenvalues of Hermitian Toeplitz Matrices Generated by Simple-loop Symbols with Relaxed Smoothness. Operator Theory: Advances and Applications, Volume 259, 2017, 179–212 pp.

Main results-Simple Loop case

For $\alpha \geq 0$, we denote by W^{α} the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_{\alpha} := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^{\alpha} < \infty.$$

Put $g(\varphi) := a(e^{i\varphi}), \ \varphi \in [0, 2\pi]$.

(I) a is real-valued;

(II) the range of g is a closed interval $[0, \mu]$ with $\mu > 0$, $g(0) = g(2\pi) = 0$, $g''(0) = g''(2\pi) > 0$, there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = \mu$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

Symbols in the class SL^α are known as *simple-loop symbols*. (I) is equivalent to the condition that all matrices $T_n(a)$ ($n \in \mathbb{Z}_+$) are Hermitian (self-adjoint). If $a \in W^\alpha$, then $g \in C^{\lfloor \alpha \rfloor}[0, 2\pi]$ where $\lfloor \alpha \rfloor$ is the integer part of α. So, the condition $a \in SL^\alpha$ with $\alpha \geq 1$ implies, in particular, that g belongs to $C^1[0, 2\pi]$.

In this work, for every $\alpha \geq 1$, we introduce a new class of symbols MSL^α (the modified simple loop class). Namely, $a \in \mathrm{MSL}^\alpha$ if $a \in \mathrm{SL}^\alpha$ and

(III) there exist functions $q_1, q_2 \in W^\alpha$ satisfying

$$a(t) = (t-1)q_1(t) \quad \text{and} \quad a(t) - a(e^{i\varphi_0}) = (t - e^{i\varphi_0})q_2(t). \quad (1)$$

It is easy to proof that, if $a \in W^\alpha$, then q_1 and q_2 both belong to $W^{\alpha-1}$, but we require the stronger condition (III) instead.

Symmetric case

$$g(s) = g(2\pi - s).$$

Let further $\lambda = g(s)$

$$\beta(\sigma, s) := \frac{(g(\sigma) - g(s))e^{is}}{(e^{i\sigma} - e^{is})(e^{-i\sigma} - e^{is})}$$
$$= \frac{g(s) - g(\sigma)}{4 \sin \frac{\sigma - s}{2} \sin \frac{\sigma + s}{2}}.$$

We will show that β is a continuous and positive function on $[0, 2\pi] \times [0, \pi]$. We define the function $\eta : [0, \pi] \to \mathbb{R}$ by

$$\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - s}{2}} \, d\sigma - \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma + s}{2}} \, d\sigma,$$

the integrals taken in the principal-value sense.

Theorem (3)

Let $\alpha \geq 1$ and $a \in \mathrm{MSL}^\alpha$. Then for every $n \geq 1$:

(I) the eigenvalues of $T_n(a)$ are all distinct: $\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}$;

(II) the numbers $s_j^{(n)}$, such that $\lambda_j^{(n)} = g(s_j^n)$ $(j = 1, \ldots, n)$ satisfy

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j), \qquad (2)$$

where $\Delta_1^{(n)}(j) = o\left(\frac{1}{n^{\alpha-1}}\right)$ as $n \to \infty$, uniformly in j;

(III) for every sufficiently large n, (2) has exactly one solution $s_j^{(n)} \in [0, \pi]$ for each $j = 1, \ldots, n$.

