A note on the Rees algebra of a bipartite graph

ISIDORO GITLER[1], CARLOS VALENCIA.
and
Rafael H. Villarreal
Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14–740
07000 México City, D.F.
e-mail vila@esfm.ipn.mx

Dedicated to Wolmer Vasconcelos in his 65th birthday

Abstract

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be an ideal of R generated by a set $x^{\alpha_1}, \ldots, x^{\alpha_q}$ of square-free monomials of degree two such that the graph G defined by those monomials is bipartite. We study the Rees algebra $\mathcal{R}(I)$ of I, by studying both the Rees cone $\mathbb{R}_+\mathcal{A}'$ generated by the set $\mathcal{A}' = \{e_1, \ldots, e_n, (\alpha_1, 1), \ldots, (\alpha_q, 1)\}$ and the matrix C whose columns are the vectors in $\mathcal{A}'$. It is shown that C is totally unimodular. We determine the irreducible representation of the Rees cone in terms of the minimal vertex covers of G. Then we compute the a-invariant of $\mathcal{R}(I)$.

1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let G be a bipartite graph with vertex set $V = V(G) = \{v_1, \ldots, v_n\}$ and edge set $E = E(G)$. The *edge ideal* of G is the square-free monomial ideal of R given by

$$I = I(G) = (\{x_i x_j |\, \{v_i, v_j\} \text{ is an edge of } G\}) \subset R\,,$$

and the *Rees algebra* of I is the K-subalgebra:

$$\mathcal{R}(I) = K[\{x_i x_j t |\, v_i \text{ is adjacent to } v_j\} \cup \{x_1, \ldots, x_n\}] \subset R[t],$$

where t is a new variable. Consider the set of vectors

$$\mathcal{A}' = \{e_i + e_j + e_{n+1} |\, v_i \text{ is adjacent to } v_j\} \cup \{e_1, \ldots, e_n\} \subset \mathbb{R}^{n+1},$$

[0]2000 *Mathematics Subject Classification.* Primary 13H10; Secondary 13F20, 13A30, 52B20.

[1]The authors were partially supported by CONACyT grant 40201F and SNI.

where e_i is the *ith* unit vector. Here we study $\mathcal{R}(I)$ by looking closely at the matrix C whose columns are the vector in $\mathcal{A}'$. One of our results proves that C is totally unimodular, then as a consequence we derive that the presentation ideal of $\mathcal{R}(I)$ is generated by square-free binomials. As another consequence we give a simple proof of the fact that $\mathcal{R}(I)$ is a normal domain [7].

We are able to determine the irreducible representation of the polyhedral *Rees cone* $\mathbb{R}_+\mathcal{A}'$ generated by $\mathcal{A}'$, see Corollary 4.3. This turns out to be related to the minimal vertex covers of the graph G and yields a description of the canonical module of $\mathcal{R}(I)$.

By assigning $\deg(x_i) = 1$ and $\deg(t) = -1$, the Rees algebra $\mathcal{R}(I)$ becomes a standard graded K-algebra, that is, it is generated as a K-algebra by elements of degree 1. Another of our results proves that the a-invariant of $\mathcal{R}(I)$, with respect to this grading, is equal to $-(\beta_0+1)$, where β_0 is the independence number of G. In order to compute this invariant we use the irreducible representation of $\mathbb{R}_+\mathcal{A}'$ together with a formula of Danilov-Stanley for the canonical module of $\mathcal{R}(I)$.

2 Preliminaries

Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials of R and let $A = (a_{ij})$ be the matrix of order $n \times q$ whose columns are the vectors $\alpha_1, \ldots, \alpha_q$. We say that the matrix A is *unimodular* if all its nonzero $r \times r$ minors have absolute value equal to 1, where r is the rank of A.

Recall that the monomial subring $K[F] \subset R$ is *normal* if $K[F] = \overline{K[F]}$, where $\overline{K[F]}$ is the integral closure of $K[F]$ in its field of fractions. The following expression for the integral closure is well known:

$$\overline{K[F]} = K[\{x^a \mid a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}], \tag{1}$$

where $\mathbb{Z}\mathcal{A}$ is the subgroup spanned by $\mathcal{A}$ and $\mathbb{R}_+\mathcal{A}$ is the *polyhedral cone*

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i\alpha_i \,\middle|\, a_i \in \mathbb{R}_+ \text{ for all } i \right\}$$

generated by $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$. Here $\mathbb{R}_+$ denotes the set of non negative real numbers. See [2, 3] and [12, Chapter 7] for a thorough discussion of the integral closure of a monomial subring and how it can be computed. For the related problem of computing the integral closure of an affine domain see [9, 10].

