Reaction-diffusion Equations on Complex Networks and Turing Patterns, via p-Adic Analysis. II.

W. A. Zúñiga-Galindo

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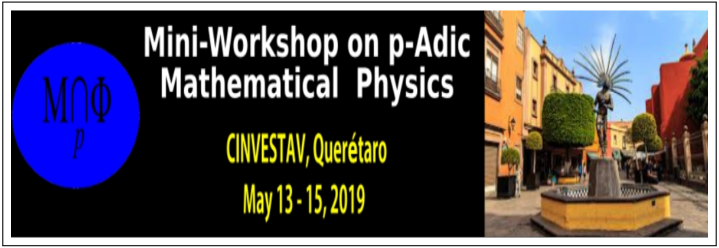


Some additional function spaces and operators

Let M be a positive integer satisfying $M \geq N$. We fix a system of representatives I_js for the quotient

$$G_I^M := \left(I + p^N \mathbb{Z}_p\right) / p^M \mathbb{Z}_p.$$

This means that

$$B_{-N}(I) = \bigsqcup_{I_j \in G_I^M} B_{-M}\left(I_j\right),$$

where $B_{-L}(J) = \left\{x \in \mathbb{Q}_p ; |x - J|_p \leq p^{-L}\right\}$. Now, we set

$$G_N^M := \bigsqcup_{I \in G_N^0} G_I^M.$$

Some additional function spaces and operators

Since $\mathcal{K}_N$ is the disjoint union of the $I + p^N\mathbb{Z}_p$, for $I \in G_N^0$,

$$\mathcal{K}_N = \bigsqcup_{I \in G_N^0} \bigsqcup_{I_j \in G_I^M} I_j + p^M\mathbb{Z}_p = \bigsqcup_{I_j \in G_N^M} I_j + p^M\mathbb{Z}_p.$$

We set X_M, $M \geq N$, to be the $\mathbb{R}$-vector space of all the test functions supported in $\mathcal{K}_N$ of the form

$$\varphi(x) = \sum_{I_j \in G_N^M} \varphi(I_j)\,\Omega\left(p^M |x - I_j|_p\right), \ \varphi(I_j) \in \mathbb{R},$$

endowed with the $\|\cdot\|_\infty$-norm. This is a real Banach space.

From now on, we set $X_\infty := C(\mathcal{K}_N, \mathbb{R})$ endowed with the $\|\cdot\|_\infty$-norm. This is also a real Banach space.

Some additional function spaces and operators

For $M \geq N$, we define $\mathbf{P}_M \in \mathfrak{B}(X_\infty, X_M)$, the bounded linear operators from X_∞ into X_M, as

$$\mathbf{P}_M \varphi(x) = \sum_{I_j \in G_N^M} \varphi(I_j)\, \Omega\left(p^M |x - I_j|_p\right). \tag{1}$$

We denote by $\mathbf{E}_M : X_M \hookrightarrow X_\infty$, $M \geq N$, the natural continuous embedding, notice that $\|\mathbf{E}_M\| \leq 1$, and that $\mathbf{P}_M \mathbf{E}_M \varphi = \varphi$ for $\varphi \in X_M$, $M \geq N$.
Whenever be possible, we will omit in our formulas operator $\mathbf{E}_M$, instead we will use the fact that $X_M \hookrightarrow X_\infty$, $M \geq N$.

Lemma

With the above notation, the following assertions hold: (i) $\|\mathbf{P}_M\| \leq 1$*; (ii)* $\lim_{M\to\infty} \|\mathbf{P}_M\varphi - \varphi\|_\infty = 0$ *for* $\varphi \in X_\infty$*.*

We now consider the real Banach spaces $X_\infty \oplus X_\infty$, $X_M \oplus X_M$ for $M \geq N$, endowed with the norm $\|u \oplus v\| := \max\{\|u\|_\infty, \|v\|_\infty\}$. We will identify $u \oplus v$ with the column vector $\begin{bmatrix} u \\ v \end{bmatrix}$.

Conditions on the nonlinearity

With respect to the nonlinearity we assume the following. We fix a, $b \in \mathbb{R}$, with $a < b$, and assume that

$$\begin{cases} \text{(i)} & f, g : (a, b) \times (a, b) \to \mathbb{R}; \\ \text{(ii)} & f, g \in C^1\left((a, b) \times (a, b)\right); \\ \text{(iii)} & \nabla f(x, y) \neq 0 \text{ and } \nabla g(x, y) \neq 0 \text{ for any } (x, y) \in (a, b) \times (a, b). \end{cases} \quad \text{(Hypothesis 1)}$$

Conditions on the nonlinearity

Now we define

$$U = \{v \in X_\infty ; a < v(x) < b \text{ for any } x \in \mathcal{K}_N\}. \tag{2}$$

Notice that U is an open set in X_∞. Indeed, take $\delta > 0$ sufficiently small and $v \in U$, if

$$h \in B(v, \delta) = \{h \in X_\infty ; \|v - h\|_\infty < \delta\},$$

then

$$a < -\delta + \min_{x \in \mathcal{K}_N} v(x) < h(x) < \delta + \max_{x \in \mathcal{K}_N} v(x) < b,$$

for δ sufficiently small.