Theorem (4)

Let $d_j^{(n)} = \dfrac{\pi j}{n+1}$, then under the conditions of previous Theorem

$$\lambda_j^{(n)} = g(d_j^{(n)}) + \sum_{k=1}^{\lfloor \alpha \rfloor} \frac{r_k(d_j^{(n)})}{(n+1)^k} + \Delta_3^{(n)}(j),$$

where $\Delta_3^{(n)}$ is $o\left(\frac{1}{n^\alpha}(d_j^{(n)}(\pi - d_j^{(n)}))^{\alpha-1}\right)$ if $1 \leq \alpha < 2$ and $o\left(\frac{d_j^{(n)}}{n^\alpha}(\pi - d_j^{(n)})\right)$ if $\alpha \geq 2$ as $n \to \infty$, uniformly in j. The coefficients r_k can be calculated explicitly; in particular,

$$r_1(s) = -g'(s)\eta(s) \quad \text{and} \quad r_2(s) = \frac{1}{2}g''(s)\eta^2(s) + g'(s)\eta(s)\eta'(s).$$

Publications. Case of symbols with minimum of 4 order

1. M. Barrera and S.M. Grudsky. Asymptotics of Eigenvalues for Pentadiagonal Symmetric Toeplitz Matrices. Operator Theory: Advances and Applications, Volume 259, 2017, 51–77 pp.

2. M. Barrera and S.M. Grudsky. Asymptotics of eigenvalues for Toeplitz matrices with rational symbols that have a minimum of the 4th order. Complex Variable and Elliptic Equation, 2021, accepted.

Asymptotics of eigenvalues for Pentadiagonal Symmetric Toeplitz matrices

Symbol is

$$a(t) = \left(\frac{1}{t} + t - 2\right)^2 \quad (t = e^{i\varphi} \in \mathbb{T}), \tag{3}$$

and the generating function

$$g(\varphi) := a(e^{i\varphi}) = 4(\cos\varphi - 1)^2 = M\sin^4\left(\frac{\varphi}{2}\right), \tag{4}$$

where $\varphi \in [-\pi, \pi]$ and $M := 16$.

The function g has the following properties:

i) For all $\varphi \in [-\pi, \pi]$ we obtain that the range of g is a segment $[0, M]$ with $M := g(\pi) > 0$.

ii) $g(0) = 0$, $g(-\pi) = g(\pi) = M$, $g'(-\pi) = g'(\pi) = 0$, $g''(-\pi) = g''(\pi) < 0$, $g'(0) = g''(0) = g'''(0) = 0$ and $g^{(4)}(0) > 0$.

Introduce the parameters $d_1 := d_{2j}^{(n)} = \frac{2\pi j}{n+2}$ and $d_2 := d_{2j+1}^{(n)} = \frac{(2j+1)\pi}{n+2}$

$$u = -\arctan(f(d_1)\tanh(\Psi_j^{(1)}(u))), \ u \in \left[-\frac{\pi}{2}, 0\right], \tag{5}$$

and

$$w = -\operatorname{arccot}\left(\frac{1}{f(d_2)}\tanh(\Psi_j^{(2)}(w))\right), \ w \in \left[-\frac{\pi}{2}, 0\right], \tag{6}$$

where $\Psi_j^{(1)}(u) := \left(\frac{n+2}{2}\right)\beta(d_1) + \beta'(d_1)u$ and
$\Psi_j^{(2)}(w) := \left(\frac{n+2}{2}\right)\beta(d_2) + \beta'(d_2)w$.

$$f(\varphi) := \frac{\cos(\frac{\varphi}{2})}{\sqrt{1+\sin^2(\frac{\varphi}{2})}}, \quad \beta(\varphi) := 2\ln\left(\sin\left(\frac{\varphi}{2}\right) + \sqrt{1+\sin^2\left(\frac{\varphi}{2}\right)}\right)$$

Theorem

Let $a(t) = \left(\frac{1}{t} + t - 2\right)^2$ be the symbol of $T_n(a)$ and $\lambda_j^{(n)} = g\left(\varphi_j^{(n)}\right)$.
Then, it is obtained the following asymptotic formulas as $n \to \infty$

i)

$$\varphi_{2j-1}^{(n)} = d_1 + \frac{2u_{1,j}}{n+2} + \frac{4u_{2,j}}{(n+2)^2} + O\left(\frac{1}{(n+2)^3}\right), \tag{7}$$

where the rest does not depend of n and j, for $j = 1, \ldots, [\frac{n}{2}] + \frac{1-(-1)^n}{2}$.
Furthermore, the coefficient $u_{1,j}$ is the unique solution on $[-\frac{\pi}{2}, 0]$ of
the equation (5).
The coefficient $u_{2,j}$ is defined in terms of $u_{1,j}$.

