Ring graphs and complete intersection toric ideals

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Abstract

We study the family of graphs whose number of primitive cycles equals its cycle rank. It is shown that this family is precisely the family of ring graphs. Then we study the complete intersection property of toric ideals of bipartite graphs and oriented graphs. An interesting application is that complete intersection toric ideals of bipartite graphs correspond to ring graphs and that these ideals are minimally generated by Gröbner bases. We prove that any graph can be oriented such that its toric ideal is a complete intersection with a universal Gröbner basis determined by the cycles. It turns out that bipartite ring graphs are exactly the bipartite graphs that have complete intersection toric ideals for any orientation.

1 Introduction

Let G be a graph (no loops or multiple edges) with n vertices and q edges, and let $\mathrm{frank}(G)$ be the number of primitive cycles of G, i.e., cycles without chords. The number $\mathrm{frank}(G)$ is called the *free rank* of G and the number $\mathrm{rank}(G) = q-n+r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. These two numbers satisfy $\mathrm{rank}(G) \leq \mathrm{frank}(G)$, as is seen in Proposition 2.2. The aim of this paper is to study and classify the family of graphs where the equality occurs. It will turn out that this family is precisely the family of ring graphs. The precise definition of a ring graph can be found in Section 2. Roughly speaking *ring graphs* can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices.

The contents of this paper are as follows. Before stating our main results, recall that a graph G has the *primitive cycle property* (PCP) if any two primitive

[0]2000 *Mathematics Subject Classification.* Primary 05C75; Secondary 05C85, 05C20, 13H10.
[1]Partially supported by COFAA-IPN.
[2]This work was partially supported by CONACyT grants 49251-F, 49835-F, and SNI.

cycles intersect in at most one edge. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. As usual we denote the complete graph on n vertices by $\mathcal{K}_n$. In Section 2, which is the core of the paper, we prove the following implications for any graph G:

$$\begin{array}{ccccccc} \text{outerplanar} & \Rightarrow & \text{ring graph} & \Leftrightarrow & \text{PCP} + \text{contains no} & & \\ & & \Updownarrow & & \text{subdivision of } \mathcal{K}_4 & \Rightarrow & \text{planar} \\ & & \text{rank} = \text{frank} & & \text{as a subgraph} & & \end{array}$$

These purely graph theoretical results are applied in Sections 3 and 4, where graphs with complete intersection toric ideals are studied, both in the oriented and unoriented case. For bipartite graphs the equality $\text{rank}(G) = \text{frank}(G)$ is related to these special types of toric ideals as we explain below.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G) = \{t_1, \ldots, t_q\}$. The *edge subring* of G is the k-subalgebra of R:

$$k[G] = k[\{x_i x_j \,|\, x_i \text{ is adjacent to } x_j\}] \subset R.$$

There is an epimorphism of k-algebras

$$\varphi\colon k[t_1, \ldots, t_q] \longrightarrow k[G], \quad \{x, y\} \longmapsto xy,$$

where $k[t_1, \ldots, t_q]$ is a polynomial ring. The kernel of φ, denoted by $P(G)$, is called the *toric ideal* of G. Toric ideals of graphs are studied in Section 3. The height of $P(G)$ is equal to $g = q - \text{rank}(A_G)$, where A_G is the incidence matrix of G. By a result of Krull [2] the ideal $P(G)$ cannot be generated by less than g polynomials. The toric ideal of G is called a *complete intersection* if it can be generated by g polynomials. The complete intersection property of $P(G)$ was first studied in [6, 19], and later in [8, 13].

An interesting result of Simis [19] shows that if G is a bipartite graph, then $\text{rank}(G) = \text{frank}(G)$ if and only if $P(G)$ is a complete intersection. Thus by describing the graphs where equality occurs, we are in particular describing the toric ideals of bipartite graphs that are complete intersections (see Corollary 3.4). We prove that complete intersection toric ideals of 2-connected bipartite graphs are minimally generated by Gröbner bases (see Corollary 3.7).