Conditions on the nonlinearity

By $\begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}$, with $u \oplus v \in U \oplus U$, we mean the mapping

$$\begin{array}{rccc} \begin{bmatrix} f \\ g \end{bmatrix}: & U \oplus U & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\ & & & \\ & u \oplus v & \longrightarrow & f(u,v) \oplus g(u,v). \end{array} \tag{3}$$

Two Cauchy problems

We denote by $\varepsilon\mathbf{L}\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ the operator acting on $X_\infty \oplus X_\infty$ as

$$\varepsilon\mathbf{L}\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \varepsilon\mathbf{L}u \\ \varepsilon d\mathbf{L}v \end{bmatrix}.$$

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} f(u(t), v(t)) \\ g(u(t), v(t)) \end{bmatrix} + \begin{bmatrix} \varepsilon\mathbf{L}u(t) \\ \varepsilon d\mathbf{L}v(t) \end{bmatrix}, \\ t \in [0, \tau), \; x \in \mathcal{K}_N; \\ u(0) \oplus v(0) = u_0 \oplus v_0 \in U \oplus U. \end{cases} \tag{4}$$

Two Cauchy problems

The Cauchy problem for the following discretization of (4), with $\mathbf{L}_M = \mathbf{L}\mid_{X_M}$:

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u^{(M)}(t) \\ v^{(M)}(t) \end{bmatrix} = \begin{bmatrix} f(u^{(M)}(t), v^{(M)}(t)) \\ g(u^{(M)}(t), v^{(M)}(t)) \end{bmatrix} + \begin{bmatrix} \varepsilon \mathbf{L}_M u^{(M)}(t) \\ \varepsilon d \mathbf{L}_M v^{(M)}(t) \end{bmatrix}, \\ t \in [0, \tau), \ x \in \mathcal{K}_N; \\ u^{(M)}(0) \oplus v^{(M)}(0) \in U \cap X_M \oplus U \cap X_M. \end{cases} \tag{5}$$

The Cauchy problem in X.

We use the following notation:

$$X_\bullet := \begin{cases} X_\infty & \text{if} \quad \bullet = \infty \\ X_M & \text{if} \quad \bullet = M \end{cases}, \qquad \mathbf{L}_\bullet := \begin{cases} \mathbf{L} & \text{if} \quad \bullet = \infty \\ \mathbf{L}_M & \text{if} \quad \bullet = M \end{cases},$$

and $u^{(\bullet)}(t) \oplus v^{(\bullet)}(t)$ means $u(t) \oplus v(t)$ if $\bullet = \infty$. By using this notation the Cauchy problems (4)-(5), with initial data in $U \cap X_N \oplus U \cap X_N$, can be written as

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u^{(\bullet)}(t) \\ v^{(\bullet)}(t) \end{bmatrix} = \begin{bmatrix} f(u^{(\bullet)}(t), v^{(\bullet)}(t)) \\ g(u^{(\bullet)}(t), v^{(\bullet)}(t)) \end{bmatrix} + \begin{bmatrix} \varepsilon \mathbf{L}_\bullet u^{(\bullet)}(t) \\ \varepsilon d \mathbf{L}_\bullet v^{(\bullet)}(t) \end{bmatrix}, \\ t \in [0, \tau), \, x \in \mathcal{K}_N; \\ u^{(\bullet)}(0) \oplus v^{(\bullet)}(0) \in U \cap X_N \oplus U \cap X_N. \end{cases} \tag{6}$$

Formal definition [edit]

A **strongly continuous semigroup** on a Banach space X is a map $T : \mathbb{R}_+ \to L(X)$ such that

1. $T(0) = I$, (identity operator on X)
2. $\forall t, s \geq 0 : \ T(t+s) = T(t)T(s)$
3. $\forall x_0 \in X : \ \|T(t)x_0 - x_0\| \to 0$, as $t \downarrow 0$.

The first two axioms are algebraic, and state that T is a representation of the semigroup $(\mathbb{R}_+, +)$; the last is topological, and states that the map T is continuous in the strong operator topology.

Infinitesimal generator [edit]

The **infinitesimal generator** A of a strongly continuous semigroup T is defined by

$$A\,x = \lim_{t\downarrow 0} \frac{1}{t}\,(T(t) - I)\,x$$

whenever the limit exists. The domain of A, $D(A)$, is the set of $x \in X$ for which this limit does exist; $D(A)$ is a linear subspace and A is linear on this domain.[1] The operator A is closed, although not necessarily bounded, and the domain is dense in X.[2]

The strongly continuous semigroup T with generator A is often denoted by the symbol e^{At}. This notation is compatible with the notation for matrix exponentials, and for functions of an operator defined via functional calculus (for example, via the spectral theorem).