Theorem

ii)

$$\varphi_{2j}^{(n)} = d_2 + \frac{2w_{1,j}}{n+2} + \frac{4w_{2,j}}{(n+2)^2} + O\left(\frac{1}{(n+2)^3}\right) \tag{8}$$

where the rest does not depend of n and j, for $j = 1, \ldots, [\frac{n}{2}]$.
Furthermore, the coefficient $w_{1,j}$ is the unique solution on $[-\frac{\pi}{2}, 0]$ of the equation (6).
The coefficient $w_{2,j}$ is defined in terms of $w_{1,j}$.

Theorem

Let $a(t) = \left(\frac{1}{t} + t - 2\right)^2$ be the symbol of $T_n(a)$.

i) If $d_1 \to 0$ as $n \to \infty$, then

$$\lambda_{2j-1}^{(n)} = \frac{(2\pi j + 2u_{1,j})^4}{(n+2)^4} + \frac{16u_{2,j}(2\pi j + 2u_{1,j})^3}{(n+2)^5} + O\left(\left(\frac{d_1}{n+2}\right)^3\right) + O\left(d_1^8\right), \tag{9}$$

where the rest does not depend of n and j. Furthermore, $u_{1,j}$ is the unique solution on $[-\frac{\pi}{2}, 0]$ of the equation (5), and the coefficient $u_{2,j}$ is defined in terms of $u_{1,j}$.

Theorem

ii) If $d_2 \to 0$ as $n \to \infty$, then

$$\lambda_{2j}^{(n)} = \frac{((2j+1)\pi + 2w_{1,j})^4}{(n+2)^4} + \frac{16w_{2,j}((2j+1)\pi + 2w_{1,j})^3}{(n+2)^5} + O\left(\left(\frac{d_2}{n+2}\right)^3\right) \tag{10}$$

$$+ O\left(d_2^8\right),$$

where the rest does not depend of n and j. Furthermore, $w_{1,j}$ is the unique solution on $[-\frac{\pi}{2}, 0]$ of the equation (6), and the coefficient $w_{2,j}$ is defined in terms of $w_{1,j}$.

Symmetric complexvalue symbol. Publications.

1. A.A. Batalshchikov, S.M. Grudsky, E. Ramírez de Arellano, V.A. Stukopin. Asymptotics of Eigenvectors of Large Symmetric Banded Toeplitz Matrices. Integral Equations and Operator Theory, Volume 83, Issue 3, 2015, 301-330 pp.

2. A. Batalshchikov, S. Grudsky, S. Mihalkovich, I. Malisheva, E. Ramírez de Arellano, and V. Stukopin. Asymptotics of eigenvalues of large symmetric Toeplitz matrices with smooth simple-loop symbols. Linea Algebra and Applications, Volume 580 (2019), 292-335 pp.

Symmetric complexvalue symbol

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \ a_j = a_{-j}.$$

$$a_1(t) = c_1 \sin(c_0 t^2) + \frac{1}{20} \left((1+t)^{5/2} + (1-t)^{5/2} \right), \ t \in [-\pi; \pi],$$

$$c_0 = \frac{1}{5} - \frac{1}{6}i, \ c_1 = \frac{(1-\pi)^{3/2} - (\pi+1)^{3/2}}{16\pi c_0 \cos(\pi^2 c_0)}$$

