Asymptotics of Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols

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This talk is based on joint work with
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The Talk presents higher-order asymptotic formulas for the eigenvalues of large Hermitian Toeplitz matrices with moderately smooth symbols which traceout a simple loop on the real line. The formulas are established not only for the extreme eigenvalues, but also for the inner eigenvalues. The results extend and make more precise existing results, which so far pertain to banded matrices or to matrices with infinitely differentiable symbols. Also given is a fixed-point equation for the eigenvalues which may be solved numerically by an iteration method.

We start with a complex-valued function a in L_1 on the unit circle $\mathbb{T}$, compute its Fourier coefficients

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\sigma}) e^{-ik\sigma} \, d\sigma, \quad k \in \mathbb{Z},$$

and consider the sequence $\{T_n(a)\}_{n=1}^{\infty}$ of the $n \times n$ Toeplitz matrices defined by $T_n(a) = (a_{j-k})_{j,k=0}^{n-1}$. The function a is referred to as the symbol of the sequence $\{T_n(a)\}_{n=1}^{\infty}$.

This talk addresses the asymptotic behavior of the eigenvalues of $T_n(a)$ as n goes to infinity. This is a topic which has attracted mathematicians and physicists for a century. Toeplitz matrices and their relatives emerge in particular in many problems of statistical physics, and there questions about the asymptotics of their spectral characteristics, especially their determinants, eigenvalues, and eigenvectors, are always at the heart of the matter.

$$n = 10^7 - 10^{12}$$

The collective asymptotic behavior of the eigenvalues of Hermitian Toeplitz matrices is described by the first Szegő limit theorem. This theorem says that, in a sense, the eigenvalues are asymptotically distributed as the values of a

$$a \in L_\infty, \ a \in L_1$$

In the Hermitian case, extensive work has also been done on the extreme eigenvalues of the matrices $T_n(a)$.

Despite the long-standing efforts and the flourishing interest in Toeplitz eigenvalues, many problems are still open, and the purpose of this talk is to present a nearly final solution of one of these problems: the individual eigenvalue asymptotics for Hermitian Toeplitz matrices with simple-loop symbols under reasonable smoothness requirements.

Results on the individual asymptotic behavior of the inner eigenvalues of Hermitian Toeplitz matrices were obtained only quite recently. These results were established under an assumption which will also be the basic assumption in this paper. Namely, we assume that a is a (real-valued) smooth function which traces out a simple loop only, that is, when t moves along $\mathbb{T}$, then $a(t)$ moves strictly monotonically from its minimum to its maximum and then strictly monotonically back to its minimum, without any rests in the minimum and the maximum (which includes that the second derivative at these points is non-zero).

I. Two parameters:

n- dimensions of matrices;

j- number of eigenvalue

$$1 \leq j \leq n$$

Asymptotics by n uniformly in j.

II. Distance between λ_j and λ_{j+1} is small:

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n}\right) \text{ – normal case}$$

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n^\gamma}\right) \text{ – special case}$$

$$\lambda_j = \lambda_{j+1} \qquad \text{– exceptional case}$$

We refer to [4] and [9], where, in addition, a is required to be a Laurent polynomial and a C^∞ function, respectively.

4. Böttcher, A., Grudsky, S., and Maksimenko, E.A.: Inside the eigenvalues of certain Hermitian Toeplitz band matrices. J. Comput. Appl. Math. 233, 2245–2264 (2010)
9. Deift, P., Its, A., and Krasovsky, I.: Eigenvalues of Toeplitz matrices in the bulk of the spectrum. Bull. Inst. Math. Acad. Sin. (N.S.) 7, 437–461 (2012)

The smoothness we will need here is significantly weaker. Moreover, our asymptotic formulas will contain precise estimates for the error terms, which show that the $o(1)$ in [9] actually is $o(1/n^k)$ for arbitrarily large k.

