p-Adic Model of Fluid in the Porous Medium

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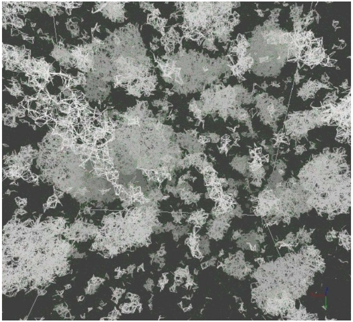
Introduction

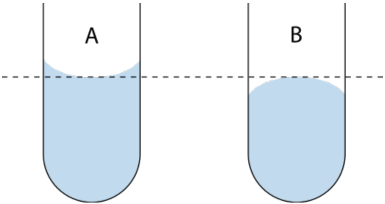
The dynamics of capillary flow have practical aspects in connection with the **movement of water or oil through soils**, the impregnation of wood and other porous materials with liquids.



Pic. 1. Tree-like capillary networks are common in variety of geological structures: images extracted from oil-saturated rocks at Mexican oil-fields.
(*Photo is given by K. Oleshko*)

Capillary Effect

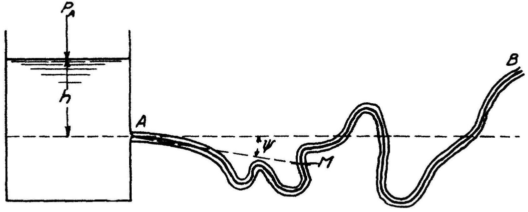
Capillary action (sometimes capillarity, capillary motion, capillary effect, or wicking) is the **ability of a liquid** to flow in narrow spaces without the assistance of, or **even in opposition** to, external forces like gravity.



The **meniscus** is the curve front surface of a liquid close to the surface of the container or another object. The curvature of the meniscus is caused by **surface tension** of the liquid.

Dynamics in Single Capillary

Dynamical problems connected with the rise of liquids in capillary tubes have been investigated by Edward W. Washburn in 1921.



Pic. 1. E.W. Washburn The dynamics of capillary flow, Phys. Rev., XVII, N3, p. 273 (1921)

Edward W. Washburn'1921

The rate of penetration into a small capillary of radius r is shown to be

$$\frac{d\ell}{dt} = \frac{P(r^2 + 4\varepsilon)}{8\eta\ell},$$

P is the **driving pressure**, ε the coefficient of **slip** and η the **viscosity**.

Washburn's argument

Starting with the **Poiseuille's** law which takes the following form, if to neglect for the moment any air resistance:

$$dV = \frac{\pi \sum P}{8\eta\,\ell}(r^4 + 4\varepsilon\, r^3)dt,$$

where dV is the **volume** of liquid which in time dt flows **through any cross-section** of the capillary;
ℓ is the **length of the column of liquid** in the capillary at the time t;
η is the **viscosity** of the liquid;
ε is its coefficient of **slip**;
$\sum P$ is the total **effective pressure** which is acting to force the liquid along the capillary.

Taking into account:

$$dV = \pi\, r^2 d\,\ell$$

Washburn received the following the **velocity of the moving meniscus**:

$$\frac{d\ell}{dt} = \frac{\sum P}{8r^2 \cdot \eta \cdot \ell}(r^4 + 4\varepsilon\, r^3),$$

where

$$\sum P = P_A + P_h + P_s$$

P_A – the unbalanced **atmospheric pressure**;
P_h – the **hydrostatic** pressure;
P_s – the **capillary pressure**.

$$P_h = h \cdot g \cdot D - \ell_s \cdot g \cdot D \sin\psi;$$
$$P_s = \frac{2\gamma}{r}\cos\theta$$

ℓ_s – the **linear distance** from point A to M on Pic. 1.
g – the gravity acceleration constant;
D – the liquid **density**;
γ – the **surface tension** of the liquid;
θ – the **contact angle** between the meniscus and the wall of the tube.

