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Centro de Investigación en Matemáticas, A.C.

Iteration of rational functions over the complex p-adic numbers

Víctor Nopal Coello

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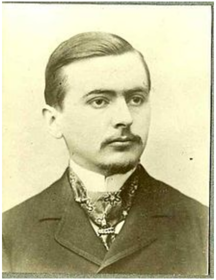
INTRODUCTION

History

The theory of iteration of rational functions over the Riemann sphere emerged in the papers produced, independently, by Pierre Fatou and Gaston Julia almost 100 years ago.



(a) Pierre Fatou

(b) Gaston Julia

History

The subject went into a quiet period for about 50 years, until around 1980 a number of leading present day mathematicians such as Mandelbrot, Douady, Hubbard, Sullivan, Milnor, Thurston and others began taking up the questions where Fatou and Julia had left off.

(c) Benoît Mandelbrot

(d) John Milnor

History

One of the main theorems in the theory of iteration of rational function is the **no-wandering-domain theorem**, given by Sullivan.

Theorem

Every component of the Fatou set of a rational map is eventually periodic.

(e) Dennis Sullivan

History

In the past thirty years several experts such as R. Benedetto, M. Baker, R. Rumely, J. Silverman, J. Rivera-Letelier, J. Kiwi, L.-C. Hsia and others reinterpreted the theory of iteration of rational functions with coefficients over any complete and algebraically closed non-archimedean field (for example, over $\mathbb{C}_p$).

(f) R. Benedetto

(g) J. Rivera-Letelier

PART I
Dynamics over the projective space $\mathbb{P}(\mathbb{C}_p)$

Notation

For the rest of this talk we use the following notation.

$\mathbb{C}_p$	the field of the p-adic complex numbers.
$\lvert \cdot \rvert$	the p-adic norm in $\mathbb{C}_p$.
$\mathcal{O}_{\mathbb{C}_p}$	the ring of integers, which is defined as $\{z \in \mathbb{C}_p : \lvert z\rvert \leq 1\}$.
$\mathcal{M}_{\mathbb{C}_p}$	the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, which is defined as $\{z \in \mathbb{C}_p : \lvert z\rvert < 1\}$.
$\widetilde{\mathbb{C}}_p$	the residue field of $\mathbb{C}_p$, which is defined as $\widetilde{\mathbb{C}}_p := \mathcal{O}_{\mathbb{C}_p}/\mathcal{M}_{\mathbb{C}_p}$.
$\lvert\mathbb{C}_p^{\times}\rvert$	the value group of $\mathbb{C}_p$, defined as $\{\lvert a\rvert : a \in \mathbb{C}_p^{\times}\}$.

The projective space $\mathbb{P}(\mathbb{C}_p)$

Let us denote by $\mathbb{P}(\mathbb{C}_p)$ the projective space of $\mathbb{C}_p$, which is the set of lines in $\mathbb{C}_p \times \mathbb{C}_p$ that contains the origin. There is a natural identification between $\mathbb{P}(\mathbb{C}_p)$ and $\mathbb{C}_p \cup \{\infty\}$.
The chordal metric in $\mathbb{P}(\mathbb{C}_p)$ is defined by

$$\mathrm{d}(z_0, z_1) = \frac{|z_0 - z_1|}{\max\{|z_0|, 1\} \cdot \max\{|z_1|, 1\}}, \text{ for } z_0, z_1 \in \mathbb{C}_p$$

and $\mathrm{d}(z_0, \infty) := \lim_{z_1 \to \infty} d(z_0, z_1)$.

Disks and affinoids

Given $r > 0$ and $a \in \mathbb{C}_p$, define the sets

$$D_r(a) = \{z \in \mathbb{C}_p : |z - a| < r\} \text{ and } B_r(a) = \{z \in \mathbb{C}_p : |z - a| \leq r\}.$$

We say that $D_r(a)$ is an **open ball** (or **disk**) of $\mathbb{C}_p$ and that $B_r(a)$ a **closed ball** (or simply a **ball**) of $\mathbb{C}_p$. If $r \notin |\mathbb{C}_p^\times|$ then $D_r(a) = B_r(a)$ and it is called **irrational ball**.

