p-adic CFT

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Introduction

Why?

- Interest on its own, philosophically, etc.
- It is an environment where you can exactly calculate.
- It is a p-adic regulator of the archimedian string:
  - Gerasimov - Shatashvili show that the effective potential of p-adic str. th. tends to the one of open string field theory when $p \to 1$.
  - Bocardo Gaspar - Compeán - Zuñiga show the above result for correlation functions.

Classical String

$X : \Sigma \longrightarrow \mathbb{R}^d$, $x = (x^1, \dots x^d)$

Σ is a Riemann surface. We will consider $\Sigma = H$, the hyperbolic plane.

Polyakov action in conformal gauge:

$$S[x] = \frac{T}{2} \int d\sigma \, \partial_+ x^\mu \partial_- x_\mu$$

$$Z = \int_{\Sigma \to \mathbb{R}^d} dx \, e^{iS[x]}$$

p-adic World sheet

H is a homogeneous space:

$$H \cong PSL(2,\mathbb{R}) / SO(2)$$. $SO(2)$ is a maximal compact subgroup

$$\mathcal{T}_p := PGL(2,\mathbb{Q}_p) / PGL(2,\mathbb{Z}_p)$$. $PGL(2,\mathbb{Z}_p)$ is a maximal compact subgroup

Lem. $\mathcal{T}_p$ is a discrete metric space with $PGL(2,\mathbb{Q}_p)$-invariant metric such that, adding an edge between pairs of points whose distance is one, $\mathcal{T}_p$ becomes a $(p+1)$-valent tree. This is called the Bruhat-Tits tree.

Def. Choose a root $c \in \mathcal{T}_p$ and define the boundary

$$\partial \mathcal{T}_p := \{ \text{geodesic from } c \}$$

Lem. $\partial \mathcal{T}_p \cong \mathbb{P}^1(\mathbb{Q}_p)$ and $PGL(2,\mathbb{Q}_p)$ acts on it.

B. Spokoiny, 1988: He takes the worldsheet as $\mathbb{Q}_p$ and the field x^μ as a real locally constant test function. In particular, the direct generalization doesn't make sense for the derivative is zero.

$$S[x] = T \int_{\mathbb{Q}_p} d\mu \; x^\mu (A x_\mu) .$$

A is the Vladimirov operator:

$$A \varphi(t) = \int_{\mathbb{Q}_p} dt' \; \frac{\varphi(t) - \varphi(t')}{|t-t'|^2}$$

He gives a sketch or physical proof that this action reproduces the p-adic analog of the Koba-Nielsen amplitude:

$$A_N(k_1, \dots k_N) = g^{N-2} \int dt_1 \dots dt_{N-3} \prod_{i=1}^{N-3} |t_i|^{\alpha k_i k_N} |1-t_i|^{\alpha k_i k_{N-1}}$$

$$\prod_{1 \le i < j \le N-3} |t_i - t_j|^{\alpha k_i k_j}$$

This is the zeroth order approximation of the n-point correlation function.

Zabrodin, 1989: He treats the worldsheet as the Bruhat-Tits tree and the field as a real valued function on the tree.

$$S[x] = T \sum_{\langle a,b \rangle} (x_a^\mu - x_b^\mu)(x_{a,\mu} - x_{b,\mu})$$

Again, he gives a sketch to reproduce the Koba-Nielsen amplitude. He also shows that this action is equivalent to the Spokoiny action.

This is pre AdS/CFT revolution!

CFT

String theory is a model of Conformal field theory. There are plenty of variations of CFT. We will give the original one (almost) (Polyakov-Zamolodchikov)

Ax 1. There is a Hilbert space H that is a highest weight Representation of two copies of the Virasoro algebra; i.e. a Verma module of $Vir \times \overline{Vir}$. H is the space of states

Ax 2. There is a state-field correspondence (the operator vertex algebra) such that highest weight vectors give primary fields.

