Heat equation on a p-adic ball

Anatoly N. Kochubei
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3, Kiev, 01024 Ukraine,
E-mail: kochubei@imath.kiev.ua

Harmonic analysis on $\mathbb{Q}_p$: complex-valued functions

Denote by dx the Haar measure on the additive group of $\mathbb{Q}_p$ normalized by the equality $\int_{\mathbb{Z}_p} dx = 1$. The Fourier transform of a complex-valued function $f \in L^1(\mathbb{Q}_p)$ is again a function on $\mathbb{Q}_p$ defined as

$$\widetilde{f}(\xi) = (\mathcal{F}f)(\xi) = \int\limits_{\mathbb{Q}_p} \chi(x\xi) f(x)\, dx$$

where χ is the canonical additive character.
If $\mathcal{F}f \in L_1(\mathbb{Q}_p)$, then we have the inversion formula

$$f(x) = \int\limits_{K} \chi(-x\xi) \widetilde{f}(\xi)\, d\xi.$$

It is possible to extend $\mathcal{F}$ from $L_1(\mathbb{Q}_p) \cap L_2(\mathbb{Q}_p)$ to a unitary operator on $L_2(\mathbb{Q}_p)$, so that the Plancherel identity holds in this case.

In order to define distributions on $\mathbb{Q}_p$, we need a class of test functions. A function $f: \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if there exists such an integer $l \geq 0$ that for any $x \in \mathbb{Q}_p$

$$f(x + x') = f(x) \quad \text{if } \|x'\| \leq p^{-l}.$$

The smallest number l with this property is called the exponent of local constancy of the function f.
Typical examples of locally constant functions are additive characters, and also cutoff functions like

$$\Omega(x) = \begin{cases} 1, & \text{if } \|x\| \leq 1; \\ 0, & \text{if } \|x\| > 1. \end{cases}$$

In particular, Ω is continuous, which is an expression of the non-Archimedean properties of $\mathbb{Q}_p$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the vector space of all locally constant functions with compact supports. Note that $\mathcal{D}(\mathbb{Q}_p)$ is dense in $L^q(\mathbb{Q}_p)$ for each $q \in [1, \infty)$. In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, consider first the subspace $D_N^l \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in a ball

$$B_N = \{x \in \mathbb{Q}_p : \ |x|_p \leq p^N\}, \quad N \in \mathbb{Z},$$

and the exponents of local constancy $\leq l$. This space is finite-dimensional and possesses a natural direct product topology. Then the topology in $\mathcal{D}(\mathbb{Q}_p)$ is defined as the double inductive limit topology, so that

$$\mathcal{D}(\mathbb{Q}_p) = \varinjlim_{N \to \infty} \varinjlim_{l \to \infty} D_N^l.$$

If $V \subset \mathbb{Q}_p$ is an open set, the space $\mathcal{D}(V)$ of test functions on V is defined as a subspace of $\mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in V.

The space $\mathcal{D}'(\mathbb{Q}_p)$ of Bruhat-Schwartz distributions on $\mathbb{Q}_p$ is defined as a strong conjugate space to $\mathcal{D}(\mathbb{Q}_p)$. In contrast to the classical situation, the Fourier transform is a linear automorphism of the space $\mathcal{D}(\mathbb{Q}_p)$. By duality, $\mathcal{F}$ is extended to a linear automorphism of $\mathcal{D}'(\mathbb{Q}_p)$. There exists a detailed theory of convolutions and direct products of distributions on $\mathbb{Q}_p$ closely connected with the theory of their Fourier transforms.

The Vladimirov operator D^α, $\alpha > 0$, of fractional differentiation, is defined first as a pseudo-differential operator with the symbol $|\xi_p^\alpha$:

$$(D^\alpha u)(x) = \mathcal{F}^{-1}_{\xi \to x} \left[|\xi|_p^\alpha \mathcal{F}_{y \to \xi} u \right], \quad u \in \mathcal{D}(\mathbb{Q}_p),$$

where we show arguments of functions and their direct/inverse Fourier transforms. There is also a hypersingular integral representation giving the same result on $\mathcal{D}(\mathbb{Q}_p)$ but making sense on much wider classes of functions (for example, bounded locally constant functions):

$$(D^\alpha u)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int\limits_{\mathbb{Q}_p} |y|_p^{-\alpha-1} [u(x-y) - u(x)]\, dy.$$