Some Wikipedia

Abstract Cauchy problems [edit]

Consider the abstract Cauchy problem:

$$u'(t) = Au(t), \quad u(0) = x,$$

where A is a closed operator on a Banach space X and $x \in X$. There are two concepts of solution of this problem:

- a continuously differentiable function $u:[0,\infty)\to X$ is called a **classical solution** of the Cauchy problem if $u(t) \in D(A)$ for all $t > 0$ and it satisfies the initial value problem,
- a continuous function $u:[0,\infty) \to X$ is called a **mild solution** of the Cauchy problem if

$$\int_0^t u(s)\,ds \in D(A) \text{ and } A\int_0^t u(s)\,ds = u(t) - x.$$

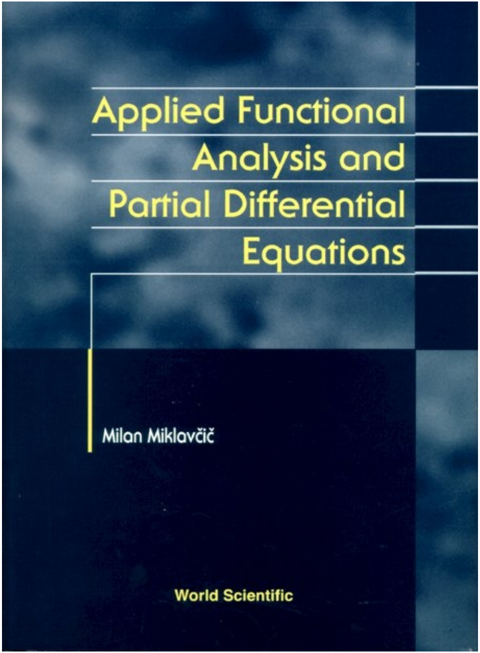
Any classical solution is a mild solution. A mild solution is a classical solution if and only if it is continuously differentiable.[4]

The following theorem connects abstract Cauchy problems and strongly continuous semigroups.

Theorem[5] Let A be a closed operator on a Banach space X. The following assertions are equivalent:

1. for all $x \in X$ there exists a unique mild solution of the abstract Cauchy problem,
2. the operator A generates a strongly continuous semigroup,
3. the resolvent set of A is nonempty and for all $x \in D(A)$ there exists a unique classical solution of the Cauchy problem.

When these assertions hold, the solution of the Cauchy problem is given by $u(t) = T(t)x$ with T the strongly continuous semigroup generated by A.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Mild solutions

We use the following conditions:

Condition AS1

$X_\bullet \oplus X_\bullet$ is a real Banach space.

Condition AS2

The operator $\begin{bmatrix} \varepsilon \mathbf{L}_\bullet \\ \varepsilon d \mathbf{L}_\bullet \end{bmatrix}$ is the generator of a strongly continuous semigroup $\left\{ e^{\varepsilon t \mathbf{L}_\bullet} \right\}_{t \geq 0} \oplus \left\{ e^{\varepsilon d t \mathbf{L}_\bullet} \right\}_{t \geq 0}$ satisfying

$$\left\| e^{\varepsilon t \mathbf{L}_\bullet} \oplus e^{\varepsilon d t \mathbf{L}_\bullet} \right\| \leq 1 \text{ for } t \geq 0,$$

Condition AS3

Let $U \subset X_\infty$ be the open set defined in (2), and let

$$\begin{bmatrix} f \\ g \end{bmatrix} : (U \oplus U) \to X_\infty \oplus X_\infty$$

be the continuous mapping defined in (3). Then for each $u_0 \oplus v_0 \in U \oplus U$, there exist $\delta > 0$ and $L < \infty$ such that

$$\left\| \begin{bmatrix} f(u_1, v_1) \\ \\ g(u_1, v_1) \end{bmatrix} - \begin{bmatrix} f(u_2, v_2) \\ \\ g(u_2, v_2) \end{bmatrix} \right\| \leq L \left\| (u_1 - u_2) \oplus (v_1 - v_2) \right\|, \qquad (7)$$

for $u_1 \oplus v_1$, $u_2 \oplus v_2$ in the ball $B(u_0 \oplus v_0, \delta)$.

Take

$$\begin{bmatrix} f \\ g \end{bmatrix} : (U \cap X_M \oplus U \cap X_M) \to X_M \oplus X_M, \qquad (8)$$

since $X_M \hookrightarrow X_\infty$, condition (7) holds for map (8).

Mild solutions

Definition

For $\tau_0 \in (0, \tau]$, let $\mathcal{S}_{\text{Mild}}\left(\tau_0, X_\bullet \oplus X_\bullet\right)$ be the collection of all $u^{(\bullet)} \oplus v^{(\bullet)} \in C\left([0, \tau_0), U \cap X_\bullet \oplus U \cap X_\bullet\right)$ which satisfy

$$\int_0^t u^{(\bullet)}(s)\, ds \in Dom\left(\varepsilon \mathbf{L}_\bullet\right) = X_\bullet \quad \text{and} \quad \int_0^t v^{(\bullet)}(s)\, ds \in Dom\left(\varepsilon d \mathbf{L}_\bullet\right) = X_\bullet,$$

and

$$\begin{cases} u^{(\bullet)}(t) - u^{(\bullet)}(0) + \varepsilon \mathbf{L}_\bullet \int_0^t u^{(\bullet)}(s)\, ds &= \int_0^t f\left(u^{(\bullet)}(s), v^{(\bullet)}(s)\right) ds \\ v^{(\bullet)}(t) - v^{(\bullet)}(0) + \varepsilon d \mathbf{L}_\bullet \int_0^t v^{(\bullet)}(s)\, ds &= \int_0^t g\left(u^{(\bullet)}(s), v^{(\bullet)}(s)\right) ds, \end{cases}$$

for $t \in [0, \tau_0)$. The elements of $\mathcal{S}_{\text{Mild}}\left(\tau_0, X_\bullet \oplus X_\bullet\right)$ are the called mild solutions of (6).