Main results

For $\alpha \geq 0$, we denote by W^α the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty.$$

Let m be the entire part of α. It is readily seen that if $a \in W^\alpha$ then the function g defined by $g(\sigma) := a(e^{i\sigma})$ is a 2π-periodic C^m function on $\mathbb{R}$. In what follows we consider real-valued simple-loop functions in W^α. To be more precise, for $\alpha \geq 2$, we let SL^α denote the set of all $a \in W^\alpha$ such that g has the following properties: the range of g is a segment $[0, M]$ with $M > 0$, $g(0) = g(2\pi) = 0$, $g''(0) = g''(2\pi) > 0$, and there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = M$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

Let $a \in SL^{\alpha}$. Then for each $\lambda \in [0, M]$, there are exactly one $\varphi_1(\lambda) \in [0, \varphi_0]$ such that $g(\varphi_1(\lambda)) = \lambda$ and exactly one $\varphi_2(\lambda) \in [\varphi_0, 2\pi]$ satisfying $g(\varphi_2(\lambda)) = \lambda$. For each $\lambda \in [0, M]$, the function g takes values less than or equal to λ on the segments $[0, \varphi_1(\lambda)]$ and $[\varphi_2(\lambda), 2\pi]$. Denote by $\varphi(\lambda)$ the arithmetic mean of the lengths of these two segments,

$$\varphi(\lambda) := \frac{1}{2}(\varphi_1(\lambda) - \varphi_2(\lambda)) + \pi = \frac{1}{2}\mu\{\sigma \in [0, 2\pi] : \ g(\sigma) \leq \lambda\},$$

where μ is the Lebesgue measure on $[0, 2\pi]$. The function $\varphi : [0, M] \to [0, \pi]$ is continuous and bijective. We let $\psi : [0, \pi] \to [0, M]$ stand for the inverse function.

Put

$$\sigma_1(s) = \varphi_1(\psi(s)) \text{ and } \sigma_2(s) = \varphi_2(\psi(s)).$$

Then

$$g(\sigma_1(s)) = g(\sigma_2(s)) = \psi(s).$$

Let further

$$\beta(\sigma, s) := \frac{(g(\sigma) - \psi(s))e^{is}}{(e^{i\sigma} - e^{i\sigma_1(s)})(e^{-i\sigma} - e^{-i\sigma_2(s)})} = \frac{\psi(s) - g(\sigma)}{4 \sin \frac{\sigma - \sigma_1(s)}{2} \sin \frac{\sigma - \sigma_2(s)}{2}}.$$

We will show that β is a continuous and positive function on $[0, 2\pi] \times [0, \pi]$. We define the function $\eta : [0, \pi] \to \mathbb{R}$ by

$$\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \sigma_2(s)}{2}} \, d\sigma - \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - \sigma_1(s)}{2}} \, d\sigma,$$

the integrals taken in the principal-value sense.

Theorem

Let $a \in SL^\alpha$ *with* $\alpha \geq 3$ *and let* $\lambda_1^{(n)} \leq \ldots \leq \lambda_n^{(n)}$ *be the eigenvalues of* $T_n(a)$*. If* n *is sufficiently large, then*

(i) *the eigenvalues of* $T_n(a)$ *are all distinct, i.e.,* $\lambda_1^{(n)} < \lambda_2^{(n)} < \ldots < \lambda_n^{(n)}$,

(ii) *the numbers* $s_j^{(n)} := \psi(\lambda_j^{(n)})$ $(j = 1, \ldots, n)$ *satisfy*

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j)$$

with $\Delta_1^{(n)}(j) = o(1/n^{\alpha-2})$ *as* $n \to \infty$*, uniformly with respect to* j*,*

(iii) *this equation has exactly one solution* $s_j^{(n)} \in [0, \pi]$ *for each* $j = 1, \ldots, n$*.*

To write down the individual asymptotics of the eigenvalues, we introduce the parameter

$$d := \frac{\pi j}{n+1}.$$

Note that the dependence of d on j and n is suppressed.