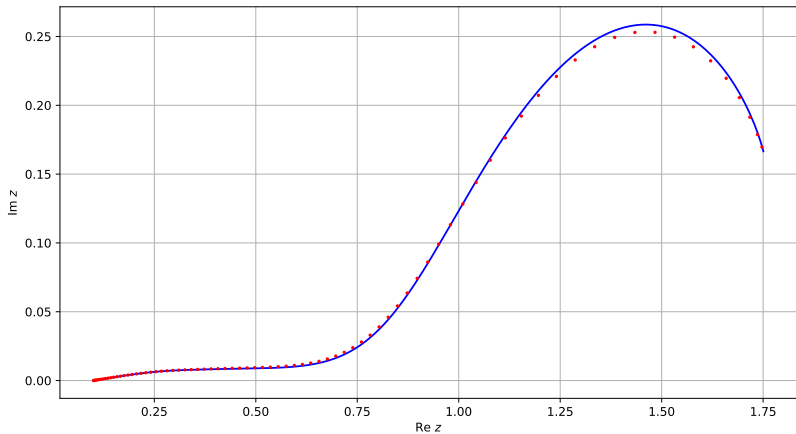Figure: Image of the symbol $a_1(t)$ and eigenvalues of the matrix $T_{80}(a_1)$

$$a(t) \in W^{\alpha}, \ \alpha \geq 2, \ g(\varphi) = a\left(e^{i\varphi}\right), \ \varphi \in [0, 2\pi].$$

$$\|a\|_{\alpha} := \sum_{j=-\infty}^{\infty} |a_j| \, (|j| + 1)^{\alpha} < \infty.$$

I. $g(\varphi) = g(2\pi - \varphi)$ – symmetric.

II. $\mathcal{R}(a)$ – is simple curve without self–intersection
$\mathcal{R}(a) = (M_0, M_1), \ M_0 \neq M_1$.

III. $g'(\varphi) \neq 0, \ \varphi \in (0, \pi)$.

IV. $g''(0) = g''(2\pi) \neq 0, \ g''(\pi) \neq 0$.

Main Idea

$$T_n(a) = T_n(a_n) \quad !$$

$$a_n(t) = \sum_{j=-(n-1)}^{n-1} a_j t^j, \ g_n(\psi) = a_n\left(e^{i\psi}\right),$$

where

$$\psi \in \Pi_n := \left\{ \psi = \varphi + i\delta \,|\, \varphi \in [c/n, \ \pi - c/n], \ \delta \in [-C/n, \ C/n] \right\}$$

with $c > 0$ and $C > 0$.

$$\mathcal{R}_n(a) =: \left\{ g_n(\psi) : \psi \in \Pi_n \right\}$$

Lemma (1)

Let $a(t) \in W^{\alpha}$, $\alpha \geq 2$, and satisfies condition 1.-2.-3.-4. Then the map

$$g_n(\psi) : \Pi_n \to \mathcal{R}_n(a) \text{ is bijection for } n \text{ large enough.}$$

Lemma (2)

Let $\psi = (\varphi + i\delta) \in \Pi_n$, $a(t) \in W^{\alpha}$, $\alpha \geq 0$, and $m = [\alpha]$. Then

$$g_n(\psi) = g(\varphi) + \sum_{k=1}^{m} \frac{g^{(k)}(\varphi)}{k!}(i\delta)^k + \sum_{k=0}^{m+1} \alpha_{n,k}(\varphi)(i\delta)^k,$$

where $\alpha_{n,k}(\varphi) \in W^0$, and

$$\|\alpha_{n,k}\|_0 = o(n^{k-\alpha}), \ k = 0, 1, \ldots, m$$

and

$$\|\alpha_{n,m+1}\|_0 = O(n^{m+1-\alpha}).$$

Let $g_n(\varphi_{1,n}(\lambda)) = \lambda$, then introduce following functions

$$\hat{b}_n(t,\lambda) = \frac{(a_n(t) - \lambda)e^{i\varphi_{1,n}(\lambda)}}{\left(t - e^{i\varphi_{1,n}(\lambda)}\right)\left(t^{-1} - e^{i\varphi_{1,n}(\lambda)}\right)}, \ \lambda \in \mathcal{R}_n(a),$$