Mild solutions

By using well-known results from semigroup theory, we have $u^{(\bullet)} \oplus v^{(\bullet)} \in \mathcal{S}_{\text{Mild}}(\tau_0, X_\bullet \oplus X_\bullet)$ if and only if

$$u^{(\bullet)} \oplus v^{(\bullet)} \in C\left([0, \tau_0), U \cap X_\bullet \oplus U \cap X_\bullet\right) \tag{9}$$

and

$$\begin{cases} u^{(\bullet)}(t) &= e^{\varepsilon t \mathbf{L}_\bullet} u^{(\bullet)}(0) + \int_0^t e^{\varepsilon(t-s)\mathbf{L}_\bullet} f\left(u^{(\bullet)}(s), v^{(\bullet)}(s)\right) ds \\ v^{(\bullet)}(t) &= e^{\varepsilon d t \mathbf{L}_\bullet} v^{(\bullet)}(0) + \int_0^t e^{\varepsilon d(t-s)\mathbf{L}_\bullet} g\left(u^{(\bullet)}(s), v^{(\bullet)}(s)\right) ds, \end{cases} \tag{10}$$

for $t \in [0, \tau_0)$.

The following result shows that Hypothesis 1, which also implies Condition AS3, implies that any mild solution is a classical solution.

Lemma

$\mathcal{S}_{Mild}(\tau_0, X_\bullet \oplus X_\bullet) \subset C^1\left([0, \tau_0), U \cap X_\bullet \oplus U \cap X_\bullet\right)$.

Theorem

For each $u_0^{(\bullet)} \oplus v_0^{(\bullet)} \in U \cap X_N \oplus U \cap X_N$, *there exists* $\tau_{u_0^{(\bullet)} \oplus v_0^{(\bullet)}} \in (0, \tau)$ *and* $u^{(\bullet)} \oplus v^{(\bullet)} \in \mathcal{S}_{Mild}\left(\tau_{u_0^{(\bullet)} \oplus v_0^{(\bullet)}}, X_\bullet \oplus X_\bullet\right)$ *such that* $u^{(\bullet)}(0) \oplus v^{(\bullet)}(0) = u_0^{(\bullet)} \oplus v_0^{(\bullet)}$. *Furthermore,*

$$\lim_{k \to \infty} \sup_{0 \le t \le \tau_{u_0^{(\bullet)} \oplus v_0^{(\bullet)}}} \left\| u_k^{(\bullet)}(t) \oplus v_k^{(\bullet)}(t) - u^{(\bullet)}(t) \oplus v^{(\bullet)}(t) \right\| = 0,$$

where $u_k^{(\bullet)} \oplus v_k^{(\bullet)} \in C(\left[0, \tau_{u_0^{(\bullet)} \oplus v_0^{(\bullet)}}\right], U \cap X_\bullet \oplus U \cap X_\bullet)$ *are defined by* $u_1^{(\bullet)}(t) \oplus v_1^{(\bullet)}(t) = u_0^{(\bullet)} \oplus v_0^{(\bullet)}$ *and*

$$\begin{cases} u_{k+1}^{(\bullet)}(t) &= e^{\varepsilon t \mathbf{L}_\bullet} u_0^{(\bullet)} + \int_0^t e^{\varepsilon(t-s)\mathbf{L}_\bullet} f\left(u_k^{(\bullet)}(s), v_k^{(\bullet)}(s)\right) ds \\ v_{k+1}^{(\bullet)}(t) &= e^{\varepsilon d t \mathbf{L}_\bullet} v_0^{(\bullet)} + \int_0^t e^{\varepsilon d(t-s)\mathbf{L}_\bullet} g\left(u_k^{(\bullet)}(s), v_k^{(\bullet)}(s)\right) ds, \end{cases}$$

for $t \in \left[0, \tau_{u_0^{(\bullet)} \oplus v_0^{(\bullet)}}\right]$ *and* $k \in \mathbb{N} \setminus \{0\}$.

The Brusselator

Take $A > 0$ and $B > 0$, the Brusselator on $X_\bullet$ is the following reaction-diffusion system:

$$\begin{cases} u(t), v(t) \in C^1([0,\tau), X_\bullet); \\ \frac{\partial u^{(\bullet)}(x,t)}{\partial t} - \varepsilon \mathbf{L}_\bullet u(x,t) = A - (B+1)u + u^2 v \\ \frac{\partial v^{(\bullet)}(x,t)}{\partial t} - \varepsilon d \mathbf{L}_\bullet v(x,t) = Bu - u^2 v, \end{cases} \tag{11}$$

for $t \in [0, \tau)$, $x \in \mathcal{K}_N$. This system has only a homogeneous steady state: $u = A$, $v = \frac{B}{A}$. We consider $f(u,v) = A - (B+1)u + u^2 v$, $g(u,v) = Bu - u^2 v$ as functions defined on

$$(-\delta + A, \delta + A) \times \left(-\delta + \frac{B}{A}, \delta + \frac{B}{A}\right) \subset (a,b) \times (a,b),$$

for $\delta > 0$ sufficiently small so that $(0,0) \notin (a,b) \times (a,b)$.