Washburn's Law for the velocity of penetration

Actually Washburn has obtained the following law for the velocity of penetration:

$$\frac{d\ell}{dt} = \frac{\left[P_A + g \cdot D(h - \ell_s \sin\psi) + \frac{2\gamma}{r}\cos\theta\right](r^2 + 4\varepsilon\, r)}{8\eta\,\ell},$$

P_A is the unbalanced atmospheric pressure;
γ – the surface tension of the liquid;
ε – coefficient of slip;
θ – the contact angle between the liquid meniscus and the wall of the tube.

For horizontal capillary

$$\ell^2 = \left(\frac{\gamma\ \cos\theta}{2 \cdot \eta}\right) r \cdot t \quad \text{or} \quad \boxed{\ell = \sqrt{\left(\frac{\gamma\ \cos\theta}{2 \cdot \eta}\right) r \cdot t}} \tag{1}$$

Remark about vertical capillary

Moreover Washburn has shown that **for the vertical capillaries** with small internal surface the logarithmic term, which arise in the solution of the equation, may be expanded and after rejection of all the lower order terms, the equation **will coincide** with one for **horizontal** capillary. And the corresponding equation for the rate is

$$\frac{d\ell}{dt} = \frac{r \cdot \gamma}{4\ell \cdot \eta} \cos\theta.$$

Murray's Law (1927)

[Cecil D. Murray'27] *A Relationship Between Circumference and Weight in Trees and Its Bearing on Branching Angles*, J. Gen. Physiol. **10(5)**, p. 725 - 729 (1927).

Murray's law ***predicts the thickness of branches*** in transport networks, such that the ***cost for transport*** and maintenance of the transport medium is minimized.

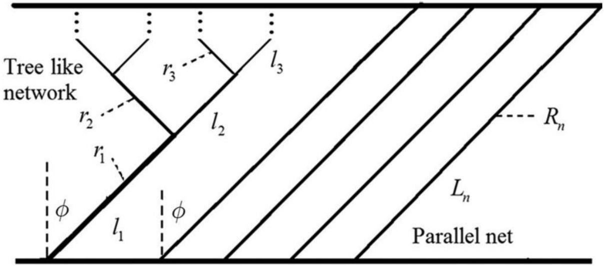
For n child branches splitting from a common parent branch, the law states that:

$$r^3 = r_1^3 + r_2^3 + r_3^3 + \ldots + r_n^3.$$

Murray's law is a basic physical principle for transfer networks.

Treelike Capillary Networks

Duhua Shou, Lin Ye, Jintu Fan (2014)



Pic. 2 Duhua Shou, Lin Ye, Jintu Fan, *Phys. Rev. E*, **89**, 053007 (2014)

The **ratio of radius between the tubes** at the $(j+1)$-th branching level and that at the j-th branching level, and corresponding the between the **lengths**:

$$\alpha = \frac{r_{j+1}}{r_j}; \qquad \beta = \frac{\ell_{j+1}}{\ell_j}.$$

The capillary flow in V-shaped treelike network

The capillary flow in all tubes of the treelike network is driven by capillary pressure. When the meniscus is in the j-th level tube, the **capillary pressure** is given by:

$$p_j = -\frac{2\gamma \cos\theta}{r_j} \tag{2}$$

The j-th level **flow rate** Q_j is obtained based on the Hagen-Poiseuille law:

$$Q_j = \pi r_j^2 u_j = -\frac{\pi r_j^4}{8\eta}\frac{\partial p}{\partial x}, \quad \text{with} \quad u_j = \frac{d\ell}{dt} \tag{3}$$

where u_j is the **spontaneous velocity** of the liquid at the j-th level.

First level of the treelike network

The time of capillary flow is obtained in terms of the **liquid penetration distance** ℓ based on Washburn's Law (1):

$$t_1(\ell) = \frac{2\eta}{\gamma \cos\theta} \frac{\ell^2}{r_1}, \quad \ell \in [0, \ell_1). \tag{4}$$

Here ℓ denotes the penetration distance for **first level tubes**, η stands for the viscosity of the liquid, γ denotes the liquid-vapor surface tension, and θ is the contact angle between liquid meniscus and the wall of the tube.