In Section 4 we introduce and study toric ideals of oriented graphs and their Gröbner bases. To the best of our knowledge these toric ideals have not been studied much except for the case of acyclic tournaments [12]. Oriented graphs share some properties with bipartite graphs. For instance in both cases their incidence matrices are totally unimodular. This is a key fact to understand the Gröbner bases of toric ideals of oriented graphs (see Lemma 4.1). We prove that the toric ideal of any oriented graph is completely determined by its primitive cycles and has a universal Gröbner basis determined by the cycles (see Proposition 4.3 and Corollary 4.5). It is shown that toric ideals of oriented ring graphs are complete intersections for any orientation. As an interesting consequence of the results of Section 2 we obtain that for bipartite graphs this property characterizes ring graphs (see Corollary 4.9). One of our main results shows that any graph has an acyclic orientation such that the corresponding toric ideal is a complete intersection (see Theorem 4.16).

The paper is essentially self contained. For unexplained terminology and notation on graph theory we refer to [5, 10]. Our main references for edge subrings are [21, 22].

2 Ring graphs

Let G be a graph with n vertices and q edges. We denote the vertex set and edge set of G by $V(G) = \{x_1, \ldots, x_n\}$ and $E(G) = \{t_1, \ldots, t_q\}$ respectively. Recall that a 0-chain (resp. 1-chain) of G is a formal linear combination $\sum a_i x_i$ (resp. $\sum b_i t_i$) of vertices (resp. edges), where $a_i \in \mathbb{Z}_2$ (resp. $b_i \in \mathbb{Z}_2$). The *boundary operator* is the linear map $\partial\colon C_1 \to C_0$ defined by

$$\partial(\{x, y\}) = x + y,$$

where C_i is the $\mathbb{Z}_2$-vector space of i-chains. A *cycle vector* is a 1-chain of the form $t_1 + \cdots + t_r$ where $t_1, \ldots, t_r$ are the edges of a cycle of G. The *cycle space* $\mathcal{Z}(G)$ of G over $\mathbb{Z}_2$ is equal to $\ker(\partial)$. The vectors in $\mathcal{Z}(G)$ can be regarded as a set of edge-disjoint cycles. A *cycle basis* for G is a basis for $\mathcal{Z}(G)$ which consists entirely of cycle vectors, such a basis can be constructed as follows:

Remark 2.1 [10, pp. 38-39] If G is connected, then G has a spanning tree T. The subgraph of G consisting of T and any edge in G not in T has exactly one cycle, the collection of all cycle vectors of cycles obtained in this way form a cycle basis for G. Hence $\dim_{\mathbb{Z}_2} \mathcal{Z}(G) = q - n + r$ if G is a graph with r connected components.

Let c be a cycle of G. A *chord* of c is any edge of G joining two non adjacent vertices of c. A cycle without chords is called *primitive*. The number $\dim_{\mathbb{Z}_2} \mathcal{Z}(G)$ is called the *cycle rank* of G and is denoted by $\operatorname{rank}(G)$. The number of primitive cycles of a graph G, denoted by $\operatorname{frank}(G)$, is called the *free rank* of G.

Proposition 2.2 *If G is a graph, then $\mathcal{Z}(G)$ is generated by cycle vectors of primitive cycles. In particular* $\operatorname{rank}(G) \le \operatorname{frank}(G)$.

Proof. Let $\mathbf{c}_1, \ldots, \mathbf{c}_r$ be a cycle basis for the cycle space of G and let $c_1, \ldots, c_r$ be the corresponding cycles of G. It suffices to notice that if some c_j has a chord, we can write $\mathbf{c}_j = \mathbf{c}'_j + \mathbf{c}''_j$, where $\mathbf{c}'_j$ and $\mathbf{c}''_j$ are cycle vectors of cycles of length smaller than that of c_j. $\Box$

Corollary 2.3 *Let G be a graph. Then the following are equivalent*:

(a) $\mathrm{rank}(G) = \mathrm{frank}(G)$.

(b) *The set of cycle vectors of primitive cycles is a basis for* $\mathcal{Z}(G)$.

