# Nonlinear parabolic equations with $p$-adic spatial variables

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Joint work with Andrei Khrennikov

# Preliminaries

*1. p-Adic numbers*

Let $p$ be a prime number. The field of $p$-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^{\nu}\frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and $m, n$ are prime to $p$. Example: $|p|_p = p^{-1}$.

$\mathbb{Q}_p$ is a locally compact topological field.

Note that by Ostrowski's theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$. We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \leq 1\}$. $\mathbb{Z}_p$, as well as all balls in $\mathbb{Q}_p$, is simultaneously open and closed.

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$
$$|xy|_p = |x|_p \cdot |y|_p;$$
$$|x+y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the ultra-metric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x-y|_p$, as well as many unusual geometric properties (Example: *two balls either do not intersect, or one of them is contained in another*). Note also the following consequence of the ultra-metric inequality:

$$|x+y|_p = \max(|x|_p, |y|_p) \quad \text{if } |x|_p \neq |y|_p.$$

The absolute value $|x|_p$ takes the discrete set of non-zero values $p^N$, $N \in \mathbb{Z}$. If $|x|_p = p^N$, then $x$ admits a (unique) canonical representation

$$x = p^{-N}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The series converges in the topology of $\mathbb{Q}_p$. For example,

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots, \quad |-1|_p = 1.$$

The canonical representation shows the hierarchical structure of $\mathbb{Q}_p$.

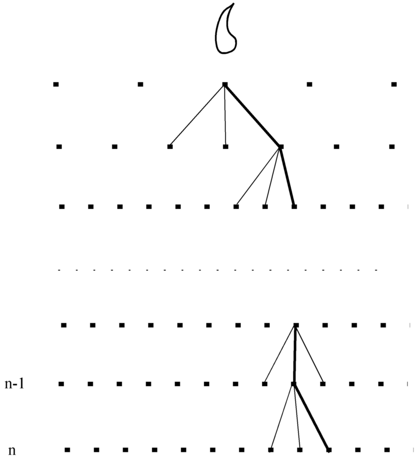


Figure: Structure of the $p$-adic tree.

Khrennikov et al (2016) - $p$-adic model of a porous medium.

*2. Field extensions*

If a field $k$ is a subfield of a field $K$, then $K$ is called an extension of $k$. In order to emphasize that $K$ is considered as an extension of $k$, we denote the extension by $K/k$.
An extension $K/k$ can be considered as a vector space over $k$. An extension $K/k$ is called finite if $K$ is a finite-dimensional vector space over $k$. Its dimension $(K : k)$ is called the degree of the extension $K/k$.
A **local field** is a non-discrete disconnected locally compact field.

**Theorem.** Every local field of characteristic 0 is a finite extension of $\mathbb{Q}_p$.

$\mathbb{Q}_p$ has finite extensions of all degrees. In particular, for each degree $n$ there exists a unique *unramified* extension $K$ possessing a basis identifying each element $x \in K$ with a vector of coefficients $(\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, so that the absolute value on $K$ equals $\|x\| = \|x\|_{\max}^n$ where

$$\|x\|_{\max} = \max_{1 \le j \le n} |\xi_j|_p,$$

## 3. Harmonic analysis on $\mathbb{Q}_p$: complex-valued functions

Denote by $dx$ the Haar measure on the additive group of $\mathbb{Q}_p$ normalized by the equality $\int_{\mathbb{Z}_p} dx = 1$. The Fourier transform of a complex-valued function $f \in L^1(\mathbb{Q}_p)$ is again a function on $\mathbb{Q}_p$ defined as

$$\widetilde{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{Q}_p} \chi(x\xi) f(x)\, dx$$

where $\chi$ is the canonical additive character.
If $\mathcal{F}f \in L_1(\mathbb{Q}_p)$, then we have the inversion formula

$$f(x) = \int_K \chi(-x\xi)\widetilde{f}(\xi)\, d\xi.$$

It is possible to extend $\mathcal{F}$ from $L_1(\mathbb{Q}_p) \cap L_2(\mathbb{Q}_p)$ to a unitary operator on $L_2(\mathbb{Q}_p)$, so that the Plancherel identity holds in this case.

