Reaction-diffusion Equations on Complex Networks and Turing Patterns, via p-Adic Analysis. I.

W. A. Zúñiga-Galindo

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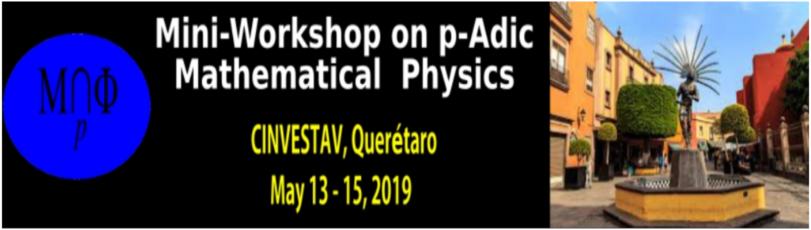

Abstract

W. A. Zúñiga-Galindo Reaction-diffusion Equations on Complex Networks and Turing Patterns, via *p*-Adic Analysis. arXiv.org $>$ math $>$ arXiv:1905.02128.

- This work aims to show that *p*-adic analysis is the natural tool to study, in a rigorous mathematical way, reaction-diffusion systems on networks and the corresponding Turing patterns.

Introduction

In 1952 A. Turing proposed that under certain conditions chemicals can react and diffuse in such way as to produce steady state heterogeneous spatial patterns of chemical (or morphogen) concentration.

Pattern-forming, reaction-diffusion systems in continuous media, are typically described by a system of PDEs of the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = & f\left(u,v\right) + \varepsilon \Delta u(x,t) \\ \frac{\partial v(x,t)}{\partial t} = & g\left(u,v\right) + \varepsilon d \Delta v(x,t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $t \geq 0$, and $u(x,t)$, $v(x,t)$ are local densities of two chemical species, the functions f and g specify the local dynamics of u and v, and ε, εd are the corresponding diffusion coefficients.

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- The Turing instability occurs when the parameter d exceeds a threshold.
- This event drives to a spontaneous development of a spatial pattern formed by alternating activator-rich and activator-poor patches.

The CIMA reaction

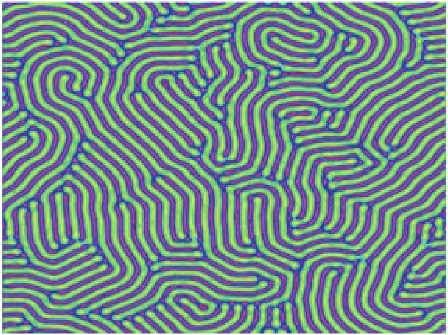
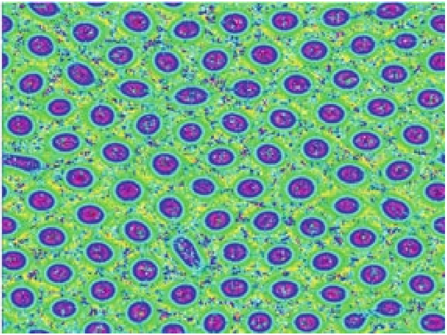
The CIMA reaction (chlorite-iodide-malonic acid) provided experimental evidence of Turing instability. It was modeled by Lengyel and Epstein.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \sigma_u \mathbf{\Delta} u\left(x,t\right) = A - u - \frac{4uv}{1+u^2} \\ \\ \frac{\partial v(x,t)}{\partial t} - \sigma_u \mathbf{\Delta} v\left(x,t\right) = BCu - \frac{Cuv}{1+u^2}. \end{cases}$$

Here u (the activator) denotes the iodide $\left(I^-\right)$ concentration and v (the inhibitor) the chlorite $\left(ClO_2^-\right)$ concentration.
We consider this system with $A > 0$, $B > 0$, $C > 0$. There is a single homogeneous steady state

$$u_0 = \frac{A}{4B+1}, \quad v_0 = B\left(1 + \frac{A^2}{\left(4B+1\right)^2}\right).$$

Labyrinth and spot patterns in the CIMA reaction

Introduction

- In the 70s, Othmer and Scriven pointed out that Turing instability can occur in network-organized systems. Since then, reaction-diffusion models on networks has been studied intensively.