Time T_1, required for the liquid to fill the single tube of the first level with the lenght ℓ_1, is equal to:

$$T_1 = \frac{2\eta}{\gamma \cos\theta} \frac{\ell_1^2}{r_1},$$

and it is the same for all the tubes of first level.

Noting that due to the **conservation of mass**:

$$m_j Q_j = Q_1,$$

where $m_j = p$ is the number of tubes at the j-th level, it follows from equation for flow rate (3) that, for example, at the second levels of the treelike network:

$$p_2 = -8\eta Q_1 \int_0^{\ell_1} \frac{dx}{\pi r_1^4} - 8\eta Q_2 \int_{\ell_1}^{\ell} \frac{dx}{\pi r_2^4},$$

with $m_2 Q_2 = Q_1 = pQ_2$. And from (2) it follows:

$$\frac{\gamma \cos\theta}{r_2} = 4p\eta Q_2 \int_0^{\ell_1} \frac{dx}{\pi r_1^4} + 4\eta Q_2 \int_{\ell_1}^{\ell} \frac{dx}{\pi r_2^4}.$$

D. Shou, L. Ye, J. Fan'2014

Shou-Ye-Fan-time:

The ***time*** T_j ***required for the liquid*** to fill all the tubes until the j-th level is equal to:

$$T_j = \frac{C}{2}\sum_{k=1}^{j}\frac{\ell_k^2}{r_k} + C\sum_{n=2}^{j}\sum_{k=2}^{n}\left(p^{n+1-k}\frac{r_n^3\ell_{k-1}}{r_{k-1}^4}\ell_n\right), \quad T_0 = 0. \qquad (5)$$

p-adic interpretation of the capillary flow in porous medium

Structure of the p-adic tree

A path, possibly infinite, on a rooted p-tree will be identified with a p-adic number $x \in \mathbb{Z}_p$ given by its canonical representation

$$x = x_0 + x_1 p + x_2 p^2 + \cdots , \quad x_n \in \{0, 1, \ldots, p-1\}.$$

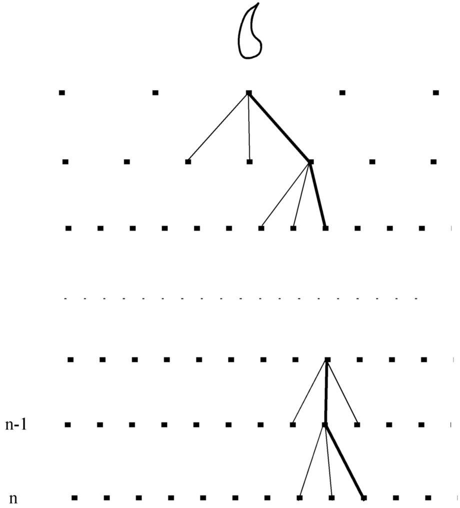
For some $a \in \mathbb{Z}_p$, $a = a_0 = a_1 p + a_2 p^2 + \cdots$ the ball

$$B_n(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leq p^{-n}\}, \quad n \geq 0$$

consists of the points

$$x = a_0 + a_1\, p + \ldots + a_n\, p^n + \underbrace{x_{n+1}\, p^{n+1} + x_{n+2}\, p^{n+2} + \ldots}_{\text{something}}, \tag{6}$$

Structure of the *p*-adic tree



Liquid volume in p-adic ball

The volume $W_m(t)$ of liquid penetrating into the set of capillaries, described as a p-adic ball $B_m(a)$, equals

$$W_m(t) = V_1(t) + \cdots + V_m(t) + \sum_{k=0}^{\infty} p^k V_{m+k}(t). \tag{7}$$

For each fixed t, this sum is in fact finite: if $T_{j-1} < t \leq T_j$, then $V_{m+k}(t) = 0$ for $k > j - m$.