In order to define distributions on $\mathbb{Q}_p$, we need a class of test functions. A function $f: \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if there exists such an integer $l \geq 0$ that for any $x \in \mathbb{Q}_p$

$$f(x + x') = f(x) \quad \text{if } \|x'\| \leq p^{-l}.$$

The smallest number $l$ with this property is called the exponent of local constancy of the function $f$.
Typical examples of locally constant functions are additive characters, and also cutoff functions like

$$\Omega(x) = \begin{cases} 1, & \text{if } \|x\| \leq 1; \\ 0, & \text{if } \|x\| > 1. \end{cases}$$

In particular, $\Omega$ is continuous, which is an expression of the non-Archimedean properties of $\mathbb{Q}_p$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the vector space of all locally constant functions with compact supports. Note that $\mathcal{D}(\mathbb{Q}_p)$ is dense in $L^q(\mathbb{Q}_p)$ for each $q \in [1, \infty)$. In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, consider first the subspace $D_N^l \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in a ball

$$B_N = \{x \in \mathbb{Q}_p : \ |x|_p \le p^N\}, \quad N \in \mathbb{Z},$$

and the exponents of local constancy $\le l$. This space is finite-dimensional and possesses a natural direct product topology. Then the topology in $\mathcal{D}(\mathbb{Q}_p)$ is defined as the double inductive limit topology, so that

$$\mathcal{D}(\mathbb{Q}_p) = \varinjlim_{N \to \infty} \varinjlim_{l \to \infty} D_N^l.$$

If $V \subset \mathbb{Q}_p$ is an open set, the space $\mathcal{D}(V)$ of test functions on $V$ is defined as a subspace of $\mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in $V$.

The space $\mathcal{D}'(\mathbb{Q}_p)$ of Bruhat-Schwartz distributions on $\mathbb{Q}_p$ is defined as a strong conjugate space to $\mathcal{D}(\mathbb{Q}_p)$. In contrast to the classical situation, the Fourier transform is a linear automorphism of the space $\mathcal{D}(\mathbb{Q}_p)$. By duality, $\mathcal{F}$ is extended to a linear automorphism of $\mathcal{D}'(\mathbb{Q}_p)$. There exists a detailed theory of convolutions and direct products of distributions on $\mathbb{Q}_p$ closely connected with the theory of their Fourier transforms.

The Vladimirov operator $D^\alpha$, $\alpha > 0$, of fractional differentiation, is defined first as a pseudo-differential operator with the symbol $|\xi_p^\alpha$:

$$(D^\alpha u)(x) = \mathcal{F}^{-1}_{\xi \to x} \left[ |\xi|_p^\alpha \mathcal{F}_{y \to \xi} u \right], \quad u \in \mathcal{D}(\mathbb{Q}_p),$$

where we show arguments of functions and their direct/inverse Fourier transforms. There is also a hypersingular integral representation giving the same result on $\mathcal{D}(\mathbb{Q}_p)$ but making sense on much wider classes of functions (for example, bounded locally constant functions):

$$(D^\alpha u)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} |y|_p^{-\alpha-1} [u(x-y) - u(x)] \, dy.$$

Parabolic equations generated by the Vladimirov operator

a) The Cauchy problem for a heat-like equation:

$$\frac{\partial u(x,t)}{\partial t} + (D^{\alpha} u)(x,t) = f(x,t), \quad x \in \mathbb{Q}_p, 0 < t \leq T,$$

$$u(x,0) = \varphi(x), \quad x \in \mathbb{Q}_p,$$

Haran, 1990
Ismagilov, 1991
K., 1991
Vladimirov, Volovich and Zelenov, 1994
Blair, 1995,
Varadarajan, 1997

Heat kernel for $D^{\alpha}$:

$$Z(t,x) = \sum_{k=-\infty}^{\infty} p^k c_k(t) \Delta_{-k}(x)$$

where $\Delta_l(x)$ is the indicator function of the ball $B_l$,

$$c_k(t) = \exp\left(-p^{k\alpha} t\right) - \exp\left(-p^{(k+1)\alpha} t\right).$$