A disk of $\mathbb{P}(\mathbb{C}_p)$ is a disk of $\mathbb{C}_p$ or the complement in $\mathbb{P}(\mathbb{C}_p)$ of a ball of $\mathbb{C}_p$.

A **connected open affinoid** is a finite intersection of disks of $\mathbb{P}(\mathbb{C}_p)$, and an **open affinoid** as a finite union of connected open affinoids.

Periodic points

Let $R \in \mathbb{C}_p(z)$ be a rational function of degree $d \geq 2$ and let $z_0 \in \mathbb{C}_p$ be a periodic point of period k. We define the multiplier of z_0 as

$$\lambda := (R^k)'(z_0) = R'(z_0) \cdot R'(R(z_0)) \cdots R'(R^{k-1}(z_0)).$$

If $|\lambda| < 1$ we say that z_0 is attracting, if $|\lambda| = 1$, we say that z_0 is indifferent, and if $|\lambda| > 1$ then z_0 is repelling.

p-adic Fatou and Julia sets

The family $\{R^n\}_{n\geq 0}$ is equicontinuous for the chordal metric at $x \in \mathbb{P}(\mathbb{C}_p)$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $z \in \mathbb{P}(\mathbb{C}_p)$ and for $n \geq 0$

$$\mathrm{d}(x, z) < \delta \text{ implies } \mathrm{d}(R^n(x), R^n(z)) < \epsilon.$$

The family $\{R^n\}_{n\geq 0}$ is equicontinuous on a subset $X \subset \mathbb{P}(\mathbb{C}_p)$ if it is equicontinuous at each point $x \in X$.

Definition

Let $R \in \mathbb{C}_p(z)$ be a rational function. The p-**adic Fatou set**, F_R, is defined as the largest open set where the family $\{R^n\}_{n\geq 1}$ is equicontinuous for the chordal metric. The p-**adic Julia set** is the complement of the Fatou set, i.e., $J_R = \mathbb{P}(\mathbb{C}_p) - F_R$.

Properties of the p-adic Fatou and Julia sets

Proposition

The p-adic Fatou and Julia sets satisfy the following properties:

- F_R *is open, whereas* J_R *is closed.*
- $R^{-1}(F_R) = R(F_R) = F_R$ *and* $R^{-1}(J_R) = R(J_R) = J_R$.
- *If* $\varphi \in \mathrm{Aut}(\mathbb{P}(\mathbb{C}_p))$, *then* $F_{\varphi\circ R\circ\varphi^{-1}} = \varphi(F_R)$ *and* $J_{\varphi\circ R\circ\varphi^{-1}} = \varphi(J_R)$.
- *The repelling periodic points are in* J_R, *whereas the non-repelling periodic points are in* F_R.
- $F_R \neq \emptyset$, *but* J_R *may be empty.*
- *If* $J_R \neq \emptyset$, *then it is perfect and has empty interior.*

Theorem of classification

The subset $\mathcal{E}(R) \subset \mathbb{P}(\mathbb{C}_p)$ is defined as the interior of the set of recurrent points, and it is called **quasiperiodic domain**.