The next result was shown in [7] if A is the incidence matrix of a bipartite graph and it was shown in [8] for general A. The proof below, in contrast to that of [8], is direct and does not make any use of Gröbner bases techniques.

Theorem 2.1 *If A is a unimodular matrix, then $K[F]$ is a normal domain.*

Proof. By Eq. (1) it suffices to prove

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}.$$

Let $b \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}$. By Carathéodory's Theorem [4, I, Theorem 2.3] there are linearly independent columns $\alpha_{i_1}, \ldots, \alpha_{i_r}$ of A, where $r = \mathrm{rank}(A)$, such that

$$b \in \mathbb{R}_+\alpha_{i_1} + \cdots + \mathbb{R}_+\alpha_{i_r}. \tag{2}$$

As A is unimodular for each j one has

$$\Delta_r([\alpha_{i_1} \cdots \alpha_{i_r}]) = \Delta_r([\alpha_{i_1} \cdots \alpha_{i_r}\ \alpha_j]) = 1,$$

where $\Delta_r(B)$ denotes the greatest common divisor of all the nonzero $r \times r$ minors of B. Hence by a classical result of I. Heger [6, p. 51] one readily obtains

$$\mathbb{Z}\mathcal{A} = \mathbb{Z}\alpha_{i_1} \oplus \cdots \oplus \mathbb{Z}\alpha_{i_r}.$$

Therefore

$$b \in \mathbb{Z}\alpha_{i_1} + \cdots + \mathbb{Z}\alpha_{i_r}. \tag{3}$$

Since $\alpha_{i_1}, \ldots, \alpha_{i_r}$ are linearly independent by comparing the coefficients of b with respect to the two representations given by (2) and (3) one derives $b \in \mathbb{N}\mathcal{A}$. Hence we have shown $\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} \subset \mathbb{N}\mathcal{A}$. The reverse containment is clear. □

3 On the defining matrix of the Rees algebra

Let G be a simple graph. The *incidence matrix* of G is the matrix whose columns are the vectors $e_i + e_j$ such that v_i is adjacent to v_j. A matrix B is *totally unimodular* if each $i \times i$ minor of B is 0 or ± 1 for all $i \geq 1$. Recall that the bipartite simple graphs are characterized as those graphs whose incidence matrix is totally unimodular [6, p. 273].

Next we present the main result of this section and two of its consequences.

Theorem 3.1 *Let G be a simple bipartite graph with n vertices and q edges and let $A = (a_{ij})$ be its incidence matrix. If $e_1, \ldots, e_n$ are the first n unit vectors in $\mathbb{R}^{n+1}$ and C is the matrix*

$$C = \begin{pmatrix} a_{11} & \ldots & a_{1q} & e_1 & \cdots & e_n \\ \vdots & \vdots & \vdots & & & \\ a_{n1} & \ldots & a_{nq} & & & \\ 1 & \ldots & 1 & & & \end{pmatrix}$$

obtained from A by adjoining a row of 1's *and the column vectors $e_1, \ldots, e_n$, then C is totally unimodular.*

Proof. Suppose that $\{1, \ldots, m\}$ and $\{m+1, \ldots, n\}$ is the bipartition of the graph G. Let C' be the matrix obtained by deleting the last $n-m$ columns from C. It suffices to show that C' is totally unimodular. First one successively subtracts the rows $1, 2, \ldots, m$ from the row $n+1$. Then one reverses the sign in the rows $m+1, \ldots, n$. These elementary row operations produce a new matrix C''. The matrix C'' is the incidence matrix of a directed graph, namely, consider G as a directed graph, and add one more vertex $n+1$, and add the edges $(i, n+1)$ for $i = 1, \ldots, m$. The matrix C'', being the incidence matrix of a directed graph, is totally unimodular [6, p. 274]. As the last m column vectors of C'' are

$$e_1 - e_{n+1}, \ldots, e_m - e_{n+1},$$

one can successively pivot on the first nonzero entry of $e_i - e_{n+1}$ for $i = 1, \ldots, m$ and reverse the sign in the rows $m+1, \ldots, n$ to obtain back the matrix C'. Here a pivot on the entry c'_{st} means transforming column t of C'' into the sth unit vector by elementary row operations. Since the operation of pivoting preserves total unimodularity [5, Lemma 2.2.20] one derive that C' is totally unimodular, and hence so is C. This proof is due to Bernd Sturmfels, it is simpler than our original proof. □

Let $\alpha_1, \ldots, \alpha_q$ be the columns of the incidence matrix of a graph G and let I be its edge ideal. There is an epimorphism of graded algebras

$$\varphi: B = K[x_1, \ldots, x_n, t_1, \ldots, t_q] \longrightarrow \mathcal{R}(I) \quad (x_i \overset{\varphi}{\mapsto} x_i), \ (t_i \overset{\varphi}{\mapsto} tx^{\alpha_i}),$$

where B is a polynomial ring with the standard grading.