The total volume penetrating through the porous medium modeled by $\mathbb{Z}_p$ is

$$W_0(t) = \sum_{k=0}^{\infty} p^k V_k(t).$$

If $T_{j-1} < t \leq T_j$, then $V_k(t) = 0$ for $k > j$.

Calculation of the capillary volume

Let $V_j(t)$ be the **volume of liquid** within a j-th level single tube at the time t. Note that it may happen that $V_j(t) = 0$ because the liquid has not reached the j-th level.

The volume $V_j(t)$ is expressed in the terms of length $\ell_j(t)$ passed by meniscus in time t:

$$V_j(t) = \pi\, r_j^2\, \ell(t)$$

We use calculations on the capillary flow dynamics given of [D. Shou, L. Ye, J. Fan'2014].

The time $t_j(\ell)$ corresponding to the liquid movement at the j-th level is calculated as:

$$t_j(\ell) = \frac{2\eta}{\gamma\cos\theta}\,\frac{\ell^2}{r_j} + \frac{4\eta}{\gamma\cos\theta}\left[r_j^3 \sum_{k=2}^{j} p^{j+1-k}\frac{\ell_{k-1}}{r_{k-1}^4}\right]\ell, \quad \ell \in [L_{j-1}, L_j), \quad (8)$$

where

$$L_j = \sum_{k=1}^{j} \ell_k.$$

In (8) the parameter ℓ corresponds to the j-th level of the tubes in the porous medium, thus ℓ is changing in the interval:

$$\sum_{k=1}^{j-1} \ell_k \le \ell < \sum_{k=1}^{j} \ell_k,$$

where ℓ_k is the length of the single tube on the k-th level.

Equation (8) **is a quadratic one with respect to ℓ.** Solving equation (8) we have:

$$\ell(t) = -r_j\, d_j + \sqrt{(r_j\, d_j)^2 + \frac{r_j\, t}{2C}}, \quad t \in [T_{j-1}, T_j),$$

where

$$C = \frac{4\eta}{\gamma \cos\theta},$$

$$d_j = \frac{\ell_0\, \alpha^{3j}}{r_0} p^j \sum_{k=2}^{j} \left(\frac{\beta}{\alpha^4 p}\right)^{k-1} = \frac{\ell_0\, \alpha^{3j}}{r_0} p^j \frac{\beta\left(1 - \left(\frac{\beta}{\alpha^4 p}\right)^j\right)}{\alpha^4 p - \beta}.$$

Therefore

$$V_j(t) = \pi\, r_j^2\, \ell(t) = \pi\, r_j^2 \left(\sqrt{(r_j\, d_j)^2 + \frac{r_j\, t}{2C}} - r_j\, d_j\right), \quad t \in [T_{j-1}, T_j), \quad (9)$$

where time T_j is Shou-Ye-Fan-time required for the liquid to fill all the tubes until the ***j*-th level** and calculated in (5).

Construction of Stochastic Process

The idea of constructing a stochastic process ξ_t **describing the capillary flow** is as follows. The state space is $\mathbb{Z}_p$, that is, in physical terms, the set of **all paths** along the rooted tree beginning at its root.

ξ_t means the path filled by the liquid at the time t. To obtain a Markov process, we need a **transition density** describing the probability that $\xi_t = y$, if it is known that $\xi_s = x, s < t$.

As the first step, we calculate the conditional probability

Conditional probabilities

$$\widetilde{\rho}(s,x;t,y) = \frac{P\Big(\{\omega : \xi_t = y\} \cap \{\omega : \xi_s = x\}\Big)}{P\Big(\{\omega : \xi_s = x\}\Big)}.$$

The probability that the process $\xi_\cdot$ at time $s \in [T_{j-1}, T_j]$ attains a point x is equal to

$$P\Big(\{\omega : \xi_s = x\}\Big) = \frac{\sum_{i=0}^{n(x)\wedge j} V_i(s)}{W_0(s)} = \frac{\sum_{i=0}^{n(x)\wedge j} V_i(s)}{\sum_{i=0}^{j} p^i V_i(s)}, \tag{10}$$

where the number $n(x)$ is finite or infinite, depending on the **"length" of the filled trajectory** x**.**