Theorem (Classification, J. Rivera-Letelier, [2])

Let $R \in \mathbb{C}_p(z)$ be a rational function. The Fatou set F_R consists of the following disjoint sets:

1. *attracting basins,*
2. $\mathcal{E}'(R) = \bigcup_{n\geq 1} R^{-n}(\mathcal{E}(R))$*, and*
3. *the union of wandering disks not attracted to an attracting periodic point.*

Fatou Components

Theorem

- *Let C be the immediate attracting basin of the attracting fixed point z_0, then*
  1. *C is a disk, or*
  2. *C is of Cantor type.*
- *Let C be a component of $\mathcal{E}(R)$, then C is an open connected affinoid, i.e.,*

$$C = \mathbb{P}(\mathbb{C}_p) - B_0 \cup \cdots \cup B_n,$$

  where $n \geq 0$ and $B_0, \ldots, B_n$ are closed balls.
  1. *If $n = 0$ then we say that C is a Siegel disk.*
  2. *If $n > 0$ then we say that C in an n-Herman ring.*

Rational functions with wandering domains

- In 2003, Rivera-Letelier conjectured that F_R does not contain wandering domains.
- In 2004, Benedetto showed that there exists $a \in \mathbb{C}_p$ such that the rational function

$$\phi_a(z) = (1-a)z^{p+1} + az^p$$

has wandering domains.

Note that $\deg(\phi_a(z)) = p + 1 \geq 3$.

Question

Do rational functions of degree 2 over $\mathbb{C}_p$ have wandering domains?

Good reduction

Recall that the residue field is $\widetilde{\mathbb{C}}_p := \mathcal{O}_{\mathbb{C}_p}/\mathcal{M}_{\mathbb{C}_p}$. Let $a \in \mathcal{O}_{\mathbb{C}_p}$, then we denote by $\widetilde{a}$ the equivalent class of a in the residue field.
Let $R \in \mathbb{C}_p(z)$ be a rational function. Then there exist polynomials $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $g(z) = b_0 + b_1 z + \cdots + b_m z^m$ such that $R(z) = \frac{f(z)}{g(z)}$ and $\max\{|a_0|, \ldots, |a_n|, |b_0|, \ldots, |b_m|\} = 1$. We define the reduction of R as

$$\widetilde{R}(z) = \frac{\widetilde{a}_0 + \widetilde{a}_1 z + \cdots + \widetilde{a}_n z^n}{\widetilde{b}_0 + \widetilde{b}_1 z + \cdots + \widetilde{b}_m z^m}$$

If $\deg(\widetilde{R}) = \deg(R)$ then we say that R has **good reduction**. Otherwise R has **bad reduction**.

Good reduction

If there exists $\varphi \in \mathrm{Aut}(\mathbb{P}(\mathbb{C}_p))$ such that $\varphi \circ R \circ \varphi^{-1}$ has good reduction, then we say that R is **simple**.

Proposition

Let $R \in \mathbb{C}_p(z)$ be a rational function of degree $\deg(R) \geq 2$, which is simple. Then

- $J_R = \emptyset$ *and* $F_R = \mathbb{P}(\mathbb{C}_p)$.
- F_R *does not contain neither wandering domains nor n-Herman rings.*

Example 1: $R(z) = z^2$

- Since $\widetilde{R} = R$, then R has good reduction.
- The fixed points of R are $0, 1$ and ∞.
- If $z \in \mathbb{C}_p$ is a periodic point of period $k \geq 2$, then $|z| = 1$.
- $R'(z) = 2z$. Hence 0 and ∞ are attracting fixed points.
- If $p = 2$, then all the periodic points are attracting and their attracting basins are disks.
- If $p > 2$, then the periodic points, different than 0 and ∞, are indifferent and they belong to Siegel disks.

Example 2: $R(z) = \frac{pz^2+z}{z+p}$

- R has bad reduction. Moreover, R is not simple.
- The fixed points of R are $0, 1$ and ∞.
- 0 and ∞ are repelling fixed points, whereas 1 is an attracting fixed point.
- The attracting basin of 1, say C, is of Cantor type and $F_R = C$.

Example 3: $R(z) = \frac{z^2 - z}{2z - p^2}$, with $p > 2$

- The fixed points of R are $0, p^2 - 1$ and ∞.
- 0 is a repelling fixed point, whereas $p^2 - 1$ and ∞ are indifferent fixed points.
- The Fatou set F_R contains only one n-Herman ring, say A, which contains the indifferent fixed points.
- The Fatou set F_R does not contain wandering domains.