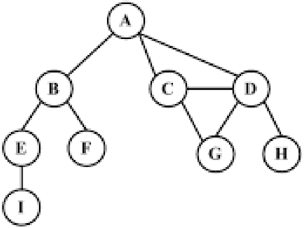
- In the discrete case, the continuous media is replaced by a network (an unoriented graph $\mathcal{G}$, which plays the role of discrete media) composed by $\#V(\mathcal{G})$ independent nodes (vertices) that interact via diffusive transport on $\#E(\mathcal{G})$ links (edges).

Introduction

- The analog of operator Δ is the Laplacian of the graph $\mathcal{G}$, which is defined as

$$[L_{JI}]_{J,I\in V(\mathcal{G})} = [A_{JI} - \gamma_I \delta_{JI}]_{J,I\in V(\mathcal{G})},$$

  where $[A_{JI}]_{J,I\in V(\mathcal{G})}$ is the adjacency matrix of $\mathcal{G}$ and γ_I is the degree of I.
- $A_{JI} = 1$ if J and I are connected, otherwise, $A_{JI} = 0$.
- γ_I is the number of conections of node I.



	A	B	C	D	E	F	G	H	I
A	0	1	1	1	0	0	0	0	0
B	1	0	0	0	1	1	0	0	0
C	1	0	0	1	0	0	1	0	0
D	1	0	1	0	0	0	1	1	0
E	0	1	0	0	0	0	0	0	1
F	0	1	0	0	0	0	0	0	0
G	0	0	1	1	0	0	0	0	0
H	0	0	0	1	0	0	0	0	0
I	0	0	0	0	1	0	0	0	0

Introduction

- The network analogue of (1) is

$$\begin{cases} \frac{\partial u_J}{\partial t} = f(u_J, v_J) + \varepsilon \sum_I L_{JI} u_I \\ \\ \frac{\partial v_J}{\partial t} = g(u_J, v_J) + \varepsilon d \sum_I L_{JI} v_I . \end{cases} \tag{2}$$

- The central goal of this work is to show that p-adic analysis is the natural tool to study, in a rigorous mathematical way, the system (2) and the corresponding Turing patterns.

Introduction

- By embedding the graph $\mathcal{G}$ into $\mathbb{Q}_p$, the field of p-adic numbers, we construct a family of continuous p-adic versions of system (2), which can be studied rigorously by using the classical semigroup theory.

- In this way, we are able to study the original system (2) and to obtain a new p-adic continuous version of it, which corresponds to a 'mean-field approximation' of the original system (2) .

Introduction

- From now on p denotes a fixed prime number. A p-adic number is a series of the form

$$x = x_{-k}p^{-k} + x_{-k+1}p^{-k+1} + \ldots + x_0 + x_1 p + \ldots, \text{ with } x_{-k} \neq 0, \tag{3}$$

where the x_js are p-adic digits, i.e. numbers in the set $\{0, 1, \ldots, p-1\}$. The set of all the possible series of form (3) constitutes the field of p-adic numbers $\mathbb{Q}_p$. There are natural field operations, sum and multiplication, on series of form (3).

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- There is also a natural norm in $\mathbb{Q}_p$ defined as $|x|_p = p^k$, for a nonzero p-adic number x of the form (3). The field of p-adic numbers with the distance induced by $|\cdot|_p$ is a complete ultrametric space.

- The ultrametric property refers to the fact that $|x-y|_p \leq \max\left\{|x-z|_p, |z-y|_p\right\}$ for any x, y, z in $\mathbb{Q}_p$.

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- We identify each vertex of $\mathcal{G}$ with a p-adic number of the form

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  where the I_js are p-adic digits.
- We denote by G_N^0 the set of all p-adic integers of the form (4) which correspond to the vertices of $\mathcal{G}$. In this way, we construct an embedding of $\mathcal{G}$ into $\mathbb{Q}_p$.

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- We attach to $\mathcal{G}$ the open compact subset $\mathcal{K}_N$ defined as the disjoint union of the balls $I + p^N\mathbb{Z}_p$ for $I \in G_N^0$, and a finite dimensional real vector space X_N generated by the functions $\left\{\Omega\left(p^N \left|x - I\right|_p\right)\right\}_{I \in G_N^0}$. This is the space of continuous functions on $\mathcal{G}$.

Introduction

There exists a kernel $J_N(x,y)$, which is a linear combination of functions of type $\Omega\left(p^N|x-I|_p\right)\Omega\left(p^N|y-J|_p\right)$, I, $J\in G_N^0$, such that the operator $\mathbf{L}_N : X_N \rightarrow X_N$ defined as

$$\mathbf{L}_N\varphi(x)=\int_{\mathcal{K}_N}(\varphi(y)-\varphi(x))\,J_N(x,y)dy, \tag{5}$$

where dy denotes the normalized Haar measure of the locally compact group $(\mathbb{Q}_p,+)$, is represented by the matrix $\left[L_{JI}\right]_{J,I\in G_N^0}$.