Corollary 3.2 *If G is a bipartite graph, then the toric ideal $J = \ker(\varphi)$ has a universal Gröbner basis consisting of square-free binomials.*

Proof. It follows from Theorem 3.1 and [8, Proposition 8.11]. □

Corollary 3.3 ([7]) *If G is a bipartite graph and I is its edge ideal, then the Rees algebra $\mathcal{R}(I)$ is normal.*

Proof. It follows from Theorem 3.1 and Theorem 2.1. □

4 The irreducible representation of the Rees cone

If $0 \neq a \in \mathbb{R}^n$, then the set H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \langle x, a \rangle = 0\}.$$

This hyperplane determines two *closed half-spaces*

$$H_a^+ = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \leq 0\}.$$

A subset $F \subset \mathbb{R}^n$ is a *proper face* of a polyhedral cone $\mathbb{R}_+\mathcal{A}$ if there is a supporting hyperplane H_a such that

(i) $F = \mathbb{R}_+\mathcal{A} \cap H_a \neq \emptyset$,

(ii) $\mathbb{R}_+\mathcal{A} \subset H_a^+$ and $\mathbb{R}_+\mathcal{A} \not\subset H_a$.

A proper face F of $\mathbb{R}_+\mathcal{A}$ is a *facet* if $\dim(F) = \dim(\mathbb{R}_+\mathcal{A}) - 1$.

The facets of the Rees cone

In the sequel G is a connected bipartite graph with edge set $E(G)$, vertex set $V = V(G) = \{v_1, \ldots, v_n\}$, and height of $I(G)$ greater or equal than 2.

The polyhedral cone of $\mathbb{R}^{n+1}$ generated by the set of vectors

$$\mathcal{A}' = \{e_i + e_j + e_{n+1} \,|\, v_i \text{ is adjacent to } v_j\} \cup \{e_i \,|\, 1 \leq i \leq n\} \subset \mathbb{R}^{n+1}$$

is called the *Rees cone* of G and it will be denoted by $\mathbb{R}_+\mathcal{A}'$. Note that the Rees cone has dimension $n+1$.

A subset $C \subset V$ is called a *minimal vertex cover* of G if the face ideal $\mathfrak{p} = (\{x_i | v_i \in C\})$ is a minimal prime of $I(G)$ and a subset $A \subset V$ is called an *independent set* of G if any two vertices in A are non adjacent. Thus A is a maximal independent set if and only if $V \setminus A$ is a minimal vertex cover.

In order to describe the facets of the Rees cone we need to introduce another graph theoretical notion. The *cone* $C(G)$ of G is the graph obtained by adding a new vertex v_{n+1} to G and joining every vertex of G to v_{n+1}. If A is an independent set of $C(G)$ we define

$$\alpha_A = \sum_{v_i \in A} e_i - \sum_{v_i \in N(A)} e_i,$$

where $N(A)$ is the neighbor set of A in $C(G)$ consisting of all vertices of $C(G)$ that are adjacent to some vertex of A.

Lemma 4.1 *Let $\mathbb{R}_+\mathcal{B}$ be the polyhedral cone in $\mathbb{R}^{n+1}$ generated by the set*

$$\mathcal{B} = \{e_i + e_j \,|\, \{v_i, v_j\} \in E(G)\} \cup \{e_i + e_{n+1} \,|\, 1 \leq i \leq n\}.$$

Then F is a facet of $\mathbb{R}_+\mathcal{B}$ if and only if

(i) $F = \mathbb{R}_+\mathcal{B} \cap H_{e_i}$, *for some* $1 \leq i \leq n$, *or*

(ii) $F = \mathbb{R}_+\mathcal{B} \cap H_{\alpha_A}$, *where A is a maximal independent set of $C(G)$.*

Proof. $\Rightarrow$) Applying [11, Theorem 3.2] to the graph $C(G)$ it follows that we can write F as in (i) or we can write $F = \mathbb{R}_+\mathcal{B} \cap H_{\alpha_A}$ for some independent set A of $C(G)$ such that the induced subgraph $\langle V \cup \{v_{n+1}\} \setminus (A \cup N(A))\rangle$ has non bipartite connected components. Since this induced subgraph is bipartite one has $V \cup \{v_{n+1}\} = A \cup N(A)$, that is, A is a maximal independent set of $C(G)$.