$$\theta_n(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\log\hat{b}_n(\tau,\lambda)}{\tau - e^{i\varphi_{1,n}(\lambda)}}d\tau - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\log\hat{b}_n(\tau,\lambda)}{\tau - e^{-i\varphi_{1,n}(\lambda)}}d\tau, \ \lambda \in \mathcal{R}_n(a)$$

$$\eta_n(s) := \theta_n(g_n(s)), \ s \in \Pi_n.$$

Consider two sequences

$$d_{j,n} = \frac{\pi j}{n+1}, \ e_{j,n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1}.$$

Introduce small domains

$$\Pi_{j,n}(a) := \left\{s \in \Pi_n(a), \ |s - e_{j,n}| \leq \frac{c_n}{n+1}\right\},$$

where $c_n \to 0$.

Theorem (5)

Let $a \in \mathrm{CSL}^\alpha$, $\alpha \geq 2$. Then for sufficiently large natural number n the following statements hold:

1. all eigenvalues $T_n(a)$ are different, and $\lambda_{j,n} \in g(\Pi_{j,n}(a))$ for $j = 1, 2, ..., n$

2. values $s_{j,n}$ such that $\lambda_{j,n} = g_n(s_{j,n})$ satisfy the equation

$$(n+1)s + \eta_n(s) = \pi j + \Delta_n(s), \qquad j = 1, 2, ..., n \qquad (11)$$

with $|\Delta_n(s)| = o(1/n^{\alpha-2})$ where $\Delta_n(s) \to 0$ as $n \to \infty$ uniformly respect to $s \in \Pi_n(a)$.

3. Equation (11) have a unique solution in the domain $\Pi_{j,n}(a)$.

Under conditions of Theorem 5

$$\lambda_j^{(n)} = g(d_{j,n}) + \sum_{k=1}^{[\alpha]-1} \frac{r_k(d_{j,n})}{(n+1)^k} + \Delta_3^{(n)}(j) \tag{12}$$

where

$$\Delta_3^{(n)}(j) = \begin{cases} o\left(\frac{d(\pi-d)}{n}\right), & \alpha = 2, \\ O\left(\frac{d(\pi-d)}{n^{\alpha-1}}\right), & \alpha > 2. \end{cases}$$

as $n \to \infty$ uniformly in j and $d = d_{j,n}$. The coefficients r_k can be calculated explicitly; in particular

$$r_1(\varphi) = -g'(\varphi)\eta(\varphi) \quad \text{and} \quad r_2(\varphi) = \frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi).$$

Remark

For small j, $\left(j^2/n \to 0\right)$ we have following asymptotics from Theorem 6, for $\alpha \geq 2$,
$$\lambda_{j,n} = g(0) + C_3 \frac{j^2}{(n+1)^2} + o\left(\frac{j^2}{(n+1)^2}\right),$$
where
$$C_3 = \frac{\pi^2 g''(0)}{2}.$$

Location the eigenvalue relative to $\mathcal{R}(a)$

$$\lambda_{j,n} = g(d_{j,n}) - \frac{g'(d_{j,n})\,\eta(d_{j,n})}{n+1} + O\left(\frac{1}{n^2}\right).$$

Let $\widetilde{e_{j,n}} = d_{j,n} + \dfrac{\operatorname{Re}\eta(d_{j,n})}{n+1}$, then

$$\lambda_{j,n} = g(\widetilde{e_{j,n}}) + i\frac{g'(\widetilde{e_{j,n}})\,(\operatorname{Im}\eta(d_{j,n}))}{n+1} + O\left(\frac{1}{n^2}\right).$$