Theorem

Let $a \in SL^\alpha$ $(\alpha \geq 3)$ and let $s_j^{(n)}$ be as in the previous theorem. Then

$$s_j^{(n)} = d - \frac{\eta(d)}{n+1} + \frac{\eta(d)\eta'(d)}{(n+1)^2} + \Delta_2^{(n)}(j),$$

where $\Delta_2^{(n)}(j) = o(1/n^2)$ as $n \to \infty$, uniformly in $j = 1, \ldots, n$.

Theorem

Let $\alpha \geq 3$ *and* $a \in SL^{\alpha}$. *Then*

$$\lambda_j^{(n)} = \psi(d) + \frac{c_1(d)}{n+1} + \frac{c_2(d)}{(n+1)^2} + \Delta_3^{(n)}(j), \tag{1}$$

where $\Delta_3^{(n)}(j) = o(d(\pi - d)/n^2)$ *as* $n \to \infty$, *uniformly in* $j = 1, \ldots, n$, *and*

$$c_1(d) = -\psi'(d)\eta(d),$$
$$c_2(d) = \psi''(d)\eta^2(d)/2 + \psi'(d)\eta(d)\eta'(d).$$

We result for $\alpha \geq 3$. If $\alpha \geq 4$ we can write 4 terms of asymptotics expansion, if $\alpha \geq 5$ - 5 terms etc. For an other hand if $\alpha \geq 2$ we have only two terms in (1) with $\Delta_3^{(n)}(j) = o\left(d\left(\pi - d\right)/n\right)$.

Here is the result for the extreme eigenvalues.

Corollary

Let $a \in SL^\alpha$ *with some* $\alpha \geq 4$.
(i) *If* $j/(n+1) \to 0$ *then*

$$\lambda_j^{(n)} = \frac{c_5 j^2}{(n+1)^2} + \frac{c_6 j^2}{(n+1)^3} + \Delta_5^{(n)}(j),$$

where $c_5 = \pi^2 g''(0)/2$, $c_6 = -\pi^2 g''(0)\eta'(0)$, *and* $\Delta_5^{(n)}(j) = o(j/n^3)$ *as* $n \to \infty$.
(ii) *If* $j/(n+1) \to 1$ *then*

$$\lambda_j^{(n)} = M + \frac{c_7 (n+1-j)^2}{(n+1)^2} + \frac{c_8 (n+1-j)^2}{(n+1)^3} + \Delta_6^{(n)}(j),$$

where $c_7 = \pi^2 g''(\varphi_0)/2$, $c_8 = -\pi^2 g''(\varphi_0)\eta'(\pi)$, *and* $\Delta_6^{(n)}(j) = o(n+1-j/n^3)$ *as* $n \to \infty$.

This theorem is close to a result by Widom 1958, who considered the case where g is an even function and j is fixed.

Ideas of the proofs

We will derive an equation for the eigenvalues of $T_n(a)$. In our work 2010, the equation was obtained using Widom's formula for the determinant of a banded Toeplitz matrix. This approach does not work in our context.

We here pursue another approach. Instead of having recourse to Toeplitz determinants, we consider the equation for the eigenvectors, $T_n(a-\lambda)X = 0$, and represent $a - \lambda = a - \psi(s)$ as a product $p(\cdot, s)b(\cdot, s)$, where $b(\cdot, s)$ is positive and separated from zero and $p(\cdot, s)$ is a certain three-term Laurent polynomial which inherits the zeros of $a - \psi(s)$. After a slight transformation of the eigenvector equation, we multiply it by the inverse of $T_{n+2}(b(\cdot, s))$. This leads to something that might remind one of the procrustean bed: we get a vector of the length $n + 2$ and a "bed" of the length n (a subspace of dimension n), and our task is to make conclusions about the "head" and the "feet" of the vector. Eventually we arrive at a homogeneous system of two linear equations with two unknowns, the extreme coefficients of $Y = T_{n+2}(a - \psi(s))\chi_1 X$. As the determinant of that system has to be zero, we obtain an exact equation for the eigenvalues.