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- Operator $\mathbf{L}_N$ has a natural extension $\mathbf{L}_M$ to X_M given by the right-hand side of formula (5).

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- In this way on each $X_\bullet$, we have an operator $\mathbf{L}_\bullet$, here the dot means N, M with $M > N$ or ∞, and a continuous version of system (2):

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u^{(\bullet)}(t) \\ v^{(\bullet)}(t) \end{bmatrix} = \begin{bmatrix} f(u^{(\bullet)}(t), v^{(\bullet)}(t)) \\ g(u^{(\bullet)}(t), v^{(\bullet)}(t)) \end{bmatrix} + \begin{bmatrix} \varepsilon \mathbf{L}_{\bullet} u^{(\bullet)}(t) \\ \varepsilon d \mathbf{L}_{\bullet} v^{(\bullet)}(t) \end{bmatrix}, \\ t \in [0, \tau), \, x \in \mathcal{K}_N. \end{cases} \tag{6}$$

- We study the Cauchy problem attached to (6), when the initial datum belongs to a sufficiently small open set containing a steady state (u_0, v_0) where $f(u_0, v_0) = g(u_0, v_0) = 0$, and assuming that $\nabla f(x) \neq 0$ and $\nabla g(x) \neq 0$ for x sufficiently close to $(u_0, v_0) \in \mathbb{R}^2$.

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- For M sufficiently large, the solution of the Cauchy problem attached to the mean-field model is arbitrarily closed to the solution of system (6) in X_M.

The matrix $\mathbb{A}^{(M)}$ of the operator $\mathbf{L}_M$ acting on X_M (after renaming the elements of the basis of X_M) is a diagonal-type matrix of the form

$$\mathbb{A}^{(M)} = \begin{bmatrix} \mathbb{A}_{N;M} & & & & \\ & \ddots & & & \\ & & \mathbb{A}_{N;M} & & \\ & & & \ddots & \\ & & & & \mathbb{A}_{N;M} \end{bmatrix}_{p^{M-N} \times p^{M-N}},$$

where

$$\mathbb{A}_{N;M} = \left[p^{N-M} A_{JI} - \gamma_I \delta_{JI} \right]_{J,I \in G_N^0} = p^{N-M} \left[A_{JI} - p^{M-N} \gamma_I \delta_{JI} \right]_{J,I \in G_N^0}.$$

Introduction

The matrix $\mathbb{A}^{(M)}$ corresponds to a network constructed by using p^{M-N} *replicas of the original network*, each these replicas correspond to a network having a diffusion operator of type $\mathbb{A}_{N;M}$ and the corresponding p-adic diffusion equation is

$$\frac{\partial f^{(N)}(x,t)}{\partial t} = \varepsilon' \mathbf{L}_{N,\lambda} f^{(N)}(x,t) \tag{7}$$

where $\varepsilon' = p^{N-M}\varepsilon$, $\lambda = p^{M-N}$, and $\mathbf{L}_{N,\lambda} : X_N \to X_N$ is defined as

$$L_{N,\lambda}\varphi(x) = \int_{\mathcal{K}_N} \{\varphi(y) - \lambda\varphi(x)\} J_N(x,y)\, dy.$$

Introduction

- The equations of type (7) form a parametric family indexed by (ε', λ), which is invariant under a scale change of type $(\varepsilon', \lambda) \rightarrow \left(\delta\varepsilon', \lambda\delta^{-1}\right)$, for $\delta \in (0, 1)$.

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- In conclusion, the mean-field approximation is the 'limit' of system (6) in X_M when M tends to infinity. In turn, any solution of system (6) in X_M $(M > N)$ is made of p^{M-N} solutions of p^{M-N} systems of type (6) in X_N, each of them is a scaled version (a scaled replica) of the original system (2).

In order to understand the 'physical contents' of the mean-field model, it is completely necessary to study its diffusion mechanism, which means to study the following Cauchy problem:

$$\begin{cases} \frac{\partial f(x,t)}{\partial t} = \varepsilon \mathbf{L} f(x,t), \quad x \in \mathcal{K}_N,\ t > 0 \\ \\ f(x,0) = f_0(x) \in X_\infty. \end{cases} \tag{8}$$

The semigroup attached to (8), $\left\{ e^{\varepsilon t \mathbf{L}} \right\}_{t \geq 0}$, is a Feller semigroup, and consequently, there is a Markov process attached to (8).