To calculate the probability of the set

$$\{\omega : \xi_s = x\} \cap \{\omega : \xi_t = y\},$$

let us first remark that points x and y in $\mathbb{Z}_p$ have the following canonical representations:

$$\begin{aligned} x &= x_0 + x_1 p + ... + x_n p^n + \cdots = \sum^{n(x)} x_i p^i; \\ y &= y_0 + y_1 p + ... + y_n p^n + \cdots = \sum^{n(y)} y_i p^i. \end{aligned} \tag{11}$$

There appear two different variants of mutual relations between s and t and the time intervals $[T_{j-1}, T_j)$, when the j-level tubes are filled.

Variant A

The times t and s, $s < t$, are located on the same time interval $[T_{j-1}, T_j)$:

$$T_{j-1} \leq s < t < T_j$$

for some j, then transition probability density $\rho(s, x; t, y) \neq 0$ only if in the canonical p-adic representations for points x and y (11) their components coincide till the level j:

$$x_0 = y_0;$$

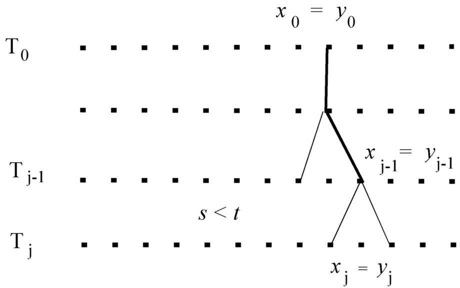
$$\ldots\ldots\ldots\ldots\ldots$$

$$x_j = y_j.$$

In this case

$$\widetilde{\rho}(s, x; t, y) = \frac{P\big(\{\omega : \xi_t = y\}\big)}{P\big(\{\omega : \xi_s = x\}\big)} = \frac{W_0(s)}{W_0(t)} \cdot \frac{\sum\limits_{i'=0}^{n(y)} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)} V_i(s)}. \qquad (12)$$

Variant A



Variant B

If instant times s and t, $s < t$, belong to different time intervals, that is:

$$0 \leq T_{k-1} \leq s < T_k \leq T_{j-1} \leq t < T_j$$

for some $k < j$, then the transition probability density $\rho(s, x, t, y) \neq 0$, only if in the canonical p-adic decomposition of points x and y their components coincide till the level $j - 1$:

$$x_0 = y_0;$$

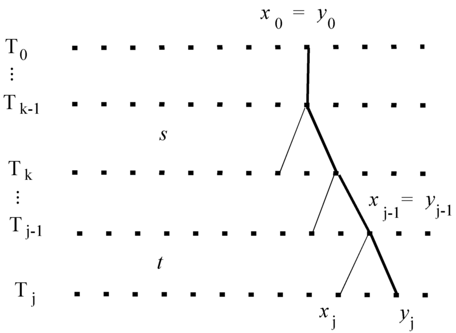
$$\ldots\ldots\ldots\ldots$$

$$x_{j-1} = y_{j-1}.$$

In this case the expression for the transition probability density $\rho(s, x; t, y)$ is given by the same formula (12):

$$\widetilde{\rho}(s, x; t, y) = \frac{P\big(\{\omega : \xi_t = y\}\big)}{P\big(\{\omega : \xi_s = x\}\big)} = \frac{W_0(s)}{W_0(t)} \cdot \frac{\sum\limits_{i'=0}^{n(y)} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)} V_i(s)}. \tag{12}$$

Variant B



If we introduce the following characteristic function:

$$\chi_j(x,y) = \begin{cases} 1, & x_0 = y_0,\ x_1 = y_1,\ \ldots,\ x_j = y_j; \\ 0, & \text{otherwise.} \end{cases}$$