Let us consider the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} + (D_N^\alpha u)(t,x) - \lambda u(t,x) = 0, \quad x \in B_N,\ t > 0;$$
$$u(0,x) = \psi(x), \quad x \in B_N,$$

where $N \in \mathbb{Z}$, $B_N = \{x \in \mathbb{Q}_p, |x|_p \le p^N\}$, $\psi \in \mathcal{D}(B_N)$, $\lambda = \dfrac{p-1}{p^{\alpha+1}-1} p^{\alpha(1-N)}$, the operator D_N^α is defined by restricting D^α to functions u_N supported in B_N and considering the resulting function $D^\alpha u_N$ only on B_N. Here and below we often identify a function on B_N with its extension by zero onto $\mathbb{Q}_p$. Note that D_N^α defines a positive definite operator on $L^2(B_N)$, λ is its smallest eigenvalue.

The solution:

$$u(x,t) = \int\limits_{B_N} Z_N(t, x-y)\psi(y)\,dy, \quad t>0, x\in B_N,$$

where

$$Z_N(t,x) = e^{\lambda t} Z(x,t) + c(t), \quad x \in B_N,$$

$$c(t) = p^{-N} - p^{-N}(1-p^{-1})e^{\lambda t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n \frac{p^{-N\alpha n}}{1-p^{-\alpha n-1}},$$

The kernel Z_N is a transition density of a Markov process on B_N.

Probabilistic interpretation. Let $\xi_\alpha(t)$ be the stochastic process with independent increments corresponding to the generator D^α. Suppose that $\xi_\alpha(0) \in B_N$. Denote by $\xi_\alpha^{(N)}(t)$ the sum of all jumps of the process $\xi_\alpha(\tau),\ \tau \in [0, t]$, whose absolute values exceed p^N. Since ξ_α is right continuous with left limits, $\xi_\alpha^{(N)}(t)$ is finite a.s., $\xi_\alpha^{(N)}(0) = 0$. Let us consider the process

$$\eta_\alpha(t) = \xi_\alpha(t) - \xi_\alpha^{(N)}(t).$$

The processes $\eta_\alpha(t)$ and $\xi_\alpha^{(N)}(t)$ are independent processes with independent increments (Gihman and Skorohod, Stochastic Processes II).
Since the jumps of η_α never exceed p^N by absolute value, this process remains a.s. in B_N (due to the ultra-metric inequality). The above Cauchy problem corresponds to this process.

Harmonic analysis on the additive group of a p-adic ball

Let us consider the p-adic ball B_N as a compact subgroup of $\mathbb{Q}_p$. Any continuous additive character of $\mathbb{Q}_p$ has the form $x \mapsto \chi(\xi x)$, $\xi \in \mathbb{Q}_p$. The annihilator $\{\xi \in \mathbb{Q}_p : \ \chi(\xi x) = 1 \text{ for all } x \in B_N\}$ coincides with the ball B_{-N}. By the Pontryagin duality theorem, the dual group $\widehat{B_N}$ to B_N is isomorphic to the discrete group $\mathbb{Q}_p/B_{-N}$ consisting of the cosets

$$p^m \left(r_0 + r_1 p + \cdots + r_{N-m-1} p^{N-m-1}\right) + B_{-N}, \quad r_j \in \{0, 1, \ldots, p-1\},$$

$m \in \mathbb{Z}, m < N$. Analytically, this isomorphism means that any nontrivial continuous character of B_N has the form $\chi(\xi x)$, $x \in B_N$, where $|\xi|_p > p^{-N}$ and $\xi \in \mathbb{Q}_p$ is considered as a representative of the class $\xi + B_{-N}$. Note that $|\xi|_p$ does not depend on the choice of a representative of the class.

The Fourier transform on B_N is given by the formula

$$(\mathcal{F}_N f)(\xi) = p^{-N} \int_{B_N} \chi(x\xi) f(x)\, dx, \quad \xi \in (\mathbb{Q}_p \setminus B_{-N}) \cup \{0\},$$

where the right-hand side, thus also $\mathcal{F}_N f$, can be understood as a function on $\mathbb{Q}_p / B_{-N}$.