$\lambda_{j,n}$ is located on the normal to curve $\mathcal{R}(a)$ in the point $z = g(\widetilde{e_{j,n}})$ with exactitude to $O\left(\frac{1}{n^2}\right)$

$$\frac{|g'(e_{j,n})\operatorname{Im}\eta(d_{j,n})|}{n+1} + O\left(\frac{1}{n^2}\right) - \text{distance between } \lambda_{j,n} \text{ and } \mathcal{R}(a).$$

Numerical example

Approximations:

$$\lambda_{1,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1},$$

$$\lambda_{2,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1} + \frac{\frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi)}{(n+1)^2}.$$

Errors:

$$\Delta_1^{(n)} = \max_j \left| \frac{\lambda_{1,j} - \lambda_j}{\lambda_j} \right|,$$

$$\Delta_2^{(n)} = \max_j \left| \frac{\lambda_{2,j} - \lambda_j}{\lambda_j} \right|,$$

$$a_1(t) \in W^{2,5-\delta}, \ \forall \quad \delta > 0$$

n	20	40	80	160	320
$\Delta_1^{(n)}$	3.2e-03	8.8e-04	2.3e-04	5.9e-05	1.5e-05
$\Delta_2^{(n)}$	3.9e-04	5.6e-05	7.2e-06	9.2e-07	1.2e-07

1. H. Dai, Z. Geary, L.P. Kadanoff. Asymptotics of eigenvalues and eigenvectors of Toeplitz matrices. Journal of Statistical Mechanics: Theory and Experiment, May, 2009, PO5012.

2. J. M. Bogoya, Albrecht Böttcher and Sergei M. Grudsky. Asymptotics of individual eigenvalues of a class of large Hessenberg Toeplitz matrices. Operator Theory: Advances and Applications, 220 (2012), 77-95 pp.

3. A. Batalshchikov, S. Grudsky, S. Mihalkovich, I. Malisheva, E. Ramírez de Arellano, and V. Stukopin. Asymptotics of eigenvalues of large symmetric Toeplitz matrices with smooth simple-loop symbols. Linea Algebra and Applications, Volume 580 (2019), 292-335 pp.
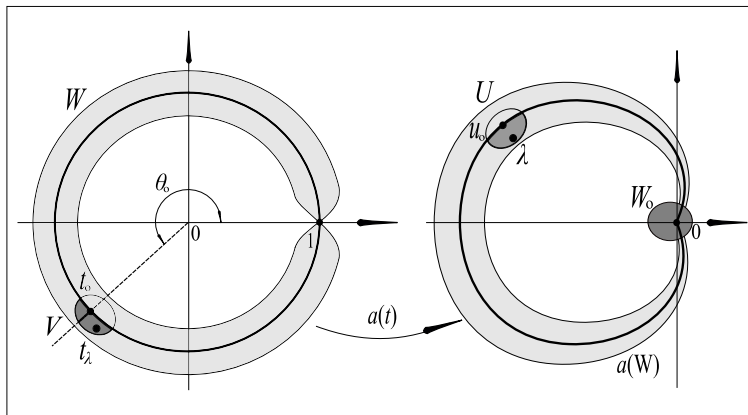
Figure: The map $a(t)$ over the unit circle.

Complex value case

$$a(t) = t^{-1}(1 - t)^{\alpha} f(t), \quad \alpha \in R_+ \backslash N$$

where

1. $f(t) \in H^{\infty} \cap C^{\infty}$.
2. f can be analytically extended to a neighborhood of $\mathbb{T} \backslash \{1\}$.
3. The range of the symbol a $\mathcal{R}(a)$ is a closed Jordan curve without loops and winding number -1 around each interior point.

Lemma (3)

Let $a(t) = t^{-1}h(t)$ be a symbol that satisfies the following conditions:

1. $h \in H^\infty$.
2. $\mathcal{R}(a)$ is a closed Jordan curve in $\mathbb{C}$ without loops.
3. $\text{wind}_\lambda(a) = -1$, for each λ in the interior of $\text{sp } T(a)$.