Then, in terms of $\chi_j(x,y)$ we may write:

$$\widetilde{\rho}(s,x;t,y) = \begin{cases} \dfrac{W_0(s)}{W_0(t)} \cdot \dfrac{\sum\limits_{i'=0}^{n(y)\wedge j} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)\wedge j} V_i(s)}\, \chi_j(x,y), & T_{j-1} \le s \le t < T_j; \\ \dfrac{W_0(s)}{W_0(t)} \cdot \dfrac{\sum\limits_{i'=0}^{n(y)\wedge j} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)\wedge j} V_i(s)}\, \chi_{j-1}(x,y), & s < T_{j-1} \le t < T_j. \end{cases}$$

Renormalization

To construction stochastic process we need to have the Chapman - Kolmogorov equation, i.e. the equality

$$\rho(s,x;u,y) = \int_{\mathbb{Z}_p} \rho(s,x;t,z)\rho(t,z;u,y)\mu(dz). \tag{13}$$

$s < t < u$

Variant A

I. First check this for $T_{j-1} \leq s < t < u < T_j$. Starting with the r.h.s. of (13) we have:

$$\int_{\mathbb{Z}_p} \widetilde{\rho}(s,x;t,z)\widetilde{\rho}(t,z;u,y)\mu(dz) =$$

$$= \int_{\mathbb{Z}_p} \frac{W_0(s)}{W_0(t)} \frac{\sum_{i'=0}^{n(z)\wedge j} V_{i'}(t)}{\sum_{i=0}^{n(x)\wedge j} V_i(s)} \cdot \frac{W_0(t)}{W_0(u)} \frac{\sum_{\ell'=0}^{n(y)\wedge j} V_{\ell'}(u)}{\sum_{\ell=0}^{n(z)\wedge j} V_i(t)} \chi_j(x,z)\,\chi_j(z,y)\,\mu(dz) =$$

$$= \frac{1}{p^j(p-1)} \widetilde{\rho}(s,x;u,y).$$

Variant B

II. For the case $s < T_{j-1} \le t \le u < T_j$ we have

$$\int_{\mathbb{Z}_p} \widetilde{\rho}(s,x;t,z)\widetilde{\rho}(t,z;u,y)\mu(dz) =$$

$$= \int_{\mathbb{Z}_p} \frac{W_0(s)}{W_0(t)} \frac{\sum_{i'=0}^{n(z)\wedge j} V_{i'}(t)}{\sum_{i=0}^{n(x)\wedge j} V_i(s)} \cdot \frac{W_0(t)}{W_0(u)} \frac{\sum_{\ell'=0}^{n(y)\wedge j} V_{\ell'}(u)}{\sum_{\ell=0}^{n(z)\wedge j} V_i(t)} \; \chi_{j-1}(x,z)\,\chi_j(z,y)\,\mu(dz) =$$

$$= \frac{1}{p^{j-1}(p-1)}\,\widetilde{\rho}\,(s,x;u,y).$$

Summing up

The above calculations gives:

$$\textbf{I}.\int_{\mathbb{Z}_p} \widetilde{\rho}(s,x;t,z)\,\widetilde{\rho}(t,z;u,y)\,\mu(dz) = \frac{1}{p^j(p-1)}\,\widetilde{\rho}(s,x;u,y) \tag{14}$$

for $T_{j-1} \leq s < t < u < T_j$ (**Variant A**) and

$$\textbf{II}.\int_{\mathbb{Z}_p} \widetilde{\rho}\,(s,x;t,z)\widetilde{\rho}\,(t,z;u,y)\mu(dz) = \frac{1}{p^{j-1}(p-1)}\,\widetilde{\rho}(s,x;u,y) \tag{15}$$

for $s < T_{j-1} \leq t \leq u < T_j$ (**Variant B**)