$\Leftarrow$) If F is as in (i), note $G \setminus \{v_i\}$ is connected and non bipartite. Hence F is a facet. Assume F is as in (ii). First note $V \cup \{v_{n+1}\} = A \cup N(A)$ because A is a maximal independent set of $C(G)$. Consider the subgraph L_1 of $C(G)$ with vertex set $A \cup N(A)$ and edge set $E(L_1) = \{z \in E(C(G)) | \, z \cap A \neq \emptyset\}$. One can rapidly verify (by considering a bipartition of G and showing that L_1 has only even cycles) that L_1 is a connected bipartite graph. Therefore F is a facet by [11, Theorem 3.2]. $\Box$

Theorem 4.2 *F is a facet of the Rees cone $\mathbb{R}_+\mathcal{A}'$ if and only if*

(a) *$F = \mathbb{R}_+\mathcal{A}' \cap H_{e_i}$ for some $1 \leq i \leq n+1$, or*

(b) *$F = \mathbb{R}_+\mathcal{A}' \cap \{x \in \mathbb{R}^{n+1} | -x_{n+1} + \sum_{v_i \in C} x_i = 0\}$ for some minimal vertex cover C of G.*

Proof. $\Rightarrow$) Since the Rees cone is of dimension $n+1$, there is a unique $a \in \mathbb{Z}^{n+1}$ with relatively prime entries such that $F = \mathbb{R}_+\mathcal{A}' \cap H_a$ and $\mathbb{R}_+\mathcal{A}' \subset H_a^+$. Hence the entries of a must satisfy $a_i \geq 0$ for $1 \leq i \leq n$. Consider the vector

$$b = (b_i) = (2a_1 + a_{n+1}, \ldots, 2a_n + a_{n+1}, -a_{n+1}).$$

Using the equalities

$$\begin{aligned} 2\langle e_i + e_j + e_{n+1}, a\rangle &= \langle e_i + e_j, b\rangle \quad \text{if} \quad \{v_i, v_j\} \in E(G), \\ 2\langle e_i, a\rangle &= \langle e_i + e_{n+1}, b\rangle \quad \text{if} \quad 1 \leq i \leq n, \end{aligned}$$

we obtain that $F' = \mathbb{R}_+\mathcal{B} \cap H_b$ is a facet of $\mathbb{R}_+\mathcal{B}$ with $\mathbb{R}_+\mathcal{B} \subset H_b^+$. Thus from Lemma (4.1) we can write b in one of the following three forms:

$$b = \begin{cases} \lambda e_i & 1 \leq i \leq n, \\ \lambda(1, \ldots, 1, -1), & \\ \lambda \alpha_A & \text{for some maximal} \\ & \text{independent set } A \text{ of } G, \end{cases}$$

for some integer $\lambda \neq 0$. In the first and second case we get $a = e_i$ with $1 \leq i \leq n$ and $a = e_{n+1}$ respectively. Now consider the case $b = \lambda\alpha_A$ with $A \subset V$ a maximal independent set of G. Note $A \cup N(A) = V \cup \{v_{n+1}\}$ and $v_{n+1} \notin A$. Hence the entries of b satisfy

$$b_i = \begin{cases} -\lambda & \text{if } v_i \in V \setminus A, \\ \lambda & \text{if } v_i \in A, \\ -\lambda & \text{if } i = n+1. \end{cases}$$

Thus $a_i = 0$ if $v_i \in A$. If $v_i \in V \setminus A$, then $a_i = -a_{n+1}$. It follows that $a_{n+1} = -1$. Therefore setting $C = V \setminus A$ we fall into case (b).

$\Leftarrow$) It follows using the same type of arguments as above. □

As a consequence we get the *irreducible representation* of the Rees cone, which is the main result of this section on polyhedral geometry:

Corollary 4.3 *$\mathbb{R}_+\mathcal{A}'$ is the intersection of the closed halfspaces given by the linear inequalities*

$$\begin{array}{rl} x_i \geq 0 & \quad i = 1, \ldots, n+1, \\ -x_{n+1} + \sum_{v_i \in C} x_i \geq 0 & \quad C \text{ is a minimal vertex cover of } G, \end{array}$$

and none of those halfspaces can be omitted from the intersection.