Lemma

Let $a \in SL^\alpha$ and $n \geq 1$. A number $\lambda = \psi(s)$ is an eigenvalue of $T_n(a)$ if and only if

$$e^{i(n+1)\sigma_2(s)}\Theta_{n+2}(e^{i\sigma_1(s)}, s)\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s) \\ -e^{i(n+1)\sigma_1(s)}\Theta_{n+2}(e^{i\sigma_2(s)}, s)\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s) = 0,$$

where, for every $k \geq 1$, the functions Θ_k and $\widehat{\Theta}_k$ are defined by

$$\Theta_k(t, s) := [T_k^{-1}(b(\cdot, s))\chi_0](t), \qquad \widehat{\Theta}_k(t, s) := [T_k^{-1}(\tilde{b}(\cdot, s))\chi_0](t^{-1}),$$

and $\tilde{b}(t, s) := b(1/t, s)$.

Proof. We are searching for all values of λ belonging to $[0, M]$ such that the equation $T_n(a)X = \lambda X$ has non-zero solutions X in $L_2^{(n)}$. Using the change of variable $\lambda = \psi(s)$ we can rewrite the latter equation as

$$T_n(a - \psi(s))X = 0. \tag{2}$$

Equation (2) is equivalent to

$$P_n b(\cdot, s) p(\cdot, s) X = 0, \tag{3}$$

where $p(t, s) := e^{-is}(t - e^{i\sigma_1(s)})(t^{-1} - e^{-i\sigma_2(s)})$. Multiply (3) by the function χ_1 to get

$$(P_{n+1} - P_1) b(\cdot, s) \chi_1 p(\cdot, s) X = 0. \tag{4}$$

Here $P_{n+1} - P_1$ is just one way to write the orthogonal projection of the space $L_2(\mathbb{T})$ onto the span of $\chi_1, \dots, \chi_n$. Note that $\chi_1 p(\cdot, s)X \in L_2^{(n+2)}$ and put

$$Y := T_{n+2}(a - \psi(s))\chi_1 X = P_{n+2} b(\cdot, s)\chi_1 p(\cdot, s)X = T_{n+2}(b(\cdot, s))\chi_1 p(\cdot, s)X.$$

Then (4) can be written as $(P_{n+1} - P_1)Y = 0$. This means that Y has the form

$$Y = y_0\chi_0 + y_{n+1}\chi_{n+1}.$$

Since $T_{n+2}(b(\cdot, s))$ is invertible, it follows that $T_{n+2}^{-1}(b(\cdot, s))Y = \chi_1 p(\cdot, s)X$, that is,

$$y_0[T_{n+2}^{-1}(b(\cdot, s))\chi_0](t) + y_{n+1}[T_{n+2}^{-1}(b(\cdot, s))\chi_{n+1}](t) = tp(t, s)X(t). \quad (5)$$

Now recall notation (5). Taking into account the identity

$$W_{n+2} T_{n+2}(b) W_{n+2} = T_{n+2}(\tilde{b}),$$

it is easy to verify that

$$[T_{n+2}^{-1}(b(\cdot, s))\chi_{n+1}](t) = t^{n+1}\widehat{\Theta}_{n+2}(t, s).$$

Therefore (5) can be written as

$$y_0 \Theta_{n+2}(t, s) + y_{n+1} t^{n+1} \widehat{\Theta}_{n+2}(t, s) = tp(t, s) X(t). \tag{6}$$

Thanks to the factor $p(t, s)$, the right-hand side vanishes at both $t = e^{i\sigma_1(s)}$ and $t = e^{i\sigma_2(s)}$. Consequently, y_0 and y_{n+1} must satisfy the homogeneous system of linear algebraic equations given by

$$\begin{aligned} \Theta_{n+2}(e^{i\sigma_1(s)}, s)y_0 + e^{i(n+1)\sigma_1(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s)y_{n+1} = 0, \\ \Theta_{n+2}(e^{i\sigma_2(s)}, s)y_0 + e^{i(n+1)\sigma_2(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s)y_{n+1} = 0. \end{aligned} \tag{7}$$