- This implies that in the mean-field model, the chemical species u, v interact via a random walk like in the classical case (1): the chemical reactions involving species u, v occur as a consequence of a random walk of the particles forming them, this random walk is produced by changes in the concentrations, which can be deterministically model by Fick's law of diffusion, the whole picture is coded into the classical heat equation.

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- A similar result holds in X_M for equation (8).

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- To achieve this goal, it is necessary to understand the spectra of operators $\left[L_{IJ}\right]_{I,J\in G_N^0}$, $\mathbf{L}_N$, $\mathbf{L}_M$ with $M > N$, $\mathbf{L}$.

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- To achieve this goal, it is necessary to understand the spectra of operators $[L_{IJ}]_{I,J\in G_N^0}$, $\mathbf{L}_N$, $\mathbf{L}_M$ with $M>N$, $\mathbf{L}$.
- The spectrum of the graph Laplacian matrix $[L_{JI}]_{J,I\in G_N^0}$ is well-understood. Since the adjacency matrix $[A_{JI}]_{J,I\in G_N^0}$ is symmetric, the eigenvalues, μ_I, $I\in G_N^0$, of $[L_{JI}]_{J,I\in G_N^0}$ are non-positive and $\mu_{I_0}=\max_{I\in G_N^0}\{\mu_I\}=0$. If λ_I, $I\in G_N^0$, are the eigenvalues of $[A_{JI}]_{J,I\in G_N^0}$, with multiplicities $mult(\lambda_I)$, then the eigenvalues of the discrete graph Laplacian are

$$\mu_I=\lambda_I-\gamma_I, \text{ with multiplicity } mult(\lambda_I), \text{ for } I\in G_N^0.$$

Introduction

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$$\{\lambda_I - \gamma_I ; I \in G_N^0 \setminus \{I_0\}\} \bigsqcup \{-\gamma_I ; I \in G_N^0\} \subset (-\infty, 0) .$$

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The space $L^2(\mathcal{K}_N, \mathbb{C})$ has an orthonormal basis formed by eigenfunctions of operator $\mathbf{L}$.
- The difference set between the spectra of $\mathbf{L}$ and $[L_{JI}]_{J,I \in G_N^0}$ is
$$\sigma(\mathbf{L}) \smallsetminus \sigma\left([L_{JI}]_{J,I \in G_N^0}\right) = \left\{-\gamma_I; I \in G_N^0 \setminus \{I_0\}\right\}.$$
For each of these spectral values there exist $p - 1$ eigenfunctions of the form
$$p^{\frac{N}{2}} \exp\left(\left\{p^{-N-1} jx\right\}_p\right) \Omega\left(p^N |x - I|_p\right),$$

In $X_\bullet$ the Turing instability criteria can be established using the classical argument. In X_∞, the Turing pattern has the form

$$\sum_{\kappa_1<\kappa<\kappa_2}\sum_{I} A_{I\kappa} e^{\lambda t}\Omega\left(p^N\left|x-I\right|_p\right) + \sum_{\kappa_1<\kappa<\kappa_2}\sum_{I,j} A_{Ij\kappa} e^{\lambda t} p^{\frac{N}{2}} \cos\left(\left\{p^{-N-1}jx\right\}_p\right)\Omega\left(p^N\left|x-I\right|_p\right) + \sum_{\kappa_1<\kappa<\kappa_2}\sum_{I,j} B_{Ij\kappa} e^{\lambda t} p^{\frac{N}{2}} \sin\left(\left\{p^{-N-1}jx\right\}_p\right)\Omega\left(p^N\left|x-I\right|_p\right) \quad (9)$$

for $t \to +\infty$, where κ runs through unstable modes.

Introduction

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- In the last fifty years, Turing patterns produced by reaction-diffusion systems on networks have been studied intensively, mainly by physicists, biologists and engineers.

- Nowadays, there is a large amount of experimental results, about the behavior of these systems, obtained mainly via computer simulations using large random networks.

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- Multistability, that is, coexistence of a number of different patterns for the same parameters values, is typically found and hysteresis phenomena are observed.
- They used mean-field approximation to understand the Turing patterns when $d > d_c$, and proposed that the mean-field approximation is the natural framework to understand the peculiar behavior of the Turing patterns on networks.