Def. - ϕ is a primary field if it is a section of $K^{\Delta} \otimes \bar{K}^{\bar{\Delta}}$:

$$\phi = \phi(z, \bar{z})\, dz^{\Delta}\, d\bar{z}^{\bar{\Delta}}.$$

- It is invariant under local conformal transformations with weights $\Delta, \bar{\Delta}$.

- $$[L_n, \Phi_i(z)] = z^{n+1} \partial_z \Phi_i(z) + \Delta_i (n+1) \Phi_i(z)$$

The other fields are descendants of the primary fields. For this talk, descendants will be derivatives of the primary fields.

Ax 3. We have the operator product expansion:

$$\langle \Phi_i(z_1) \Phi_j(z_2) \ldots \rangle = \sum_k C_{ij}^k (z_1 - z_2)^{\Delta_k - \Delta_i - \Delta_j} \langle \Phi_k(z_2) \ldots \rangle$$

Φ_i, Φ_j are primary fields and the OPE is associative.

This is the bootstrap Hipothesis.

p-adic CFT

We will give the E. Melzer, 1989, approach:

Now the fields live on $\mathbb{Q}_p$ and we have the following striking differences:

- $PGL(2, \mathbb{Q}_p)$ is the full symmetry group and we have the generators L_{-1}, L_0, L_1 only. In particular, there is no central extension.

- The theory have no descendants! All the fields are primary in the sense of item one.

- Recall that $dz\, d\bar{z}$ is the measure in $\mathbb{C}$. Then, if $\Delta = \bar{\Delta}$, a primary field is $\phi = \phi(x)\, dx^{\Delta}$ such that dx is the Haar measure ($dx(\mathbb{Z}_p) = 1$). It will be a spin zero theory.

- Now, the structure constants C_{ij}^{k} of the OPE define an associative algebra!

- The conformal blocks are constants.

Spokoiny as a CFT

$$\langle \phi(x)\, \phi(y) \rangle = -2 \ln \left| \frac{x-y}{a} \right| \qquad (a = \text{reg. cut off})$$

Then, ϕ is not a primary field. However, defining vertex operators : $V_\alpha(x) := e^{i\alpha\varphi(x)}$ / $\alpha \in R$.

$$\langle V_\alpha(x)\, V_\beta(y) \rangle = \delta(\alpha+\beta)\, |x-y|^{-2\alpha^2}$$

In particular, $\Delta_\alpha = \alpha^2$.

$$C^{\gamma}_{\alpha\beta} = \delta(\gamma-\alpha-\beta).$$

$$\langle V_{\alpha_1}(x_1) \ldots V_{\alpha_r}(x_r) \rangle = \delta(\alpha_1+\ldots\alpha_r) \prod_{i<j} |x_i - x_j|^{2\alpha_i\alpha_j}$$

Tensor Networks

Q: Given a CFT on the boundary, how is it reproduced from the bulk?

A: Tensor Network

Hung, Li, Melby-Thompson, Feb 2019:

Consider ~~s~~ the state-field correspondence:

$|a\rangle \longmapsto \phi_a$ and define the operator:

$$C : H \hookrightarrow End(H)$$

$$|c\rangle \longmapsto \sum_{a,b} |a\rangle C^a_{cb} \langle b|$$

It has the following properties:

- $C(a)\,C(b) = C(b)\,C(a)$
- $C(\bar{a}) = C(a)^\dagger$
- $C(|1\rangle) = id_H$
- Associativity

In particular, H is a C^*-algebra.

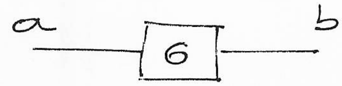
Def. $\langle 1| C(a) C(b) |1\rangle = C^1_{ac} \underbrace{C^c_{b1}}_{=\delta^c_b} = C^1_{ab} =: C_{ab}$

$$C_{abc} := C_{ae} C^e_{bc}$$

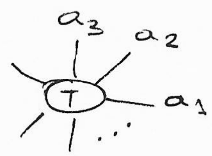
$$g^a_b := \delta^a_b\, p^{-\Delta_a} \; ; \quad g := p^{-\Delta}$$

We redefine the fields such that $C_{ab} = \delta_{ab}$.