Another expression for $Z(t,x)$, valid for $x \neq 0$, is

$$Z(t,x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \cdot \frac{1-p^{\alpha m}}{1-p^{-\alpha m-1}} t^m |x|_p^{-\alpha m-1}.$$

$Z$ is a probability density and

$$0 < Z(t,x) \leq Ct(t^{1/\alpha} + |x|_p)^{-\alpha-1}, \quad t > 0, x \in \mathbb{Q}_p.$$

**b) A general theory of parabolic equations** (K., 1991):

$$\frac{\partial u(x,t)}{\partial t} + a_0(x,t)(D^{\alpha} u)(x,t) + \sum_{k=1}^{n} a_k(x,t)(D^{\alpha_k} u)(x,t) + b(x,t)u(x,t) = f(x,t), \quad x \in \mathbb{Q}_p,\ t \in (0,T],$$

$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha$.
Parametrix method, Fundamental solutions, Cauchy problem (existence, uniqueness, stabilization, probabilistic interpretation).

Multi-dimensional problems: 1) reduction to the unramified extension (K., 2001); 2) other methods (Zuniga-Galindo and his school). See the talk by A. Antoniouk.

## c) A Heat-Like Equation on a p-Adic Ball

Let us consider the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} + (D_N^\alpha u)(t,x) - \lambda u(t,x) = 0, \quad x \in B_N,\ t > 0;$$
$$u(0,x) = \psi(x), \quad x \in B_N,$$

where $N \in \mathbb{Z}$, $B_N = \{x \in \mathbb{Q}_p, |x|_p \le p^N\}$, $\psi \in \mathcal{D}(B_N)$, $\lambda = \frac{p-1}{p^{\alpha+1}-1} p^{\alpha(1-N)}$, the operator $D_N^\alpha$ is defined by restricting $D^\alpha$ to functions $u_N$ supported in $B_N$ and considering the resulting function $D^\alpha u_N$ only on $B_N$. Here and below we often identify a function on $B_N$ with its extension by zero onto $\mathbb{Q}_p$. Note that $D_N^\alpha$ defines a positive definite operator on $L^2(B_N)$, $\lambda$ is its smallest eigenvalue.

The solution:

$$u(x,t) = \int_{B_N} Z_N(t, x-y)\psi(y)\,dy, \quad t>0, x\in B_N,$$

where

$$Z_N(t,x) = e^{\lambda t} Z(x,t) + c(t), \quad x \in B_N,$$
$$c(t) = p^{-N} - p^{-N}(1-p^{-1})e^{\lambda t}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n \frac{p^{-N\alpha n}}{1-p^{-\alpha n-1}},$$

The kernel $Z_N$ is a transition density of a Markov process on $B_N$.

Other probabilistic interpretations and an interpretation in terms of harmonic analysis on the additive group of a $p$-adic ball will be considered in a separate talk.

# Nonlinear Equations

Below we consider a $p$-adic analog of one of the most important classical nonlinear equations, the porous medium equation:

$$\frac{\partial u}{\partial t} + D^{\alpha}(\varphi(u)) = 0, \quad u = u(t,x), \quad t > 0, x \in \mathbb{Q}_p, \qquad (1)$$

where $\varphi$ is a strictly monotone increasing continuous real function, $|\varphi(s)| \leq C|s|^m$ for $s \in \mathbb{R}$ ($C > 0$, $m \geq 1$). A typical example of the latter is $\varphi(u) = u|u|^{m-1}$, $m > 1$.

Another interesting example is the $p$-adic Navier-Stokes equation for a porous medium (Oleschko, Khrennikov et al, 2017):

$$\frac{\partial u(t,x)}{\partial t} = u(t,x)(D^1 u)(t,x) - \theta(D^2 u)(t,x) + G(t,x), \quad t > 0, x \in \mathbb{Q}_p,$$

where $\theta > 0$, $G$ is a given function. Oleschko, Khrennikov et al derived this equation from the discretized model of hydrodynamics (Benzi et al, 1997).