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- By identifying the ball $I + p^N\mathbb{Z}_p$ with a cluster, we have that Turing pattern (9) is organized in a finite number of disjoint clusters, each of them supporting a stationary pattern, all these patterns are controlled by the same kinetic parameters.

- Notice that the occurrence of clusters in the Turing patterns is a direct consequence of the hierarchical structure of $\mathbb{Q}_p$: every ball is a finite disjoint union of balls of smaller radii.

- More generally, clustering (as the method of hierarchical classification of objects using trees) is deeply connected with the geometric of ultrametric spaces.

- Our results show that Turing criteria remains essentially the same as in the classical case. It is relevant to mention here, that from a mathematical perspective, it is necessary to show first, that the Cauchy problem attached to the reaction-diffusion system has a solution with initial data near to the steady state (u_0, v_0). Then, one shows that the solution has an asymptotic profile of Turing type.

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- The solutions of reaction-diffusion systems may not exist for all times or simply vanish.

- We have not found in the current literature a rigorous study of the Cauchy problem associated with reaction-diffusion systems on networks. However, the study of differential equations on graphs is nowadays a relevant mathematical matter.

We consider an arbitrary graph $\mathcal{G}$ with vertices $I \in G_N^0$, where G_N^0 is a finite set.

When there is no connection between the vertices, the dynamics on each vertex is controlled by a local interactions described as

$$\begin{cases} \frac{\partial u_J}{\partial t} = f(u_J, v_J) \\ \\ \frac{\partial v_J}{\partial t} = g(u_J, v_J), \end{cases} \tag{10}$$

for $J \in G_N^0$, where a pair $(u_J, v_J) = (u_J(t), v_J(t))$ represents some quantities in the vertex J, such as population densities of biological species or concentrations of chemical substances.

- When connection between vertices is taken into account, we assume the existence of a flux of quantities between these vertices.

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- If two vertices are not connected, there is no flux between them.
- The flux is assumed to be given by Fick's law of diffusion, which means that the flux is proportional to the difference of quantities on the two vertices.

Therefore the dynamics of u_J and v_J on vertex J is described as

$$\begin{cases} \frac{\partial u_J}{\partial t} = f(u_J, v_J) + \varepsilon \sum_{I \in G_N^0} A_{JI} \{u_I - u_J\} \\ \\ \frac{\partial v_J}{\partial t} = g(u_J, v_J) + \varepsilon d \sum_{I \in G_N^0} A_{JI} \{v_I - v_J\}, \end{cases} \tag{11}$$

for $J \in G_N^0$, where

$$A_{JI} := \begin{cases} 1 & \text{if the vertices } J \text{ and } I \text{ are connected} \\ \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $[A_{JI}]_{J,I \in G_N^0}$ is called the *adjacency matrix* of $\mathcal{G}$.

The positive constants ε and εd denote the diffusivities of u and v. The number of edges connecting to vertex I is $\gamma_I := \sum_{J \in G_N^0} A_{IJ}$. We set $\gamma_{\mathcal{G}} := \max_{I \in G_N^0} \gamma_I$.

For each $J \in G_N^0$, we can rewrite the flux term as

$$\sum_{I \in G_N^0} A_{JI} \{u_I - u_J\} = \sum_{I \in G_N^0} L_{JI} u_I,$$

where $L_{JI} = A_{JI} - \gamma_I \delta_{JI}$, here δ_{JI} denotes the Kronecker delta. The matrix $[L_{JI}]_{J,I \in G_N^0}$ is called the *Laplacian matrix of the graph* $\mathcal{G}$.

Then, system (11) can be rewritten as

$$\begin{cases} \frac{\partial u_J}{\partial t} = f(u_J, v_J) + \varepsilon \sum_{I \in G_N^0} L_{JI} u_I \\ \\ \frac{\partial v_J}{\partial t} = g(u_J, v_J) + \varepsilon d \sum_{I \in G_N^0} L_{JI} v_I, \end{cases} \quad \text{for } J \in G_N^0. \tag{12}$$

We set

$$G_N := \mathbb{Z}_p / p^N \mathbb{Z}_p \text{ for } N \geq 1.$$

We identify G_N with the set of representatives of the form

$$I = I_0 + I_1 p + \ldots + I_{N-1} p^{N-1}, \tag{13}$$

where the I_js are p-adic digits. We assume that $G_N^0 \subset G_N$.