If $y_0 = y_{n+1} = 0$, then, by (6), the function X is zero. Therefore the initial equation (2) has a non-trivial solution X if and only if the determinant of system (7) is zero. □

Recall that $b_{\pm}(\cdot, s)$ are the Wiener-Hopf factors of $b(\cdot, s)$:

$$b(t,s) = b_+(t,s)\, b_-(t,s)$$

$$b_+(t,s) = \sum_{j=0}^{\infty} u_j(s) t^j \quad \text{and} \quad b_-(t,s) = \sum_{j=0}^{\infty} v_j(s) t^{-j}$$

$$T^{-1}(b(\cdot,s)) = b_+^{-1}(\cdot,s) P b_-^{-1}(\cdot,s),$$

$$[T^{-1}(b(\cdot,s))\chi_0](t) = [b_+^{-1}(\cdot,s) P b_-^{-1}(\cdot,s)](t) = b_+^{-1}(t,s).$$

$$e^{i(n+1)(\sigma_1(s)-\sigma_2(s))} = \frac{b_-(e^{i\sigma_1(s)}, s)b_+(e^{i\sigma_2(s)}, s)}{b_-(e^{i\sigma_2(s)}, s)b_+(e^{i\sigma_1(s)}, s)}(1 + R_5^{(n)}(s)),$$

$$\|R_5^{(n)}\|_\infty = o(1/n^{\alpha-2}) \quad \text{and} \quad \|R_5^{(n)\prime}\|_\infty = o(1/n^{\alpha-3}) \quad \text{as} \quad n \to \infty.$$

Taking into account that $\sigma_1(s) - \sigma_2(s) - 2\pi = 2s$

$$\exp(2i(n+1)s) = \exp(2i(\eta(s) + R_6^{(n)}(s))),$$

$$(n+1)s + \eta(s) + R_6^{(n)}(s) = \pi j,\ 1 \le j \le n. \tag{8}$$

Nonlinear equation

$$(n+1)s + \eta(s) = \pi j. \qquad (9)$$

Theorem

Let $n \geq n_1$. Then for each j, $1 \leq j \leq n$, equation (8) has a unique solution which coincides with $s_j^{(n)}$, and equation (9) has a unique solution in $[0,\pi]$, which we denote by $\hat{s}_j^{(n)}$. These solutions form strictly increasing sequences,

$$s_1^{(n)} < s_2^{(n)} < \ldots < s_n^{(n)}, \qquad \hat{s}_1^{(n)} < \hat{s}_2^{(n)} < \ldots < \hat{s}_n^{(n)}, \qquad (10)$$

with

$$|s_j^{(n)} - \hat{s}_j^{(n)}| = o(1/n^{\alpha-1}) \qquad (11)$$

uniformly in j.

Numerical tests

Denote by $\lambda_j^{(n,p)}$ the approximation of $\lambda_j^{(n)}$ resulting from any of our formulas (1) with p terms. For example we have

$$\lambda_j^{(n,1)} := \psi(d), \quad \lambda_j^{(n,2)} := \psi(d) + \frac{c_1(d)}{n+1}, \quad \lambda_j^{(n,3)} := \psi(d) + \frac{c_1(d)}{n+1} + \frac{c_2(d)}{(n+1)^2}.$$

For each $j = 1, \ldots, n$, put $\varepsilon_j^{(n,p)} := |\lambda_j^{(n)} - \lambda_j^{(n,p)}|$ and let $\varepsilon^{(n,p)}$ be the corresponding maximal error,

$$\varepsilon^{(n,p)} := \max\{\varepsilon_j^{(n,p)} : 1 \le j \le n\}.$$

Let $\hat{\lambda}_j^{(n)}$ denote the approximation of $\lambda_j^{(n)}$ obtained by fixed-point iterations and the relation $\hat{\lambda}_j^{(n)} = \psi(\hat{s}_j^{(n)})$, and let $\hat{\varepsilon}^{(n)}$ stand for the corresponding maximal error,

$$\hat{\varepsilon}^{(n)} := \max\{|\lambda_j^{(n)} - \hat{\lambda}_j^{(n)}| : 1 \le j \le n\}.$$

To indicate that $\varepsilon^{(n,p)} = O(1/(n+1)^p)$, we also calculate the normalized maximal errors $(n+1)^p \varepsilon^{(n,p)}$.