Our strategy for studying Eq. (1) is as follows. There exists an abstract theory of the equations

$$\frac{\partial u}{\partial t} + A(\varphi(u)) = 0. \tag{2}$$

developed by Crandall and Pierre (1982) and based on the theory of stationary equations

$$u + A\varphi(u) = f \tag{3}$$

developed by Brézis and Strauss (1973). In Eq. (2) and (3), $A$ is a linear $m$-accretive operator in $L^1(\Omega)$ where $\Omega$ is a $\sigma$-finite measure space. Under some natural assumptions, the nonlinear operator$A\varphi = A \circ \varphi$ is accretive and admits an $m$-accretive extension $A_\varphi$, the generator of a contraction semigroup of nonlinear operators. This result gives information on a kind of generalized solvability of Eq. (2), though the available description of $A_\varphi$ is not quite explicit.

In order to use this method for Eq. (1), we need an $L^1$-theory of the Vladimirov operator $D^\alpha$, which is a subject of independent interest. In the classical situation where $\Omega = \mathbb{R}^n$, $A$ is the Laplacian, there are stronger results (Bénilan, Brézis and Crandall, 1975) based on the study of Eq. (3), showing that $A\varphi$ is $m$-accretive itself. This employs some delicate tools of local analysis of solutions, such as imbedding theorems for Marcinkiewicz and Sobolev spaces in bounded domains.
For our $p$-adic situation, we prove a little weaker result, namely the $m$-accretivity of the closure of the operator $A\varphi$. Our tool is the $L^1$-theory of the Vladimirov type operator on a $p$-adic ball.

## The Vladimirov Operator in $L^1(\mathbb{Q}_p)$

*The Heat-Like Equation and the Corresponding Semigroup of Operators.*

Using the fundamental solution $Z$, we define the operator family

$$(S(t)\psi)(x) = \int_{\mathbb{Q}_p} Z(t, x-\xi)\psi(\xi)\, d\xi, \quad \psi \in L^1(\mathbb{Q}_p),$$

$t > 0$. $S$ is a contraction semigroup in $L^1(\mathbb{Q}_p)$.

**Proposition**

*$S(t)$ has the $C_0$-property.*

### Definition

*We define the realization $A$ of $D^\alpha$ in $L^1(\mathbb{Q}_p)$ as the generator of the semigroup $S(t)$.*

Let $D(A)$ be the domain of the operator $A$.

### Proposition

*If $u \in \mathcal{D}(\mathbb{Q}_p)$, then $u \in D(A)$ and $Au = D^\alpha u$ where the right-hand side is understood as usual in terms of the Fourier transform or the hypersingular integral representation.*

The proof is based on the detailed analysis of actions of $D^\alpha$ and $S(t)$ on characteristic functions of open-closed sets.

# *The Green function.*

Since the operator $A$ in $L^1(\mathbb{Q}_p)$ is defined as the generator of the contraction semigroup $S(t) = e^{-tA}$, then by the Hille-Yosida theorem, we can find the resolvent $R_\mu(A) = (A + \mu I)^{-1}$, $\mu > 0$, by the formula

$$R_\mu(A)\psi = -\int_0^\infty e^{-\mu t} S(t)\psi \, dt, \quad \psi \in L^1(\mathbb{Q}_p).$$

We will consider below the case where $\alpha > 1$, in which the resolvent is an integral operator with a kernel possessing some smoothness properties. Thus, from now on,

$$\alpha > 1.$$

In this case, $R_\mu$ is a convolution operator with the continuous integral kernel $E_\mu(x-\xi)$, such that $E_\mu(x) \sim \text{const} \cdot |x|_p^{-\alpha-1}$, $|x|_p \to \infty$. The function $E_\mu$ is represented by the uniformly convergent series

$$E_\mu(x) = \sum_{N=-\infty}^{\infty} e_\mu^{(N)}(x),$$

$$e_\mu^{(N)}(x) = \int\limits_{|\xi|_p = p^N} \frac{\chi(-x\xi)}{|\xi|_p^\alpha + \mu}\, d\xi.$$

*Description of A in the distribution sense.*
Let $u \in L^1(\mathbb{Q}_p)$. Then $D^\alpha u$ can be defined as a distribution from $\mathcal{D}'(\mathbb{Q}_p)$, a convolution $u * f_{-\alpha}$, $f_{-\alpha}(x) = \dfrac{|x|_p^{-\alpha-1}}{\Gamma_p(-\alpha)}$,

$$\Gamma_p(z) = \frac{1 - p^{z-1}}{1 - p^{-z}}.$$

$f_{-\alpha}$ defines a distribution by analytic continuation.