This implies that the number of vertices $\#G_N^0$ of $\mathcal{G}$ must satisfy $\#G_N^0 \leq p^N$.

There is no a canonical way of choosing N and p. On the other hand, since the elements of the form (13) belong to $\mathbb{Z}_p \smallsetminus p^N \mathbb{Z}_p$, the assumption $G_N^0 \subset G_N$ gives rise an embedding of $\mathcal{G}$ into $\mathbb{Z}_p \smallsetminus p^N \mathbb{Z}_p$.

We define

$$\mathcal{K}_N = \bigsqcup_{I \in G_N^0} I + p^N \mathbb{Z}_p.$$

Notice that $\mathcal{K}_N$ is an open compact subset of $\mathbb{Z}_p$.

We also define

$$J_N(x,y) = p^N \sum_{J \in G_N^0} \sum_{K \in G_N^0} A_{JK} \Omega\left(p^N |x-J|_p\right) \Omega\left(p^N |y-K|_p\right), \quad (14)$$

x, $y \in \mathbb{Q}_p$, where $[A_{JI}]_{J,I \in G_N^0}$ is the adjacency matrix of graph $\mathcal{G}$. Notice that $J_N(x,y)$ is a test function from $\mathcal{D}(\mathcal{K}_N \times \mathcal{K}_N, \mathbb{R})$.

We denote by $C(\mathcal{K}_N, \mathbb{R})$ the vector space of all the continuous real-valued functions on $\mathcal{K}_N$ endowed with supremum norm, denoted as $\|\cdot\|_\infty$.

We denote by X_N, the $\mathbb{R}$-vector space consisting of all the test functions supported in $\mathcal{K}_N$ having the form

$$\varphi(x) = \sum_{J \in G_N^0} \varphi(J)\, \Omega\left(p^N |x - J|_p\right),$$

where $\varphi(J) \in \mathbb{R}$. We endow X_N with the $\|\cdot\|_\infty$-norm. Notice that $\left\{\Omega\left(p^N |x - J|_p\right)\right\}_{J \in G_N^0}$ is a basis of X_N.

Then X_N is a closed subspace of $C(\mathcal{K}_N, \mathbb{R})$, in addition,

$$X_N \simeq \left(\mathbb{R}^{\# G_N^0}, \|\cdot\|_\infty\right), \text{ as Banach spaces,}$$

where $\left\|\left(x_1, \ldots, x_{\# G_N^0}\right)\right\|_\infty := \max\left\{|x_1|, \ldots, \left|x_{\# G_N^0}\right|\right\}$ for $\left(x_1, \ldots, x_{\# G_N^0}\right) \in \mathbb{R}^{\# G_N^0}$.

$$\mathbf{L}_N\varphi\left(x\right)=\int_{\mathcal{K}_N}\left\{\varphi\left(y\right)-\varphi\left(x\right)\right\}J_N(x,y)dy,\text{ for }\varphi\in X_N.$$

$\mathbf{L}_N : X_N \rightarrow X_N$ is a linear bounded operator.

$$\begin{aligned}&\mathbf{L}_N\Omega\left(p^N\left|x-I\right|_p\right)\\&=\sum_{J\in G_N^0}A_{JI}\Omega\left(p^N\left|x-J\right|_p\right)-\left(\sum_{K\in G_N^0}A_{IK}\right)\Omega\left(p^N\left|x-I\right|_p\right)\\&=\sum_{J\in G_N^0}A_{JI}\Omega\left(p^N\left|x-J\right|_p\right)-\gamma_I\Omega\left(p^N\left|x-I\right|_p\right)\\&=\sum_{J\in G_N^0}\left\{A_{JI}-\gamma_I\delta_{JI}\right\}\Omega\left(p^N\left|x-J\right|_p\right).\end{aligned}$$

Consequently, operator $\mathbf{L}_N : X_N \rightarrow X_N$ is represented by the matrix

$$\left[A_{JI}-\gamma_I\delta_{JI}\right]_{J,I\in G_N^0}. \qquad (15)$$

The original system can be rewritten as

$$\begin{cases} u^{(N)}(\cdot,t), v^{(N)}(\cdot,t) \in C^1(\mathbb{R}_+, X_N); \\ \frac{\partial u^{(N)}(x,t)}{\partial t} = f(u^{(N)}(x,t), v^{(N)}(x,t)) + \varepsilon \mathbf{L}_N u^{(N)}(x,t) \\ \frac{\partial v^{(N)}(x,t)}{\partial t} = g(u^{(N)}(x,t), v^{(N)}(x,t)) + \varepsilon d \mathbf{L}_N v^{(N)}(x,t). \end{cases} \tag{16}$$