Renormalization Factor

To guarantee the Chapman - Kolmogorov equation (13) it is necessary to introduce renormalization factor into the definition of the transition probability density. Thus we need to define a constant λ so that

$$\rho = \lambda\widetilde{\rho}$$

would satisfy the the Chapman - Kolmogorov equation. Thus, from (14) we have the equation:

$$\frac{1}{\lambda_A} = \frac{1}{p^j(p-1)}\frac{1}{\lambda_A}$$

therefore

$$\lambda_A = p^j(p-1),$$

correspondingly

$$\lambda_B = p^{j-1}(p-1).$$

Renormalized definition of probability density

The final definition of the probability density for the process under construction is as follows:

$$\rho\,(s,x;t,y) = \begin{cases} p^{j}(p-1)\dfrac{W_0(s)}{W_0(t)}\cdot\dfrac{\sum\limits_{i'=0}^{n(y)\wedge j} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)\wedge j} V_i(s)}\,\chi_j(x,y), & \text{Variant A} \\ p^{j-1}(p-1)\dfrac{W_0(s)}{W_0(t)}\cdot\dfrac{\sum\limits_{i'=0}^{n(y)\wedge j} V_{i'}(t)}{\sum\limits_{i=0}^{n(x)\wedge j} V_i(s)}\,\chi_{j-1}(x,y), & \text{Variant B.} \end{cases}$$

and

$$\rho(s,x;u,y) = \int\limits_{\mathbb{Z}_p} \rho(s,x;t,z)\rho(t,z;u,y)\,\mu(dz).$$

Transition Probability

Lemma 1

The transition probability for the process ξ_t:

$$P(s,x,t,B_n(a)) = \int_{B_n(a)} \rho\,(s,x,t,z)\,\mu(dz)$$

is equal:

$$P(s,x;t,B_n(a)) = p^{-n}\frac{W_0(s)}{W_0(t)} \cdot \frac{\sum_{i'=0}^{j} V_{i'}(t)}{\sum_{i=0}^{n(x)\wedge j} V_i(s)}.$$

for both Variants **A** *and* **B**.

Lemma 2

$$P(s,x;t,\mathbb{Z}_p) \leq 1.$$

Inhomogeneous Markov processes

Thus we may use the analytic definition of a Markov process, that is a definition of a transition probability.

Suppose that $(E, \mathcal{E})$ is a measurable space. A family of real-valued non-negative functions $P(s, x; t, \Gamma)$, $s < t$, $x \in E, \Gamma \in \mathcal{E}$, such that $P(s, x; t, \cdot)$ is **a measure** on $\mathcal{E}$ and $P(s, x; t, \Gamma) \leq 1$, is called a ***transition probability***, if the Kolmogorov-Chapman equality

$$\int_E P(s, x; t, dy) P(t, y; u, \Gamma) = P(s, x; u, \Gamma)$$

holds whenever $s < t < u$, $x \in E$, $\Gamma \in \mathcal{E}$.

A transition probability P is called **normal**, if for any $s > 0$, $x \in E$,

$$\lim_{t \downarrow s} P(s, x; t, E) = 1.$$

Evolution Family

An object related to the transition probability is the evolution family, a biparametric family $U(s,t)$ of operators acting on the space $B(\mathcal{E})$ of **bounded $\mathcal{E}$-measurable** functions on E:

$$(U(s,t)f)(x) = \int_E f(y)P(s,x;t,dy).$$

These operators are positivity-preserving and $U(s,t)1 \leq 1$.

Theorem 3

The linear operators $U(s,t)$ are positivity preserving and satisfy:

(i) $U(s,t)\,1 \leq 1$.

(ii) $U(s,s) = \mathrm{Id}$.

(iii) $U(s,t) = U(s,\tau)U(\tau,t)$.

The relation (ii) means that P is a **normal transition probability**.