## Proposition

*The operator A defined as a semigroup generator has the domain* $D(A) = \{u \in L^1(\mathbb{Q}_p) : \; D^\alpha u \in L^1(\mathbb{Q}_p)\}$ *where* $Au = D^\alpha u$ *(understood in the distribution sense).*

*The Heat-Like Semigroup.*

On a ball $B_N$, $N \in \mathbb{Z}$, we consider the Cauchy problem (4)-(5). Its fundamental solution $Z_N$ defines a contraction semigroup

$$(T_N(t)u)(x) = \int_{B_N} Z_N(t, x-\xi)u(\xi)\, d\xi$$

on $L^1(B_N)$.

**Proposition**

*The semigroup $T_N$ is strongly continuous.*

*The Generator.*

Denote by $A_N$ the generator of the contraction semigroup $T_N$ on $L^1(B_N)$. By the Hille-Yosida theorem, $A_N$ has a bounded resolvent $(A_N + \mu I)^{-1}$ for each $\mu > 0$. In order to study the domain $D(A_N)$, we need the following auxiliary result.

### Proposition

*Let the support of a function $u \in L^1(\mathbb{Q}_p)$ be contained in $\mathbb{Q}_p \setminus B_N$. Then the restriction to $B_N$ of the distribution $D^\alpha u \in \mathcal{D}'(\mathbb{Q}_p)$ coincides with the constant*

$$R_N = R_N(u) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int\limits_{|x|_p > p^N} |x|_p^{-\alpha-1} u(x)\, dx.$$

The following main result of this section is based on this property. As before, $A$ denotes the generator of the semigroup $S(t)$ on $L^1(\mathbb{Q}_p)$.

## Proposition

*If $\psi \in D(A)$, then the restriction $\psi_N$ of the function $\psi$ to $B_N$ belongs to $D(A_N)$, and $A_N \psi_N = (D_N^\alpha - \lambda)\,\psi_N$ where $D_N^\alpha \psi_N$ is understood in the sense of $\mathcal{D}'(B_N)$, that is $\psi_N$ is extended by zero to a function on $\mathbb{Q}_p$, $D^\alpha$ is applied to it in the distribution sense, and the resulting distribution is restricted to $B_N$.*

In the study of nonlinear equations, this result makes it possible to use the operator $A_N$ in the investigation of local properties of functions. This is a substitute for the local Sobolev and Marcinkiewicz spaces used in the classical literature.

## Nonlinear Equations: the Main Result

Let us return to Eq. (1) interpreted as Eq. (2) on $L^1(\mathbb{Q}_p)$, where the linear operator $A$ is a generator of the semigroup $S(t)$, $\varphi$ is a strictly monotone increasing continuous real function, $|\varphi(s)| \leq C|s|^m$, $m \geq 1$. Below we re-interpret Eq. (1) as the equation

$$\frac{\partial u}{\partial t} + \overline{A\varphi}(u) = 0 \tag{4}$$

where $\overline{A\varphi}$ is the closure of $A\varphi$.

Recall that a mild solution of the Cauchy problem for a nonlinear equation with the initial condition $u(0,x) = u_0(x)$ is defined as a function given by a limit, uniformly on compact time intervals, of solutions of the problem for the difference equations approximating the differential one. This the usual "nonlinear version" of the notion of a generalized solution.

Theorem

*The operator $\overline{A\varphi}$ is m-accretive, so that, for any initial function $u_0 \in L^1(\mathbb{Q}_p)$, the Cauchy problem for Eq. (6) has a unique mild solution.*

*Idea of Proof.* By the general results of Crandall and Pierre, the operator $A\varphi$ is accretive. Therefore it is sufficient to show that $I + A\varphi$ has a dense range. This property is proved using a priori estimates by Brézis and Strauss, relative local compactness criterion for subsets of $L^1(\mathbb{Q}_p)$ and the local compactness considerations based on properties of the operator $A_N$.