Remark 4.4 Below we will give applications of Corollary 4.3. One noteworthy consequence is that we can use this result to compute the minimal vertex covers of G using linear programming. Normaliz [2] can in practice be used to determine the facets of the Rees cone.

The canonical module and the a-invariant

Let $I = I(G)$ be the edge ideal of G. Since the Rees algebra $\mathcal{R}(I)$ is a normal domain and a standard graded K-algebra, according to a formula of Danilov-Stanley [1, Theorem 6.3.5] its *canonical module* is the ideal of $\mathcal{R}(I)$ given by

$$\omega_{\mathcal{R}(I)} = (\{x_1^{a_1} \cdots x_n^{a_n} t^{a_{n+1}} \,|\, a = (a_i) \in (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}} \cap \mathbb{Z}^{n+1}\}),$$

where $(\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$ is the topological interior of the Rees cone. Thus Corollary 4.3 yields a description of the canonical module of $\mathcal{R}(I)$ in terms of halfspaces.

For use below β_0 will denote the maximal size of an independent set of G and α_0 will denote the height of $I(G)$. Thus $n = \alpha_0 + \beta_0$. The integer β_0 is called the *independence number* of G. In algebraic terms β_0 is the Krull dimension of the edge ring $R/I(G)$.

Proposition 4.5 *If $a(\mathcal{R}(I))$ is the a-invariant of $\mathcal{R}(I)$ with respect to the grading induced by $\deg(x_i) = 1$ and $\deg(t) = -1$, then*

$$a(\mathcal{R}(I)) = -(\beta_0 + 1).$$

Proof. The a-invariant of $\mathcal{R}(I)$ can be expressed as

$$a(\mathcal{R}(I)) = -\min\{\, i \,|\, (\omega_{\mathcal{R}(I)})_i \neq 0\},$$

see [1]. Let $a=(a_i)$ be an arbitrary vector in $(\mathbb{R}_+\mathcal{A}')^{\rm o}\cap\mathbb{Z}^{n+1}$. By Corollary 4.3 a satisfies $a_i\geq 1$ for $1\leq i\leq n+1$ and

$$-a_{n+1}+\textstyle\sum_{v_i\in C}a_i\geq 1$$

for any minimal vertex cover C of G. Let C be a vertex cover of G with α_0 elements and let $A=V\setminus C$. Note $\beta_0=|A|$. Hence if $m=x_1^{a_1}\cdots x_n^{a_n}t^{a_{n+1}}$, then

$$\begin{aligned}\deg(m) &= a_1+\cdots+a_n-a_{n+1}\\ &= \sum_{v_i\in A}a_i+\sum_{v_i\in C}a_i-a_{n+1}\geq\beta_0+1.\end{aligned}$$

This proves $a(\mathcal{R}(I))\leq-(\beta_0+1)$. On the other hand using Corollary 4.3 and the assumption $\alpha_0\geq 2$ we get that the monomial $m=x_1\cdots x_nt^{\alpha_0-1}$ is in $\omega_{\mathcal{R}(I)}$ and has degree β_0+1. Thus $a(\mathcal{R}(I))\geq-(\beta_0+1)$. $\Box$

References

[1] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, Revised Edition, 1997.

[2] W. Bruns and R. Koch, Normaliz – a program for computing normalizations of affine semigroups, 1998. Available via anonymous ftp from `ftp.mathematik.Uni-Osnabrueck.DE/pub/osm/kommalg/software`.

[3] W. Bruns and R. Koch, Computing the integral closure of an affine semigroup. Effective methods in algebraic and analytic geometry, 2000 (Kraków). Univ. Iagel. Acta Math. **39** (2001), 59–70.

[4] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics **168**, Springer-Verlag, New York, 1996.

[5] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[6] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1986.

[7] A. Simis, W. V. Vasconcelos and R. Villarreal, On the ideal theory of graphs, J. Algebra, **167** (1994), 389–416.

[8] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series **8**, American Mathematical Society, Rhode Island, 1996.

[9] W. V. Vasconcelos, Computing the integral closure of an affine domain, Proc. Amer. Math. Soc. **113** (1991), 633–638.

[10] W. V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer–Verlag, 1998.

[11] R. Villarreal, On the equations of the edge cone of a graph and some applications, Manuscripta Math. **97** (1998), 309–317.

[12] R. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, Inc., New York, 2001.