Example 1.

Consider the non-rational symbol

$$a(e^{i\sigma}) = g(\sigma) = g_2\sigma^2 + g_3\sigma^3 + g_4\sigma^{4+\beta} + g_5\sigma^5 + g_6\sigma^6 + g_7\sigma^7, \quad \sigma \in [0, 2\pi],$$

where $\beta \in [0, 1)$ and the coefficients $g_2, \dots, g_7$ are chosen in such a manner that

$$g(2\pi) = g'(2\pi) = 0 \qquad \text{and} \qquad g^{(k)}(2\pi) = g^{(k)}(0) \quad \text{for} \quad k = 2, 3, 4.$$

Elementary computations yield

$$\begin{aligned}
g_2 &= (24 - 38\beta + 13\beta^2 + 2\beta^3 - \beta^4)/(2\pi)^2, \\
g_3 &= (24 - 50\beta + 35\beta^2 - 10\beta^3 + \beta^4)/(2\pi)^3, \\
g_4 &= 240/(2\pi)^{4+\beta}, \\
g_5 &= (360 + 42\beta - 201\beta^2 + 42\beta^3 + 3\beta^4)/(2\pi)^5, \\
g_6 &= (-216 + 66\beta + 209\beta^2 - 54\beta^3 - 5\beta^4)/(2\pi)^6, \\
g_7 &= (48 - 20\beta - 50\beta^2 + 20\beta^3 + 2\beta^4)/(2\pi)^7.
\end{aligned}$$

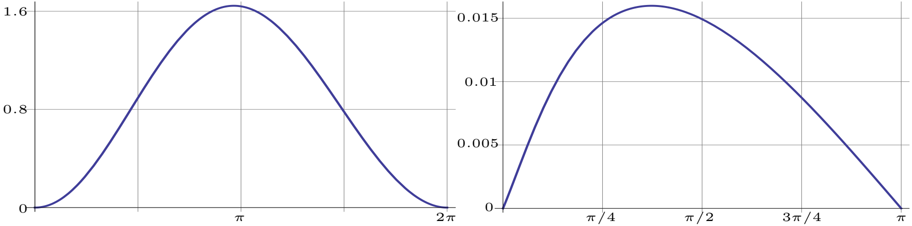

Figure: Graph of $g(\sigma) = a(e^{i\sigma})$ (left), and $\eta(s)$ (right) for $\beta = 1/5$.

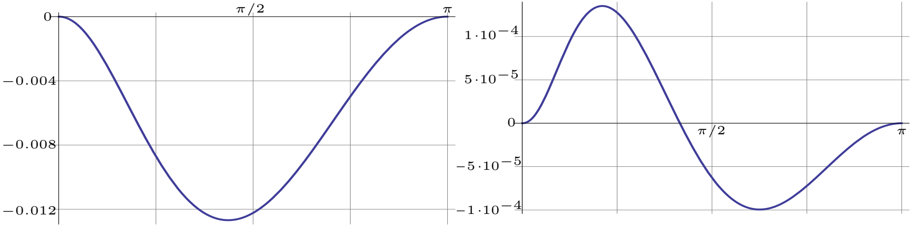

Figure: The functions $c_1(d)$ (left) and $c_2(d)$ (right).