## Explicit Solution: an Example

Let us consider Eq. (1) with $\alpha > 0$, $\varphi(u) = |u|^m$, $m > 1$. We look for a solution of the form

$$u(t,x) = \rho\left(\frac{|x|_p^\gamma}{t_0 - t}\right)^\nu, \quad 0 < t < t_0, x \in \mathbb{Q}_p,$$

where $t_0 > 0$, $\gamma > 0$, $\nu > 0$, $0 \neq \rho \in \mathbb{R}$.

After investigating possible values of parameters, we come to the solution

$$u(t,x) = \rho(t_0 - t)^{-\frac{1}{m-1}}|x|_p^{\frac{\alpha}{m-1}}$$

where

$$\rho = -\left[\frac{\Gamma_p(1+\frac{\alpha}{m-1})}{(m-1)\Gamma_p(1+\frac{\alpha m}{m-1})}\right]^{\frac{1}{m-1}}.$$

In a similar way, we can obtain another solution

$$u(t,x) = \mu(t_0 + t)^{-\frac{1}{m-1}} |x|_p^{\frac{\alpha}{m-1}}, \quad t > 0, x \in \mathbb{Q}_p,$$

where $\mu = -\rho$.

## $p$-Adic Navier-Stokes Equation

A $p$-adic model of propagation of fluids through the capillary structure of a porous medium was suggested by Khrennikov et al (2016). In this model an idealized fragment of a porous medium is identified with the $p$-adic ball interpreted as the set of (generally infinite) paths of a homogeneous rooted tree of valence $p+1$. Here $p$ is a fixed prime number.

Natural developments prompted by this idea include both new mathematical models of percolation phenomena and purely mathematical works dealing with $p$-adic analogs of equations of mathematical hydrodynamics, such as the porous medium equation.

The $p$-adic Navier-Stokes equation (Khrennikov et al, 2017) is a pseudo-differential evolution equation on the field $\mathbb{Q}_p$ of $p$-adic numbers describing average velocity of a fluid moving through the $p$-tree of capillaries. This nonlinear equation deduced from the discretized model of hydrodynamics (Benzi et al, 1997) has the form

$$\frac{\partial u(t,x)}{\partial t} = u(t,x)(D^1 u)(t,x) - \theta(D^2 u)(t,x) \tag{5}$$

where $\theta > 0$, $D^\alpha$ $(\alpha > 0)$ is the Vladimirov fractional differentiation operator on $\mathbb{Q}_p$.

In this work we initiate the mathematical theory of the equation (5). Note that there exists a well-developed theory of linear pseudo-differential equations on $\mathbb{Q}_p$. The study of nonlinear equations of this kind is only beginning.
Our method of investigating the equation (5) is based on abstract results by von Wahl (1985) who found sufficient conditions of local solvability of the Cauchy problem for the equation

$$v'(t) + Av(t) + M(v(t)) = 0, \quad t > 0, \tag{6}$$

where $A$ is the generator of an analytic semigroup $e^{-tA}$ in a Banach space $\mathfrak{B}$, $M$ is a nonlinear operator subordinated to $A^{1-\rho}$, $0 < \rho < 1$.

In our situation, where we study the local solvability of (5) on a bounded domain, $\mathfrak{B} = L^q(B_N)$ where $1 < q < \infty$, $B_N = \left\{ x \in \mathbb{Q}_p : \ |x|_p \le p^N \right\}$ is a $p$-adic ball, the operators $A$ and $M$ are constructed from the Vladimirov operators $D^2$ and $D^1$ on the ball. Note that the latter operators are nonlocal, which makes even the definition of an operator on a bounded domain nontrivial. In addition, there is an $L^2$-theory of the Vladimirov operator and the initial steps towards its $L^1$-theory. Here we have to develop its $L^q$-theory, in particular to prove certain inequalities for various $L^q$-norms involving $D^\alpha u$ and $D^\beta u$, $0 < \alpha < \beta$.