Notice that

$$\nabla f(u,v) = (0,0) \Leftrightarrow (u,v) \in \{0\} \times \mathbb{R} \text{ and } \nabla g(u,v) \neq (0,0) \text{ for any } (u,v)$$

Then, there exist $a, b \in \mathbb{R}$ such that $\nabla f \mid_{(a,b)\times(a,b)} \neq (0,0)$ and $\nabla g \mid_{(a,b)\times(a,b)} \neq (0,0)$, and consequently Hypothesis 1 holds. Now, we take the subset

$$\mathcal{U} := \{r \in X_\infty; \|r - A\|_\infty < \delta\} \oplus \left\{s \in X_\infty; \left\|h - \frac{B}{A}\right\|_\infty < \delta\right\} \subset U \oplus U.$$

Then for any initial datum in $\mathcal{U} \cap X_\bullet \oplus X_\bullet$, system (11) has a unique solution, cf. Theorem 4 and Lemma 3.

Existence of good approximations

Theorem

Take $u_0 \oplus v_0 \in U \oplus U$. *Let* $u \oplus v$ *be the mild solution of (4), and let* $u^{(M)} \oplus v^{(M)}$ *be the mild solution of (5) with initial datum* $u^{(M)}(0) \oplus v^{(M)}(0) = (P_M \oplus P_M)(u_0 \oplus v_0)$. *Then*

$$\lim_{M \to \infty} \sup_{0 \leq t \leq \tau} \left\| u^{(M)}(t) \oplus v^{(M)}(t) - u(t) \oplus v(t) \right\| = 0,$$

where $\tau < \tau_{\max}$, *and* $\tau_{\max}$ *is the maximal interval of existence for the solution* $u(t) \oplus v(t)$ *with initial datum* $u_0 \oplus v_0$.

The spectrum of operator L

- From now on, we assume that $\mathcal{G}$ is an unoriented graph, with a symmetric adjacency matrix $\left[A_{JI}\right]_{J,I\in G_N^0}$ such that its diagonal contains zeros.

The spectrum of operator L

- From now on, we assume that $\mathcal{G}$ is an unoriented graph, with a symmetric adjacency matrix $[A_{JI}]_{J,I\in G_N^0}$ such that its diagonal contains zeros.

- The eigenvalues, μ_I, $I \in G_N^0$, of $[L_{JI}]_{J,I\in G_N^0}$ are non-positive and $\max_{I\in G_N^0}\{\mu_I\} = 0$. If λ_I, $I \in G_N^0$, are the eigenvalues of $[A_{JI}]_{J,I\in G_N^0}$, with multiplicities $mult(\lambda_I)$, then the eigenvalues of the discrete Laplacian are

$$\mu_I = \lambda_I - \gamma_I, \text{ with multiplicity } mult(\lambda_I), \text{ for } I \in G_N^0.$$

We set $X_\bullet \otimes \mathbb{C}$ for the complexification of $X_\bullet$. In particular,

$X_\infty \otimes \mathbb{C} = C(\mathcal{K}_N, \mathbb{C})$, with the L^∞-norm. Then $\mathbf{L} : X_\infty \otimes \mathbb{C} \to X_\infty \otimes \mathbb{C}$ is linear bounded operator. We set $\mathbf{L}_M := \mathbf{L}\mid_{X_M \otimes \mathbb{C}}$.

The spectrum of operator L

Lemma

The operator $\mathbf{L}$ *has a unique extension to* $L^2(\mathcal{K}_N, \mathbb{C})$ *as a bounded linear operator.*

Lemma

The operator $\mathbf{L} : L^2(\mathcal{K}_N, \mathbb{C}) \rightarrow L^2(\mathcal{K}_N, \mathbb{C})$ *is compact.*

Since $\mathbf{L}$ is a compact operator on $L^2(\mathcal{K}_N, \mathbb{C})$, every spectral value $\kappa \neq 0$ of $\mathbf{L}$ (if it exists) is an eigenvalue. For $\kappa \neq 0$ the dimension of any eigenspace of $\mathbf{L}$ is finite.

The spectrum of operator L

- Let λ_I, $I \in G_N^0$ be the eigenvalues of the matrix $\left[A_{JI}\right]_{J,I\in G_N^0}$, in this list repetitions may occur, with multiplicity $\mathrm{mult}(\lambda_I)$. Then the eigenvalues of $\mathbf{L}\mid_{X_N\otimes\mathbb{C}}=\mathbf{L}_N$ are exactly the eigenvalues of the matrix $\left[A_{JI}-\gamma_I\delta_{JI}\right]_{J,I\in G_N^0}$, which are

$$\mu_I := \lambda_I - \gamma_I, \text{ for } I \in G_N^0, \text{ with multiplicity } \mathrm{mult}\left(\lambda_I\right).$$

The spectrum of operator L

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$$\mu_I := \lambda_I - \gamma_I, \text{ for } I \in G_N^0, \text{ with multiplicity mult}\left(\lambda_I\right).$$

- The eigenvalues, μ_I, $I \in G_N^0$, of $[L_{JI}]_{J,I\in G_N^0}$ are non-positive and $\max_{I\in G_N^0}\{\mu_I\} = 0$. We denote the eigenfunctions of $[L_{JI}]_{J,I\in G_N^0}$ as φ_I, $I \in G_N^0$.