Then, for each λ in the interior of $\text{sp } T(a)$, we have the equality

$$D_n(a - \lambda) = (-1)^n h_o^{n+1} \left[\frac{1}{h(t) - \lambda t} \right]_n,$$

for every $n \in \mathbb{N}$.

Theorem (7)

We have the following asymptotic expression for λ_j:

$$\lambda_j = a(\omega_j) + (\alpha + 1)\omega_j a'(\omega_j)\frac{\log(n)}{n} + \frac{\omega_j a'(\omega_j)}{n}\log\left(\frac{a^2(\omega_j)}{c_o a'(\omega_j)\omega_j^2}\right)$$
$$+ \mathcal{O}\left(\frac{\log(n)}{n}\right)^2, \ n \to \infty,$$

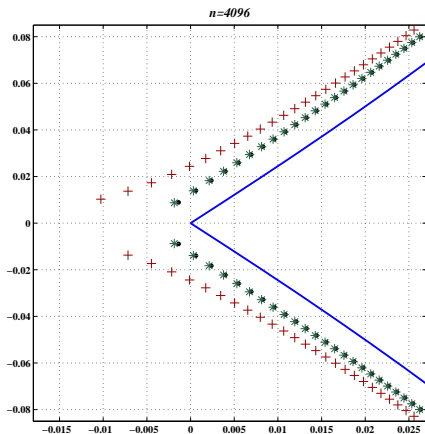where $\omega_j = \exp\left(-i\,\frac{2\pi j}{n}\right)$.

$$n = 4096$$

Figure: The solid blue line is the range of a. The black dots are sp $T_n(a)$ calculated by *Matlab*. The red crosses and the green stars are the approximations, for 1 and 2 terms respectively. Here we took $\alpha = 3/4$.

Eigenvectores of Toeplitz Matrices

1. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. On the Structure of the Eigenvectors of Large Hermitian Toeplitz Band Matrices. Operator Theory: Advances and Applications, 210 (2010), 15-36 pp.

2. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvectors of Hessenberg Toeplitz matrices and a problem by Dai Geary and Kadanoff. Linea Algebra and its Applications, 436, 2012, 3480-3492 pp.

3. A.A. Batalshchikov, S.M. Grudsky, E. Ramírez de Arellano, V.A. Stukopin. Asymptotics of Eigenvectors of Large Symmetric Banded Toeplitz Matrices. Integral Equations and Operator Theory, Volume 83, Issue 3, 2015, 301-330 pp.

4. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvectors of Hermitian Toeplitz matrices with smooth simple-loop symbols. Linear Algebra and its Applications, Volume 493, 2016, 606–637 pp.

Eigenvectors

Let $\lambda_j^{(n)}(j = 1, \ldots, n)$ be the eigenvalues of the Toeplitz matrix $T_n(a)$. For each j, there corresponds a unique $s_j^{(n)} \in [0, \pi]$ such that $\lambda_j^{(n)} = \psi\left(s_j^{(n)}\right)$. Put

$$z_1 := e^{i\sigma_1(s_j^{(n)})}, \quad z_2 := e^{i\sigma_2(s_j^{(n)})}. \tag{13}$$