The Riesz kernel

$$f_{\alpha}^{(N)}(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |x|_p^{\alpha-1}, \quad \operatorname{Re}\alpha > 0, \quad \alpha \not\equiv 1 \pmod{\frac{2\pi i}{\log p}\mathbb{Z}}.$$

generates a distribution on B_N extended analytically in α to

$$\left\langle f_{-\alpha}^{(N)}, \varphi \right\rangle = \lambda\varphi(0) + \frac{1 - p^{\alpha}}{1 - p^{-\alpha-1}} \int_{B_N} [\varphi(x) - \varphi(0)] |x|_p^{-\alpha-1} dx.$$

The emergence of λ in the last formula "explains" its role in the probabilistic construction of a process on B_N.

Theorem

The operator D_N^α, $\alpha > 0$, *acts from* $\mathcal{D}(B_N)$ *to* $\mathcal{D}(B_N)$ *and admits, for each* $\varphi \in \mathcal{D}(B_N)$, *the representations:*

(i) $D_N^\alpha \varphi = f_{-\alpha}^{(N)} * \varphi$ *where the convolution is understood in the sense of harmonic analysis on the additive group of* B_N;

(ii)

$$\left(D_N^\alpha \varphi\right)(x) = \lambda\varphi(x) + \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int_{B_N} |y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy.$$

(iii) *On* $\mathcal{D}(B_N)$, $D_N^\alpha - \lambda I$ *coincides with the pseudo-differential operator* $\varphi \mapsto \mathcal{F}_N^{-1}(P_{N,\alpha}\mathcal{F}_N\varphi)$ *where*

$$P_{N,\alpha}(\xi) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int_{B_N} |y|_p^{-\alpha-1}[\chi(y\xi)-1]\,dy.$$

This symbol is extended uniquely from $(\mathbb{Q}_p \setminus B_{-N}) \cup \{0\}$ *onto* $\mathbb{Q}_p / B_{-N}$.

L^1-Theory of the Vladimirov Type Operator on a p-Adic Ball

The Heat-Like Semigroup.

On a ball B_N, $N \in \mathbb{Z}$, we consider the Cauchy problem (4)-(5). Its fundamental solution Z_N defines a contraction semigroup

$$(T_N(t)u)(x) = \int_{B_N} Z_N(t, x - \xi) u(\xi)\, d\xi$$

on $L^1(B_N)$.

Proposition

The semigroup T_N is strongly continuous.

The Generator.

Denote by A_N the generator of the contraction semigroup T_N on $L^1(B_N)$. By the Hille-Yosida theorem, A_N has a bounded resolvent $(A_N+\mu I)^{-1}$ for each $\mu>0$. In order to study the domain $D(A_N)$, we need the following auxiliary result.

Proposition

Let the support of a function $u\in L^1(\mathbb{Q}_p)$ be contained in $\mathbb{Q}_p\setminus B_N$. Then the restriction to B_N of the distribution $D^\alpha u\in\mathcal{D}'(\mathbb{Q}_p)$ coincides with the constant

$$R_N=R_N(u)=\frac{1-p^\alpha}{1-p^{-\alpha-1}}\int\limits_{|x|_p>p^N}|x|_p^{-\alpha-1}u(x)\,dx.$$

The following main result of this section is based on this property. As before, A denotes the generator of the semigroup $S(t)$ on $L^1(\mathbb{Q}_p)$.

Proposition

If $\psi \in D(A)$, then the restriction ψ_N of the function ψ to B_N belongs to $D(A_N)$, and $A_N \psi_N = (D_N^\alpha - \lambda)\,\psi_N$ where $D_N^\alpha \psi_N$ is understood in the sense of $\mathcal{D}'(B_N)$, that is ψ_N is extended by zero to a function on $\mathbb{Q}_p$, D^α is applied to it in the distribution sense, and the resulting distribution is restricted to B_N.

In the study of nonlinear equations, this result makes it possible to use the operator A_N in the investigation of local properties of functions. This is a substitute for the local Sobolev and Marcinkiewicz spaces used in the classical literature.

Some Publications

1. A. N. Kochubei, Parabolic equations over the field of *p*-adic numbers, *Math. USSR Izvestiya*, **39** (1992), 1263–1280.
2. A. N. Kochubei, Heat equation in a *p*-adic ball, *Meth. Funct. Anal. and Topol.*, **2**, No. 3-4 (1996), 53–58.
3. A. N. Kochubei, *Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields*, Marcel Dekker, New York, 2001.
4. A. N. Kochubei, Linear and nonlinear heat equations on a p-adic ball, *Ukrainian Math. J.*, **70**, No. 2 (2018), 217–231.