n	64	128	512	1024	2048	4096
$\varepsilon^{(n,1)}$	$2.0\cdot 10^{-4}$	$9.8\cdot 10^{-5}$	$2.5\cdot 10^{-5}$	$1.2\cdot 10^{-5}$	$6.2\cdot 10^{-6}$	$3.1\cdot 10^{-6}$
$(n+1)\varepsilon^{(n,1)}$	$1.3\cdot 10^{-2}$	$1.3\cdot 10^{-2}$	$1.3\cdot 10^{-2}$	$1.3\cdot 10^{-2}$	$1.3\cdot 10^{-2}$	$1.3\cdot 10^{-2}$
$\varepsilon^{(n,2)}$	$3.2\cdot 10^{-8}$	$8.1\cdot 10^{-9}$	$5.1\cdot 10^{-10}$	$1.3\cdot 10^{-10}$	$3.2\cdot 10^{-11}$	$8.1\cdot 10^{-12}$
$(n+1)^2\varepsilon^{(n,2)}$	$1.4\cdot 10^{-4}$	$1.4\cdot 10^{-4}$	$1.4\cdot 10^{-4}$	$1.4\cdot 10^{-4}$	$1.4\cdot 10^{-4}$	$1.4\cdot 10^{-4}$
$\varepsilon^{(n,3)}$	$2.3\cdot 10^{-10}$	$1.3\cdot 10^{-11}$	$4.1\cdot 10^{-14}$	$2.2\cdot 10^{-15}$	$2.4\cdot 10^{-16}$	$3.0\cdot 10^{-17}$
$(n+1)^3\varepsilon^{(n,3)}$	$6.2\cdot 10^{-5}$	$2.8\cdot 10^{-5}$	$5.5\cdot 10^{-6}$	$2.4\cdot 10^{-6}$	$2.1\cdot 10^{-6}$	$2.1\cdot 10^{-6}$
$\hat{\varepsilon}^{(n)}$	$2.3\cdot 10^{-10}$	$1.3\cdot 10^{-11}$	$4.1\cdot 10^{-14}$	$2.2\cdot 10^{-15}$	$1.2\cdot 10^{-16}$	$6.7\cdot 10^{-18}$
$(n+1)^{4.2}\hat{\varepsilon}^{(n)}$	$9.3\cdot 10^{-3}$	$9.6\cdot 10^{-3}$	$9.8\cdot 10^{-3}$	$9.8\cdot 10^{-3}$	$9.9\cdot 10^{-3}$	$9.9\cdot 10^{-3}$

Table: Maximum errors and normalized maximum errors for the eigenvalues of $T_n(a)$ obtained with our formula (1), $\varepsilon^{(n,p)}$ with $p = 1, 2, 3$, and by fixed-point iterations, $\hat{\varepsilon}^{(n)}$, for different values of n. The data were obtained by comparison with the solutions given by *Wolfram Mathematica*.

Note that Table 1 shows that $\hat{\varepsilon}^{(n)} = O(1/(n+1)^{4.2})$ as $n \to \infty$.

Example 2.

In this example the generating symbol g is a symmetric polynomial spline.

$$\mathcal{L}(x) = \frac{315}{128}\,x^2 - \frac{105}{32}\,x^4 + \frac{189}{64}\,x^6 - \frac{45}{32}\,x^8 + \frac{35}{128}\,x^{10}.$$

It luckily turns out that $\mathcal{L}'(x) > 0$ for all $x \in (0,1)$. Let $\rho \in (0,1)$ be a fixed parameter and define $g\colon [0,2\pi] \to \mathbb{R}$ by

$$g(\sigma) = \begin{cases} \rho\,\mathcal{L}\left(\frac{\sigma}{\rho\pi}\right), & 0 \le \sigma < \rho\pi, \\ 1 - (1-\rho)\mathcal{L}\left(\frac{\sigma-\pi}{(1-\rho)\pi}\right), & \rho\pi \le \sigma < (2-\rho)\pi, \\ \rho\,\mathcal{L}\left(\frac{\sigma-2\pi}{\rho\pi}\right), & (2-\rho)\pi \le \sigma \le 2\pi. \end{cases}$$