Generator of Evolution Family

Let us fix some $T > 0$ and denote by $\mathfrak{D}$ the class of functions from $B(\mathfrak{B})$ for which for any $s \in (0, T)$ the next limit exists:

$$\lim_{h \downarrow 0} \frac{U(s-h,s)f(x) - f(x)}{h} =: A(s)f(x) \tag{16}$$

and

$$\lim_{h \downarrow 0} U(t-h,t)f(x) = f(x).$$

Theorem 4

The generator $A(s)$ defined of evolution family $U(t,s)$ of the inhomogeneous Markov process has the explicit representation:

$$(A(s)f)(x) = p^j(p-1) \int\limits_{\substack{\mathbb{Z}_p,\ y_0 = x_0 \\ \vdots \\ y_j = x_j}} f(y) \left(\log \frac{\sum_{k=0}^{j} p^k V_k(s)}{\sum_{i=0}^{n(x)\wedge j} V_i(s)} \right)'_s \mu(dy).$$

Sketch of the proof

We start with the calculation of the difference in the l.h.s. of (16).

$$\frac{1}{h}\Big(U(s-h,s)f(x)-f(x)\Big)=$$
$$=\frac{1}{h}\Big(\int\limits_{\mathbb{Z}_p} f(y)\,\rho(s-h,x,s,y)\mu(dy)-f(x)\Big)=$$
$$=\frac{1}{h}\int\limits_{\mathbb{Z}_p} f(y)\,\Big(\rho(s-h,x,s,y)-\rho\,(s,x;s,y)\Big)\,\mu(dy).$$

Thus we need to calculate the difference:

$$\boxed{\Delta=\frac{1}{h}\Big(\rho(s-h,x;s,y)-\rho(s,x;s,y)\Big)}$$

Noting that

$$\rho(s,x;t,y) = p^j(p-1)\frac{W_0(s)}{W_0(t)} \cdot \frac{\sum_{i'=0}^{n(y)\wedge j} V_{i'}(t)}{\sum_{i=0}^{n(x)\wedge j} V_i(s)} \chi_j(x,y),$$

if we introduce the notation:

$$\hat{V}_{x,j}(s) := \sum_{i'=0}^{n(x)\wedge j} V_{i'}(s)$$

then we have

$$\Delta = \frac{p^j(p-1)\,\chi_j(x,y)}{h}\frac{\hat{V}_{y,j}(s)}{W_0(s)}\left(\frac{W_0(s-h)\hat{V}_{x,j}(s) - W_0(s)\hat{V}_{x,j}(s-h)}{\hat{V}_{x,j}(s-h)\cdot\hat{V}_{x,j}(s)}\right).$$

Thus

$$\lim_{h\downarrow 0}\Delta = p^j(p-1)\,\chi_j(x,y)\frac{\hat{V}_{y,j}(s)}{W_0(s)}\frac{W_0'(s)\hat{V}_{x,j}(s) - W_0(s)\hat{V}_{x,j}'(s)}{\hat{V}_{x,j}^2(s)} =$$

$$= p^j(p-1)\,\chi_j(x,y)\frac{\hat{V}_{y,j}(s)}{W_0(s)\cdot\hat{V}_{x,j}^2(s)}\begin{vmatrix}\hat{V}_{x,j}(s) & W_0(s)\\ \hat{V}_{x,j}'(s) & W_0'(s)\end{vmatrix}.$$

$$A(s)f(x) = \int\limits_{\mathbb{Z}_p} f(y)p^j(p-1)\,\chi_j(x,y)\frac{\hat{V}_{y,j}(s)}{W_0(s)\cdot\hat{V}^2_{x,j}(s)}\begin{vmatrix}\hat{V}_{x,j}(s) & W_0(s)\\ \hat{V}'_{x,j}(s) & W'_0(s)\end{vmatrix}\,\mu(dy)$$

where

$$\chi_j(x,y) = \begin{cases} 1, & x_0 = y_0,\ x_1 = y_1,\ \ldots,\ x_j = y_j; \\ 0, & \text{otherwise.} \end{cases}$$

And this gives exactly

$$A(s)f(x) = p^j(p-1) \int\limits_{\mathbb{Z}_p,\ \substack{y_0 = x_0 \\ \vdots \\ y_j = x_j}} f(y)\Big(\log\frac{W_0(s)}{\hat{V}_{x,j}(s)}\Big)'_s\,\mu(dy).\ \square$$