PART II
Dynamics over the Berkovich projective line $\mathbb{P}_B$

The Berkovich projective line $\mathbb{P}_B$

- $\mathbb{P}_B$ is a compact, Hausdorff and path-connected topological space.
- $\mathbb{P}_B$ contains four types of points, namely points of type I, II, III or IV.
- $\mathbb{P}(\mathbb{C}_p) \subset \mathbb{P}_B$ is the set of points of type I, which is dense in $\mathbb{P}_B$.
- The points of type II correspond to rational balls.
- The points of type III correspond to irrational balls.
- The points of type IV correspond to sequences of balls $\{B_n\}_{n\geq 0}$, such that $B_0 \supset B_1 \supset B_2 \supset \cdots$ and $\bigcap_{n\geq 0} B_n = \emptyset$.

Berkovich Disks and Berkovich affinoids

Definition

Let $X \subset \mathbb{P}_B$ be a finite subset of points of type II and let $C \subset \mathbb{P}_B \setminus X$ be connected component. If ∂C consists of a single point, then we say that C is a **Berkovich disk**, otherwise we say that C is a **Berkovich open connected affinoid**.

Remark

- *If C is a Berkovich disk then $C \cap \mathbb{P}(\mathbb{C}_p)$ is a disk of $\mathbb{P}(\mathbb{C}_p)$. Analogously, if C is a Berkovich open connected affinoid then $C \cap \mathbb{P}(\mathbb{C}_p)$ is an open connected affinoid.*
- *The set of Berkovich open affionoids and Berkovich disks, form a basis for the topology of $\mathbb{P}_B$*

Rational functions

Let $R \in \mathbb{C}_p(z)$ be a rational function, then the action R in $\mathbb{P}(\mathbb{C}_p)$ extends continuously to an action $R : \mathbb{P}_B \to \mathbb{P}_B$ as follows:
Let $x \in \mathbb{P}_B$ be a point of type II or III which correspond to the ball $B_r(a)$, and suppose that $R(a) \neq \infty$. Then there exists $0 < r' < r$ close enough to r, such that

$$R(B_r(a) \setminus B_{r'}(a)) = B_t(b) \setminus B_{t'}(b)$$

for some $t, t' \in \mathbb{R}_+$ and some $b \in \mathbb{C}_p$.
Let $y \in \mathbb{P}_B$ be the point which correspond to the ball $B_t(b)$. Then we define $R(x) = y$.

Proposition

R takes points of each type to points of the same type.

Periodic points

Let $x \in \mathbb{P}_B \setminus \mathbb{P}(\mathbb{C}_p)$ be a periodic point of period $k \geq 1$, then there is a number $\deg_R(x) \in \{1, 2, \ldots, \deg(R)\}$ called the **local degree** of R in x.

If $\deg_R(x) = 1$ the we say that x is an indifferent periodic point, whereas if $\deg_R(x) > 1$ then we say that x is a repelling periodic point.

Rivera-Letelier proved that if x is a periodic point of type III or IV then it is indifferent. A periodic point of type II can be indifferent or repelling.

Berkovich Fatou and Julia sets

Definition (Definition 8.1 in [1])

Let $R \in \mathbb{C}_p(z)$ be a rational function. An open set $U \subset \mathbb{P}_B$ is dynamically stable under R if $\bigcup_{n\geq 0} R^n(U)$ omits infinitely many points on $\mathbb{P}_B$.
The **Berkovich Fatou** set of R, denoted $F_{B,R}$, is the subset of $\mathbb{P}_B$ consisting of all points $x \in \mathbb{P}_B$ having a dynamically stable neighborhood.
The **Berkovich Julia** set of R, denoted $J_{B,R}$, is the complement, $\mathbb{P}_B \setminus F_{B,R}$, of the Berkovich Fatou set.