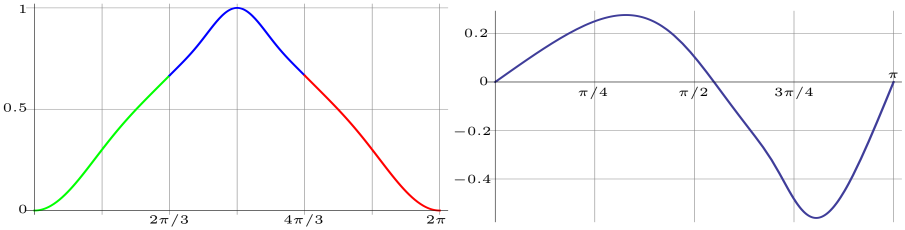

Figure: Graph of $g(\sigma) = a(e^{i\sigma})$ (left), and $\eta(s)$ (right) for $\rho = 2/3$. The green, blue, and red curves represent the first, second, and third pieces, respectively.

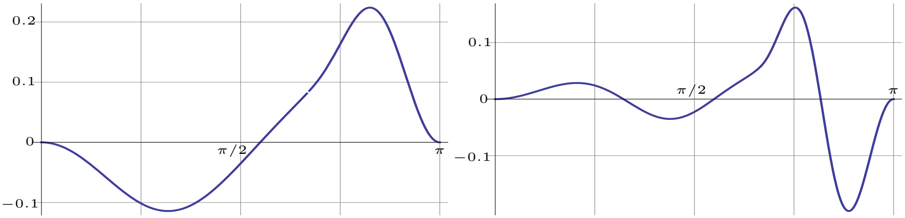

Figure: Functions $c_1(d)$ (left) and $c_2(d)$ (right).

n	64	128	512	1024	2048	4096
$\varepsilon^{(n,1)}$	$6.8\cdot10^{-3}$	$3.4\cdot10^{-3}$	$1.7\cdot10^{-3}$	$4.4\cdot10^{-4}$	$2.2\cdot10^{-4}$	$1.1\cdot10^{-4}$
$(n+1)\varepsilon^{(n,1)}$	$2.2\cdot10^{-1}$	$2.2\cdot10^{-1}$	$2.2\cdot10^{-1}$	$2.2\cdot10^{-1}$	$2.2\cdot10^{-1}$	$2.2\cdot10^{-1}$
$\varepsilon^{(n,2)}$	$1.8\cdot10^{-4}$	$4.6\cdot10^{-5}$	$1.2\cdot10^{-5}$	$7.5\cdot10^{-7}$	$1.9\cdot10^{-7}$	$4.7\cdot10^{-8}$
$(n+1)^2\varepsilon^{(n,2)}$	$1.9\cdot10^{-1}$	$2.0\cdot10^{-1}$	$2.0\cdot10^{-1}$	$2.0\cdot10^{-1}$	$2.0\cdot10^{-1}$	$2.0\cdot10^{-1}$
$\varepsilon^{(n,3)}$	$9.8\cdot10^{-6}$	$1.3\cdot10^{-6}$	$1.7\cdot10^{-7}$	$2.7\cdot10^{-9}$	$3.4\cdot10^{-10}$	$4.2\cdot10^{-11}$
$(n+1)^3\varepsilon^{(n,3)}$	$3.5\cdot10^{-1}$	$3.6\cdot10^{-1}$	$3.6\cdot10^{-1}$	$3.6\cdot10^{-1}$	$3.6\cdot10^{-1}$	$3.6\cdot10^{-1}$
$\hat{\varepsilon}^{(n)}$	$7.5\cdot10^{-8}$	$1.9\cdot10^{-9}$	$9.2\cdot10^{-11}$	$9.4\cdot10^{-14}$	$2.0\cdot10^{-15}$	$9.3\cdot10^{-17}$
$(n+1)^5\hat{\varepsilon}^{(n)}$	3.0	2.2	3.3	3.3	2.2	3.3

Table: Maximum errors and normalized maximum errors for the eigenvalues of $T_n(a)$ obtained with our formula (1), $\varepsilon^{(n,p)}$ with $p = 1, 2, 3$, and by fixed-point iterations, $\hat{\varepsilon}^{(n)}$, for different values of n. The data were obtained by comparison with the solutions given by *Wolfram Mathematica*.