Notice that for $\varphi \in C(\mathcal{K}_N, \mathbb{R})$, the function

$$\mathbf{L}\varphi(x) = \int_{\mathcal{K}_N} \{\varphi(y) - \varphi(x)\} J_N(x, y)\, dy \tag{17}$$

belongs to $C(\mathcal{K}_N, \mathbb{R})$, and that operator $\mathbf{L}$ is a linear continuous operator satisfying

$$\|\mathbf{L}\| \leq 2\gamma_{\mathcal{G}} \text{ and } \mathbf{L}_N = \mathbf{L}\mid_{X_N}.$$

By using the fact that operator $\mathbf{L}$ as an extension of $\mathbf{L}_N$, result natural to postulate that the system

$$\begin{cases} u(\cdot, t), v(\cdot, t) \in C^1(\mathbb{R}_+, C(\mathcal{K}_N, \mathbb{R})); \\ \frac{\partial u(x,t)}{\partial t} = f(u, v) + \varepsilon \mathbf{L} u(x, t) \\ \frac{\partial v(x,t)}{\partial t} = g(u, v) + \varepsilon d \mathbf{L} v(x, t), \end{cases} \tag{18}$$

is a '*p*-adic analog' of system (16).

We study the following Cauchy problem:

$$\begin{cases} h(x,t) \in C^1((0,\infty), C(\mathcal{K}_N, \mathbb{R})); \\ \frac{\partial h(x,t)}{\partial t} = \varepsilon \mathbf{L} h(x,t), \quad x \in \mathcal{K}_N, \ t > 0; \\ h(x,0) = h_0(x) \in C(\mathcal{K}_N, \mathbb{R}), \end{cases} \tag{19}$$

where $\mathbf{L} : C(\mathcal{K}_N, \mathbb{R}) \to C(\mathcal{K}_N, \mathbb{R})$ is the operator defined above.

We show that equation (19) is a 'p-adic heat equation,' which means that the semigroup attached to it is a Feller semigroup, and consequently the differential equation in (19) is associated with a p-adic diffusion process in $\mathcal{K}_N$.

Yosida-Hille-Ray theorem and Feller semigroups

A semigroup $\{\mathbf{Q}(t)\}_{t\geq 0}$ on $C(\mathcal{K}_N, \mathbb{R})$ is said to be *positive* if $\mathbf{Q}(t)$ is a positive operator for each $t \geq 0$, i.e. it maps non-negative functions to non-negative functions.

An operator $(\mathbf{A}, Dom(\mathbf{A}))$ on $C(\mathcal{K}_N, \mathbb{R})$ is said to satisfy the *positive maximum principle* if whenever $h \in Dom(\mathbf{A}) \subseteq C(\mathcal{K}_N, \mathbb{R})$, $x_0 \in \mathbb{Q}_p$, and $\sup_{x\in\mathbb{Q}_p} h(x) = h(x_0) \geq 0$ we have $\mathbf{A}h(x_0) \leq 0$.

Theorem (Hille-Yosida-Ray Theorem)

Let $(\mathbf{A}, Dom(\mathbf{A}))$ be a linear operator on $C(\mathcal{K}_N, \mathbb{R})$. The closure $\overline{\mathbf{A}}$ of $\mathbf{A}$ on $C(\mathcal{K}_N, \mathbb{R})$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{\mathbf{Q}_t\}_{t\geq 0}$ on $C(\mathcal{K}_N, \mathbb{R})$ if and only if:
(i) $Dom(\mathbf{A})$ is dense in $C(\mathcal{K}_N, \mathbb{R})$;
(ii) $\mathbf{A}$ satisfies the positive maximum principle;
(iii) $Rank(\eta\mathbf{I} - \mathbf{A})$ is dense in $C(\mathcal{K}_N, \mathbb{R})$ for some $\eta > 0$.