Let $\left[c_J^I\right]_{J\in G_N^0}$ be an eigenvector corresponding to μ_I, by identifying it with the function

$$\varphi_I\left(x\right):=\sum_{J\in G_N^0}c_J^I\Omega\left(p^N\left|x-J\right|_p\right)\in X_N\otimes\mathbb{C},\ c_J^I\in\mathbb{C},$$

and since $X_N\otimes\mathbb{C}\hookrightarrow X_\infty\otimes\mathbb{C}$ and $\mathbf{L}:X_N\otimes\mathbb{C}\rightarrow X_N\otimes\mathbb{C}$, we have

$$\begin{cases}\varphi_I\in X_\infty\otimes\mathbb{C};\\ \\ \mathbf{L}\varphi_I=\mu_I\varphi_I.\end{cases}$$

The φ_Is form a $\mathbb{C}$-vector space of dimension $\mathrm{mult}(\lambda_I)$.

The spectrum of operator L

We now recall that the set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{\frac{-r}{2}} \chi_p\left(p^{r-1} jx\right) \Omega\left(\left|p^r x - n\right|_p\right), \tag{12}$$

where $r \in \mathbb{Z}$, $j \in \{1, \cdots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$.

Furthermore,

$$\int_{\mathbb{Q}_p} \Psi_{rnj}(x)\, dx = 0. \tag{13}$$

This result is due to S. Kozyrev.

The functions of the form

$$\Psi_{-N(p^{-N}I)j}(x) = p^{\frac{N}{2}}\chi_p\left(p^{-N-1}jx\right)\Omega\left(p^N|x-I|_p\right), \tag{14}$$

for $I \in G_N^0$, $j \in \{1, \cdots, p-1\}$ are the functions in Kozyrev's basis supported in $\mathcal{K}_N = \bigsqcup_{I \in G_N^0} I + p^N\mathbb{Z}_p$.

A direct calculation using (13) shows that

$$\mathbf{L}\Psi_{-N(p^{-N}I)j}(x) = -\gamma_I\Psi_{-N(p^{-N}I)j} \tag{15}$$

for any $I \in G_N^0$, $j \in \{1, \cdots, p-1\}$.

Theorem

The operator $\mathbf{L} : L^2(\mathcal{K}_N, \mathbb{C}) \to L^2(\mathcal{K}_N, \mathbb{C})$ *is compact. The elements of the set:*

$$\{\lambda_I - \gamma_I; I \in G_N^0 \setminus \{I_0\}\} \bigsqcup \{-\gamma_I; I \in G_N^0\} \subset (-\infty, 0),$$

where $\{\lambda_I - \gamma_I\}_{I \in G_N^0 \setminus \{I_0\}}$ *are the non-zero eigenvalues of matrix* $[L_{JI}]_{J,I \in G_N^0}$*, are the non-zero eigenvalues of* $\mathbf{L}$*. The corresponding eigenfunctions are*

$$\left\{\frac{\varphi_I}{\|\varphi_I\|_2}; I \in G_N^0\right\} \bigsqcup \left\{\Psi_{-N(p^{-N}I)j}; I \in G_N^0, j \in \{1, \cdots, p-1\}\right\}. \quad (16)$$

Furthermore, the set (16) is an orthonormal basis of $L^2(\mathcal{K}_N, \mathbb{C})$*, and*

$$L^2(\mathcal{K}_N, \mathbb{C}) = X_N \otimes \mathbb{C} \oplus \mathcal{L}_0^2(\mathcal{K}_N, \mathbb{C}), \quad (17)$$

where $\mathcal{L}_0^2(\mathcal{K}_N, \mathbb{C}) := \left\{f \in L^2(\mathcal{K}_N, \mathbb{C}); \int_{\mathcal{K}_N} f dx = 0\right\}$.

Turing Criteria

We now consider a homogeneous steady state (u_0, v_0), which is a nonnegative solution of

$$f(u, v) = g(u, v) = 0. \tag{18}$$

Since u, v are real-valued functions, to study the linear stability of (u_0, v_0), we can use the classical results.

Following Turing, in the absence of any spatial variation, the homogeneous state must be linearly stable. With no spatial variation u, v satisfy

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = f(u,v) \\ \frac{\partial v}{\partial t}(x,t) = g(u,v). \end{cases} \tag{19}$$

Notice that (19) is an ordinary system of differential equations in $\mathbb{R}^2$.

Turing Criteria

Now, for $\delta > 0$ sufficiently small and (u_0, v_0) as in (18), we define

$$U_{\delta,u_0} \oplus U_{\delta,v_0} = \left\{ u_1 \oplus u_2 \in C(\mathcal{K}_N, \mathbb{R}) \oplus C(\mathcal{K}_N, \mathbb{R}) ; \|u_1 - u_0\|_\infty < \delta, \ \|v_1 - v_0\|_\infty < \delta \right\}.$$

Then, the Cauchy problem:

$$\begin{cases} u \oplus v \in C^1\left([0, \tau_0), U_{\delta,u_0} \oplus U_{\delta,v_0}\right); \\ \frac{\partial}{\partial t} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} f(u(t), v(t)) \\ g(u(t), v(t)) \end{bmatrix} + \varepsilon \mathbf{L} \mathbb{D} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}; \\ u(0) \oplus v(0) \in U_{\delta,u_0} \oplus U_{\delta,v_0}, \end{cases} \tag{20}$$

Turing Criteria

where

$$\mathbb{D}=\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix},$$

has a classical solution.