For $\gamma \in \mathbb{R}$, we define z_k^γ as $e^{i\sigma_k(s_j^{(n)})\gamma}$. Given a function $f : \mathbb{T} \to \mathbb{C}$, let f_p be its p-th Fourier coefficient, and for a vector X, let X_p stand for its p-th component. Let $\theta = (\theta_p)_{p=0}^{n+1}$ be the vector in the first column of the matrix $T_{n+2}^{-1}\left(b\left(\cdot, s_j^{(n)}\right)\right)$. For $t \in \mathbb{C}$, we put

$$\theta(t) = \theta_0 + \theta_1 t + \ldots + \theta_{n+1} t^{n+1}. \tag{14}$$

Theorem

Let $a \in SL^{\alpha}$. The vector

$$X^{(n,j)} = M^{(n,j)} + L^{(n,j)} + R^{(n,j)} \tag{15}$$

whose p-th component, $p = 0, 1, \ldots, n-1$, is given by

$$M_p^{(n,j)} := z_1^{\frac{n-1}{2}-p}|\theta(z_1)| + (-1)^{n-j}z_2^{\frac{n-1}{2}-p}|\theta(z_2)|,$$

$$L_p^{(n,j)} := -\frac{z_1^{\frac{n+1}{2}}\overline{\theta(z_1)}}{2\pi i|\theta(z_1)|}\int_{\mathbb{T}}\left(\frac{\theta(t)-\theta(z_1)}{t-z_1} - \frac{\theta(t)-\theta(z_2)}{t-z_2}\right)\frac{dt}{t^{p+1}},$$

$$R_p^{(n,j)} := \overline{L_{n-p-1}^{(n,j)}}$$

is an eigenvector of $T_n(a)$ corresponding to the eigenvalue $\lambda_j^{(n)}$. Moreover, $M^{(n,j)}$ is conjugate symmetric, i.e. $M_p^{(n,j)} = \overline{M_{n-p-1}^{(n,j)}}$.

For n large enough and every $j \in \{1, \ldots, n\}$, we denoted by $\hat{s}_j^{(n)}$ the unique solution of the equation

$$(n+1)s + \eta(s) = \pi j, \tag{16}$$

where

$$\eta(s) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\log b(e^{i\sigma}, s)}{\tan \frac{\sigma - \sigma_2(s)}{2}} d\sigma - \frac{1}{4\pi} \int_0^{2\pi} \frac{\log b(e^{i\sigma}, s)}{\tan \frac{\sigma - \sigma_1(s)}{2}} d\sigma, \tag{17}$$

$$|s_j^{(n)} - \hat{s}_j^{(n)}| = o(1/n^{\alpha-1}), \quad n \to \infty. \tag{18}$$

Theorem

Let $a \in SL^{\alpha}$. For $k = 1, 2$, let $\hat{z}_k := e^{i\sigma_k(\hat{s}_j^{(n)})}$, and, for $p = 0, 1, \ldots, n - 1$, put

$$\hat{M}_p^{(n,j)} := \frac{\hat{z}_1^{\frac{n-1}{2}-p}}{|a'(\hat{z}_1)|^{\frac{1}{2}}} + (-1)^{n-j} \frac{\hat{z}_2^{\frac{n-1}{2}-p}}{|a'(\hat{z}_2)|^{\frac{1}{2}}},$$

$$\hat{L}_p^{(n,j)} := -\frac{\hat{z}_1^{\frac{n+1}{2}} b_+(\hat{z}_1)}{2\pi i |a'(\hat{z}_1)|^{\frac{1}{2}}} \int_{\mathbb{T}} \left(\frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_1)}{t - \hat{z}_1} - \frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_2)}{t - \hat{z}_2} \right) \frac{dt}{t^{p+1}},$$

$$\hat{R}_p^{(n,j)} := \overline{\hat{L}_{n-1-p}^{(n,j)}}.$$

Theorem

Then there is a vector $\Omega_1^{(n,j)}$ such that $[\Omega_1^{(n,j)}]_p = o(1/n^{\alpha-3})$ as $n \to \infty$, uniformly j and p, and such that

$$X^{(n,j)} = \hat{M}^{(n,j)} + \hat{L}^{(n,j)} + \hat{R}^{(n,j)} + \Omega_1^{(n,j)}, \qquad (19)$$

is a eigenvector of $T_n(a)$ corresponding to the eigenvalue $\lambda_j^{(n)}$.