Definition

A family of bounded linear operators $\mathbf{P}_t : C(\mathcal{K}_N, \mathbb{R}) \rightarrow C(\mathcal{K}_N, \mathbb{R})$ is called a Feller semigroup if
(i) $\mathbf{P}_{s+t} = \mathbf{P}_s \mathbf{P}_t$ and $\mathbf{P}_0 = I$;
(ii) $\lim_{t\rightarrow 0} ||\mathbf{P}_t h - h||_\infty = 0$ for any $h \in C(\mathcal{K}_N, \mathbb{R})$;
(iii) $0 \leq \mathbf{P}_t h \leq 1$ if $0 \leq h \leq 1$, with $h \in C(\mathcal{K}_N, \mathbb{R})$ and for any $t \geq 0$.

Therefore, Theorem 1 characterizes the Feller semigroups, more precisely, if $(\mathbf{A}, Dom(\mathbf{A}))$ satisfies Theorem 1, then $\mathbf{A}$ has a closed extension which is the generator of a Feller semigroup.

Yosida-Hille-Ray theorem and Feller semigroups

Lemma

The operator $\varepsilon\mathbf{L}$ *generates a strongly continuous, positive, contraction semigroup* $\left\{e^{t\varepsilon\mathbf{L}}\right\}_{t\geq 0}$ *on* $C\left(\mathcal{K}_N, \mathbb{R}\right)$.

Theorem

There exists a probability measure $p_t\left(x,\cdot\right)$, $t\geq 0$, $x\in\mathcal{K}_N$, *on the Borel* σ*-algebra of* $\mathcal{K}_N$, *such that Cauchy problem (19) has a unique solution of the form*

$$h(x,t)=\int_{\mathcal{K}_N} h_0(y)p_t\left(x,dy\right).$$

In addition, $p_t\left(x,\cdot\right)$ *is the transition function of a Markov process* $\mathfrak{X}$ *whose paths are right continuous and have no discontinuities other than jumps.*

Self-similarity

The above theorem can be easily extended to a larger class of operators. For instance, take $J(x,y) \in L^{\infty}\left(\mathcal{K}_N \times \mathcal{K}_N, \mathbb{R}\right)$, $J(x,y) \geq 0$ and $\lambda \geq 1$, and set

$$\varepsilon L_{\lambda}\varphi\left(x\right) = \varepsilon \int\limits_{\mathcal{K}_N} \left\{\varphi(y) - \lambda\varphi\left(x\right)\right\} J\left(x,y\right) dy. \tag{20}$$

Then $L_{\lambda} : C\left(\mathcal{K}_N, \mathbb{R}\right) \rightarrow C\left(\mathcal{K}_N, \mathbb{R}\right)$ is a linear bounded operator, with $\left\|\varepsilon L_{\lambda}\right\| \leq \left(1+\lambda\right)\varepsilon \left\|J\right\|_{\infty}$.

Notice that condition $\lambda \geq 1$ is essential to assure that operator εL_{λ} satisfies the positive maximum principle.

Self-similarity

The above theorem also holds for the following Cauchy problem:

$$\begin{cases} h(x,t) \in C^1((0,\infty), C(\mathcal{K}_N, \mathbb{R})); \\ \frac{\partial h(x,t)}{\partial t} = \varepsilon \mathbf{L}_\lambda h(x,t), \quad x \in \mathcal{K}_N, \ t > 0; \\ h(x,0) = h_0(x) \in C(\mathcal{K}_N, \mathbb{R}). \end{cases} \tag{21}$$

Then, for a fixed $J(x,y) \in L^\infty(\mathcal{K}_N \times \mathcal{K}_N, \mathbb{R})$, $J(x,y) \geq 0$, (21) is a family of p-adic diffusion equations parametrized by the set

$$\mathcal{P} := \left\{ (\varepsilon, \lambda) \in \mathbb{R}_+^2; \ \varepsilon > 0, \ \lambda \geq 1 \right\}.$$

We identify family (21) with the set $\mathcal{P}$. Now, for $\sigma \in (0,1]$, we define the mapping:

$$\begin{array}{cccc} S_\sigma : & \mathcal{P} & \rightarrow & \mathcal{P} \\ & (\varepsilon,\lambda) & \rightarrow & \left(\sigma\varepsilon,\sigma^{-1}\lambda\right). \end{array}$$

The set of all S_σ for $\sigma \in (0,1]$ form naturally a monoid, under the composition of functions, denoted as $\mathcal{S}_{\mathcal{P}}$. Therefore, we have established the following result:

Theorem

The family $\mathcal{P}$ is invariant under the action of the monoid $\mathcal{S}_{\mathcal{P}}$.


To be continued...