Properties of the Berkovich Fatou and Julia sets

Proposition

The Berkovich Fatou and Julia sets satisfy the following properties:

- $F_{B,R}$ *is open, whereas* $J_{B,R}$ *is closed.*
- $R^{-1}(F_{B,R}) = R(F_{B,R}) = F_{B,R}$ *and* $R^{-1}(J_{B,R}) = R(J_{B,R}) = J_{B,R}$.
- *If* $\varphi \in \mathrm{Aut}(\mathbb{P}(\mathbb{C}_p))$, *then* $F_{B,\varphi\circ R\circ\varphi^{-1}} = \varphi(F_{B,R})$ *and* $J_{B,\varphi\circ R\circ\varphi^{-1}} = \varphi(J_{B,R})$.
- *The repelling periodic point are in* $J_{B,R}$, *whereas the non-repelling periodic point are in* $F_{B,R}$.
- $F_{B,R} \neq \emptyset$.
- $J_{B,R} \neq \emptyset$. *And if* R *is simple then* $J_{B,R} = \{x\}$ *where* x *is a repelling fixed point of type II.*
- $F_R = F_{B,R} \cap \mathbb{P}(\mathbb{C}_p)$ *and* $J_R = J_{B,R} \cap \mathbb{P}(\mathbb{C}_p)$.

Examples in $\mathbb{P}_B$

- Let $R(z) = z^2$. Since R has good reduction then $J_{B,R} = \{x\}$, where x correspond to the ball $B_1(0)$.
- Let $R(z) = \frac{pz^2+z}{z+p}$. In this case there are no periodic points in $\mathbb{P}_B \setminus \mathbb{P}(\mathbb{C}_p)$, and moreover $J_{B,R} \subset \mathbb{P}(\mathbb{C}_p)$.
- Let $R(z) = \frac{z^2-z}{2z-p^2}$, with $p > 2$. There exists a Berkovich connected affinoid $\mathcal{A} \subset \mathbb{P}_B$, such that $A = \mathcal{A} \cap \mathbb{P}(\mathbb{C}_p)$ is an n-Herman ring. The boundary $\partial\mathcal{A}$ is the orbit of repelling periodic point of type II.

Some new results

In my dissertation, we study the dynamics of rational functions with coefficients over $\mathbb{C}_K$, a complete and algebraically closed non-archimedean field.

Among our results, we provide a complete description of the Fatou and Berkovich Fatou sets for quadratic rational maps, when the residue field $\widetilde{\mathbb{C}}_K$ is algebraic over a finite field. For example, for quadratic rational maps with coefficients in $\mathbb{C}_p$.

Quadratic rational function

Theorem

Let $R \in \mathbb{C}_p(z)$ be a rational function of degree $\deg(R) = 2$. Then one and exactly one of the following holds:

1. *R is simple. In this case $J_{B,R} = \{x\}$ where x corresponds to the ball $B_1(0)$. The Berkovich Fatou set $F_{B,R}$ contains only Siegel disks and attracting basins which are disks.*
2. *R has two repelling fixed points and one attracting fixed point. In this case the Berkovich Fatou set $F_{B,R}$ consists of only one attracting basin of Cantor type, and $J_{B,R}$ is a Cantor set.*
3. *R has one repelling fixed point and two indifferent fixed points. In this case the Berkovich Fatou set contains only one n-Herman ring and does not contain wandering domains.*

Quadratic rational function

Corollary

Let $R \in \mathbb{C}_p(z)$ be a rational function of degree $\deg(R) = 2$. Then the Berkovich Fatou set $F_{B,R}$ does not contain wandering domains.

Remark

- *Some of these results can be extended to more general non-archimedean fields, for example the field of Puiseux series.*
- *If $\mathbb{C}_K$ is a complete and algebraically closed non-archimedean field such that its residue field is not algebraic over a finite field, then we can construct examples of quadratic rational functions $R \in \mathbb{C}_K(z)$ such that $F_{B,R}$ contains wandering domains.*

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Thank you!!!