**A theorem by von Wahl.**

Let us consider the equation (2), that is $v'(t) + Av(t) + M(v(t)) = 0,\ t > 0$, with the initial condition $v(0) = \varphi$, $\varphi \in D(A)$, where a linear operator $A$ is the generator of an analytic semigroup in a Banach space $\mathfrak{B}$, $M$ is a nonlinear operator in $\mathfrak{B}$. It is assumed that, for some $\rho \in (0, 1)$, $M$ acts from the domain $D(A^{1-\rho})$ to $\mathfrak{B}$ and satisfies the following condition: if $v, w \in D(A)$, $\|Av\| + \|Aw\| \le h,\ h > 0$, then there exists such a constant $k(h) > 0$ that

$$\begin{cases} \|M(u) - M(v)\| \le k(h)\|A^{1-\rho}(v - w)\|, \\ \|M(v)\| \le k(h). \end{cases}$$

Here $\|\cdot\|$ means the norm in $\mathfrak{B}$.

The von Wahl theorem states that, under the above conditions, there exists $T(\varphi) \in (0, \infty]$ with the following property. For any $T < T(\varphi)$, there exists a unique function $v \in C^1([0, T], \mathfrak{B})$, such that $Av \in C([0, T], \mathfrak{B})$, satisfying the equation (2) and the initial condition $v(0) = \varphi$.

# Inequalities

*1. Fractional powers*. Below the fractional powers are understood in the sense of the theory of generators of analytic semigroups.

**Proposition**

*Let $0 < \alpha \leq \beta$. For all $u \in \mathcal{D}(B_N)$,*

$$\|D_N^\alpha u\|_{L^q(B_N)} \leq C \left\| \left(D_N^\beta\right)^{\alpha/\beta} u \right\|_{L^q(B_N)}$$

*where $C$ does not depend on $u$.*

*2. Comparison of fractional powers.* Let $0 < \alpha < \beta \leq \alpha + 1$, $\beta > 1$, $1 \leq q \leq r < \infty$.

### Proposition

*Suppose that*

$$\alpha - \beta + \frac{1}{q} < \frac{1}{r} \leq \frac{1}{q}.$$

*Then for any $u \in \mathcal{D}(B_N)$*

$$\|D_N^{\alpha} u\|_{L^r(B_N)} \leq C \left\| D_N^{\beta} u \right\|_{L^q(B_N)}$$

*where the constant $C$ does not depend on $u$.*

# Local solvability of the $p$-adic Navier-Stokes equation

Let us consider the equation

$$\frac{\partial u(t,x)}{\partial t} = u(t,x)\left(D_N^1 u\right)(t,x) - \theta\left(D_N^2 u\right)(t,x)$$

with the initial condition

$$u(0,x) = \varphi(x).$$

We apply von Wahl's theorem with $\mathfrak{B} = L^q(B_N)$, $1 < q < \infty$, $A = \theta D_N^2$, $M(u) = u \cdot D_N^1 u$. As we know, $A$ is a generator of an analytic subgroup in $\mathfrak{B}$.

### Theorem

*For any* $\varphi \in D(A)$ *(in particular, for any* $\varphi \in \mathcal{D}(B_N)$*), there exists* $T(\varphi) \in (0, \infty]$ *with the following property. For any* $T$*,* $0 < T < T(\varphi)$*, the above Cauchy problem possesses a unique solution* $u \in C^1([0, T], \mathfrak{B})$*, such that* $Au \in C([0, T], \mathfrak{B})$ *.*

## Some Publications


1. A. Yu. Khrennikov and A. N. Kochubei. *p*-Adic Analogue of the Porous Medium Equation, *J. Fourier Anal. Appl.*, **24** (2018), 1401–1424.

2. A. N. Kochubei, Linear and nonlinear heat equations on a *p*-adic ball, *Ukrainian Math. J.*, **70**, No. 2 (2018), 217–231.

3. A. V. Antoniouk, K. Oleschko, A. N. Kochubei and A. Yu. Khrennikov, A stochastic *p*-adic model of the capillary flow in porous random medium, *Physica A*, **505** (2018), 763-777.

4. A. Yu. Khrennikov and A. N. Kochubei, On the *p*-Adic Navier-Stokes Equation, *Applicable Anal.*, published online 16 Oct 2018, DOI: 10.1080/00036811.2018.1533120.