Our goal is to give an asymptotic profile as t tends infinity of this mild solution (the Turing instability criteria). We linearize system (20) about the steady state (u_0, v_0), by setting

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}. \tag{21}$$

By using the fact that f and g are differentiable, and assuming that $\|\mathbf{w}\| = \|w_1 \oplus w_2\|$ is sufficiently small, then (19) can be approximated as

$$\frac{\partial \mathbf{w}}{\partial t}(x,t) = \mathbb{J}\mathbf{w}, \tag{22}$$

Turing Criteria

where

$$\mathbb{J}_{u_0,v_0} =: \mathbb{J} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix} (u_0, v_0) =: \begin{bmatrix} f_{u_0} & f_{v_0} \\ \\ g_{u_0} & g_{v_0} \end{bmatrix}.$$

Turing Criteria

We now look for solutions of (22) of the form

$$\mathbf{w}\left(t;\lambda\right)=e^{\lambda t}\mathbf{w}_0. \tag{23}$$

By substituting (23) in (22), the eigenvalues λ are the solutions of

$$\det\left(\mathbb{J}-\lambda\mathbb{I}\right)=0,$$

i.e.

$$\lambda^2-\left(Tr\mathbb{J}\right)\lambda+\det\mathbb{J}=0. \tag{24}$$

Consequently

$$\lambda=\frac{1}{2}\left\{Tr\mathbb{J}\ \pm\sqrt{\left(Tr\mathbb{J}\right)^2-4\det\mathbb{J}}\right\}. \tag{25}$$

The steady state $\mathbf{w}=\mathbf{0}$ is linearly stable if $\mathrm{Re}\,\lambda<0$, this last condition is guaranteed if

$$Tr\mathbb{J}<0 \quad \text{and} \quad \det\mathbb{J}>0. \tag{26}$$

Turing Criteria

We now consider the full reaction-ultradiffusion system (20). We linearize it about the steady state, which with (21) is $\mathbf{w} = \mathbf{0} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, to get

$$\begin{cases} u \oplus v \in C^1\left([0,\tau), U_{\delta,u_0} \oplus U_{\delta,v_0}\right); \\ \frac{\partial}{\partial t}\mathbf{w}(x,t) = \left(\mathbb{J} + \varepsilon \mathbf{L}\mathbb{D}\right)\mathbf{w}(x,t),\ t \in [0,\tau); \\ u(0) \oplus v(0) \in U_{\delta,u_0} \oplus U_{\delta,v_0}, \end{cases} \tag{27}$$

where $\mathbb{J} + \varepsilon\mathbf{L}\mathbb{D}$ is a strongly continuous semigroup on $C\left(\mathcal{K}_N, \mathbb{R}\right) \oplus C\left(\mathcal{K}_N, \mathbb{R}\right)$.

Furthermore, (27), has also a unique solution, when $\mathbf{L}$ is considered as an operator on $L^2\left(\mathcal{K}_N, \mathbb{C}\right)$, for this reason, we can use the orthonormal basis given in Theorem 8 to solve (27) in $L^2\left(\mathcal{K}_N, \mathbb{C}\right)$, by using the separation of variables method, then, the solution of the original problem is exactly the real part of the solution of (27) in $L^2\left(\mathcal{K}_N, \mathbb{C}\right)$.

Turing Criteria

To solve the system (27) in $L^2(\mathcal{K}_N, \mathbb{C})$, we first consider the following eigenvalue problem:

$$\begin{cases} \mathbf{L}\mathbb{D}\mathbf{w}_\kappa(x) = \kappa \mathbf{w}_\kappa(x) \\ \\ \mathbf{w}_\kappa \in L^2(\mathcal{K}_N, \mathbb{C}) \oplus L^2(\mathcal{K}_N, \mathbb{C}), \end{cases} \tag{28}$$

which has a solution $\mathbf{w}_\kappa = w_{\kappa,1} \oplus w_{\kappa,2}$ due to Theorem 8, where

$$w_{\kappa,1}, w_{\kappa,2} \in \left\{ \frac{\varphi_I}{\|\varphi_I\|_2}; I \in G_N^0 \right\} \bigsqcup \left\{ \Psi_{-N(p^{-N}I)j}; I \in G_N^0, j \in \{1, \cdots, p-1\} \right\}$$

Turing Criteria

We look for an solution of type

$$\mathbf{w}(x,t)=\sum_{I\in G_N^0}\sum_{j\in\{1,\cdots,p-1\}}\mathbf{a}_{Ij}e^{\lambda t}\Psi_{-N(p^{-N}I)j}+\sum_{I\in G_N^0}\mathbf{b}_I\varphi_I \qquad (29)$$

where the vectors $\mathbf{a}_{Ij}$, $\mathbf{b}_I$ are determined by the Fourier expansion of the initial conditions. Substituting (29) with (28) in (27), we obtain that the existence of a non-trivial solution $\mathbf{w}(x,t)$ requires that the λs satisfy

$$\det\left(\lambda\mathbb{I}-\mathbb{J}-\varepsilon\kappa\mathbb{D}\right)=0, \qquad (30)$$

i.e.,

$$\lambda^2-\left\{\left(1+d\right)\varepsilon\kappa+Tr\mathbb{J}\right\}\lambda+h\left(\kappa\right)=0, \qquad (31)$$

where

$$h\left(\kappa\right):=\varepsilon^2 d\kappa^2+\varepsilon\kappa\left(df_{u_0}+g_{v_0}\right)+\det\mathbb{J}. \qquad (32)$$

Turing Criteria

When $\kappa = 0$. The steady state (u_0, v_0) is linearly stable if both solutions of (31) have $\mathrm{Re}\,(\lambda) < 0$.