"Feynman rules":



$$= \delta^{ab} p^{-\Delta a}$$



$$= \langle 1 | C(a_1) \dots C(a_n) | 1 \rangle = T^{(n)}_{a_1 \dots a_n}$$

T is a totally symmetric tensor.

Def. If Γ is a graph, $\partial\Gamma = \{\text{hanging edges}\}$

Then,
$$T_\Gamma \in H^*_{\partial\Gamma} := \bigotimes_{e \in \partial\Gamma} H_e^* \quad ; \text{ i.e.}$$

$$T_\Gamma : H_{\partial\Gamma} \longrightarrow \mathbb{C}$$

Regularization of Bruhat-Tits tree:

$$\mathcal{E}_p^\Lambda := \{ v \in \mathcal{E}_p \, / \, |v| < \Lambda \} . \quad \text{Then,} \quad \# \partial \mathcal{E}_p^\Lambda = q^\Lambda (1 + \tfrac{1}{q})$$

$$Z_\Lambda := T_{\mathcal{E}_p^\Lambda} : \mathbb{H}_\Lambda \longrightarrow \mathbb{C} \quad , \quad \mathbb{H}_\Lambda := \bigotimes_{e \in \partial \mathcal{E}_p^\Lambda} \mathbb{H}_e$$

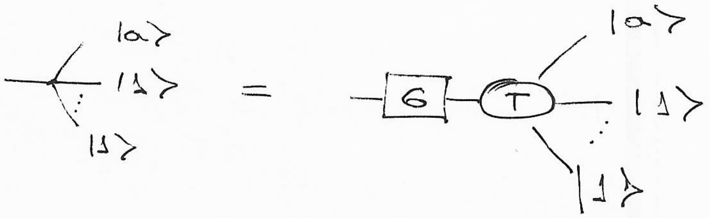
Sources:
$$|J\rangle = \bigotimes_{x \in \partial \mathcal{E}^\Lambda} |J(x)\rangle \quad , \quad |J(x)\rangle = \sum_a J_a(x) |a\rangle$$

$$Z_\Lambda[J] := Z_\Lambda(|J\rangle) \quad , \quad \text{Partition function with source.}$$

Obs.
- $Z_\Lambda = Z_\Lambda(|1\rangle_\Lambda) = 1$
- $$\langle \phi_{a_1}(x_1) \dots \phi_{a_N}(x_N) \rangle_\Lambda = \frac{1}{Z_\Lambda} \left. \frac{\delta Z_\Lambda[J]}{\delta J_{a_1}(x_1) \dots \delta J_{a_N}(x_N)} \right|_{J = |1\rangle_\Lambda}$$
$$= Z_\Lambda (|a_1\rangle_{x_1} \otimes \dots |a_N\rangle_{x_N})$$

Regularization of states:

Going one level inside the tree ...



$$= p^{-\Delta_a} |a\rangle$$

Then, $\phi_a(x)$ is renormalized to $p^{\wedge \Delta_a} |a\rangle$

2-point:

$$\langle \phi_a(x_1)\, \phi_b(x_2) \rangle = Z[J] = \frac{\delta_{ab}}{|x-y|^{2\Delta_a}}$$

$$J_a(x_1) = \delta(x-x_1), \quad J_b(x_2) = \delta(x-x_2)$$

3-point:

$$\langle \phi_a(x_1)\, \phi_b(x_2)\, \phi_c(x_3) \rangle = \frac{C_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}\, |x_{13}|^{\Delta_a+\Delta_c-\Delta_b}\, |x_{23}|^{\Delta_b+\Delta_c-\Delta_a}}$$

Then, the CFT is reproduced.