The steady state is stable in absence of spatial effects, i.e. $\mathrm{Re}\,(\lambda\;|_{\kappa=0}) < 0$.

For the steady state to be unstable to spatial disturbances we require $\mathrm{Re}\,(\lambda\,(\kappa)) > 0$ for some $\kappa \neq 0$.

This happens if if $h\,(\kappa) < 0$ for some $\kappa \neq 0$ in (32).

Turing Criteria

This is a necessary condition, but not sufficient for $\operatorname{Re}(\lambda(\kappa)) > 0$. For $h(\kappa)$ to be negative for some nonzero κ, the minimum $h_{\min}$ of $h(\kappa)$ must be negative. An elementary calculation shows that

$$h_{\min} = \left\{ \det \mathbb{J} - \frac{(df_{u_0} + g_{v_0})^2}{4d} \right\}, \tag{33}$$

and the minimum is achieved at

$$\kappa_{\min} = \frac{-(df_{u_0} + g_{v_0})}{2\varepsilon d} \tag{34}$$

Thus the condition $h(\kappa) < 0$ for some $\kappa \neq 0$ is

$$\frac{(df_{u_0} + g_{v_0})^2}{4d} > \det \mathbb{J}. \tag{35}$$

Turing Criteria

A bifurcation occurs when $h_{\min} = 0$, for fixed kinetics parameters, this condition,

$$\det \mathbb{J} = \frac{\left(d f_{u_0} + g_{v_0}\right)^2}{4d}, \tag{36}$$

defines a critical diffusion d_c, which is given as an appropriate root of

$$f_{u_0}^2 d_c^2 + 2\left(2 f_{v_0} g_{u_0} - f_{u_0} g_{v_0}\right) d_c + g_{v_0}^2 = 0. \tag{37}$$

For $d > d_c$ model ((20)) exhibits Turing instability, while for $d < d_c$ no.

Turing Criteria

When $d > d_c$, there exists a range of unstable of positive wavenumbers $\kappa_1 < \kappa < \kappa_2$, where κ_1, κ_2 are the zeros of $h(\kappa) = 0$, see (32) and (35):

$$\kappa_2 = \frac{-1}{2d\varepsilon}\left\{(df_{u_0} + g_{v_0}) - \sqrt{(df_{u_0} + g_{v_0})^2 - 4d\det\mathbb{J}}\right\} < 0,$$

$$\kappa_1 = \frac{-1}{2d\varepsilon}\left\{(df_{u_0} + g_{v_0}) + \sqrt{(df_{u_0} + g_{v_0})^2 - 4d\det\mathbb{J}}\right\} < 0.$$

In the solution $\mathbf{w}(x,t)$ given by (29), the dominant contributions as t increases are the modes for which $\mathrm{Re}\,\lambda(\kappa) > 0$ since the other modes tend to zero exponentially, thus, if

$$\{\kappa \in \sigma(L) \smallsetminus \{0\}\,;\kappa_1 < \kappa < \kappa_2\} \neq \varnothing,$$

then

$$\mathbf{w}(x,t) \sim \sum_{\kappa_1<\kappa<\kappa_2} \sum_{I} A_{I\kappa} e^{\lambda t} \Omega\left(p^N |x-I|_p\right) + \tag{38}$$

$$\sum_{\kappa_1<\kappa<\kappa_2} \sum_{I,j} A_{Ij\kappa} e^{\lambda t} p^{\frac{N}{2}} \cos\left(\left\{p^{-N-1} jx\right\}_p\right) \Omega\left(p^N |x-I|_p\right) +$$

$$\sum_{\kappa_1<\kappa<\kappa_2} \sum_{I,j} B_{Ij\kappa} e^{\lambda t} p^{\frac{N}{2}} \sin\left(\left\{p^{-N-1} jx\right\}_p\right) \Omega\left(p^N |x-I|_p\right)$$

for $t \rightarrow +\infty$. In the above expansion all the sums run through a finite number of indices.

Turing Criteria

Theorem

Consider the reaction-diffusion system (27). The steady state (u_0, v_0) *is linearly unstable (Turing unstable) if the following conditions hold:*

(T1) $Tr\mathbb{J} = f_{u_0} + g_{v_0} < 0;$

(T2) $\det \mathbb{J} = f_{u_0} g_{v_0} - f_{v_0} g_{u_0} > 0;$

(T3) $df_{u_0} + g_{v_0} > 0;$

(T4) $(df_{u_0} + g_{v_0})^2 - 4d\,(f_{u_0} g_{v_0} - f_{v_0} g_{u_0}) > 0;$

(T5) $\{\kappa \in \sigma(L) \setminus \{0\}\,; \kappa_1 < \kappa < \kappa_2\} \neq \emptyset;$

(T6) the derivatives f_{u_0} *and* g_{v_0} *must have opposite signs.*

Furthermore in (20), we can take $\tau_0 = +\infty$*, for any initial data in* $U_{\delta, u_0} \oplus U_{\delta, v_0}$.

Remark

Theorem 9 is also valid for reaction-diffusion systems on X_M*, for* $M \geq N$.


THANK
YOU
GRACIAS
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