(c) *The set of cycle vectors of primitive cycles is linearly independent.*

Proof. (a) $\Rightarrow$ (b): By Proposition 2.2 there is a basis $\mathcal{B}$ of $\mathcal{Z}(G)$ consisting of cycle vectors of primitive cycles. By hypothesis $\mathrm{rank}(G) = \mathrm{frank}(G)$. Thus $\mathcal{B}$ is the set of all cycle vectors of primitive cycles and $\mathcal{B}$ is a basis. That (b) implies (c) and (c) implies (a) are also very easy to prove. $\Box$

Let G be a graph. A vertex v (resp. an edge e) of G is called a *cutvertex* (resp. *bridge*) if the number of connected components of $G \setminus \{v\}$ (resp. $G \setminus \{e\}$) is larger than that of G. A maximal connected subgraph of G without cutvertices is called a *block*. A graph G is 2-*connected* if $|V(G)| > 2$ and G has no cutvertices. Thus a block of G is either a maximal 2-connected subgraph, a bridge or an isolated vertex. By their maximality, different blocks of G intersect in at most one vertex, which is then a cutvertex of G. Therefore every edge of G lies in a unique block, and G is the union of its blocks.

Lemma 2.4 *Let G be a graph and let $G_1, \ldots, G_r$ be its blocks. Then* $\mathrm{rank}(G) = \mathrm{frank}(G)$ *if and only if* $\mathrm{rank}(G_i) = \mathrm{frank}(G_i)$ *for all i.*

Proof. $\Rightarrow$) Let G_i be any block of G. We may assume $|V(G_i)| > 2$, otherwise $\mathrm{rank}(G_i) = \mathrm{frank}(G_i) = 0$. If c is a primitive cycle of G_i, then by the maximality condition of a block one has that c is also a primitive cycle of G. Thus by Corollary 2.3 the set of cycle vectors of primitive cycles of G_i is linearly independent and $\mathrm{rank}(G_i) = \mathrm{frank}(G_i)$.

$\Leftarrow$) Let $\mathcal{B}_i$ and $\mathcal{B}$ be the set of cycle vector of primitive cycles of G_i and G respectively. As $\cup_{i=1}^r \mathcal{B}_i$ is linearly independent, by Corollary 2.3 it suffices to prove that $\cup_{i=1}^r \mathcal{B}_i = \mathcal{B}$. In the first part of the proof we have already observed that $\cup_{i=1}^r \mathcal{B}_i \subset \mathcal{B}$. To prove the equality take any cycle vector $\mathbf{c}$ of a primitive cycle c of G. Since c is a 2-connected subgraph, it must be contained in some block of G, i.e., in some G_i. Thus c is a primitive cycle of G_i, so $\mathbf{c}$ is in $\mathcal{B}_i$. $\Box$

Definition 2.5 Given a graph H, we call a path $\mathcal{P}$ an *H-path* if $\mathcal{P}$ is non-trivial and meets H exactly in its ends.

In order to describe, in graph theoretical terms, the family of graphs satisfying the equality $\mathrm{rank}(G) = \mathrm{frank}(G)$ we need to introduce another notion.

Definition 2.6 A graph G is a *ring graph* if each block of G which is not a bridge or a vertex can be constructed from a cycle by successively adding H-paths of length at least 2 that meet graphs H already constructed in two adjacent vertices.

Families of ring graphs include forests and cycles. These graphs are planar by construction.

Remark 2.7 Let G be a 2-connected ring graph and let c be a fixed primitive cycle of G, then G can be constructed from c by successively adding H-paths of length at least 2 that meet graphs H already constructed in two adjacent vertices.

A graph H is called a *subdivision* of a graph G if $H = G$ or H arises from G by replacing edges by paths.

Lemma 2.8 [1, Lemma 7.78, p. 387] *Let G be a graph with vertex set V. If G is 2-connected and $\deg(v) \geq 3$ for all $v \in V$, then G contains a subdivision of $\mathcal{K}_4$ as a subgraph.*

Lemma 2.9 *Let G be a graph. If $\text{rank}(G) = \text{frank}(G)$ and x, y are two non adjacent vertices of G, then there are at most two vertex disjoint paths joining x and y.*

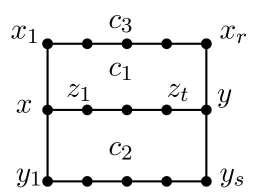
Proof. Assume that there are three vertex disjoint paths joining x and y:

$$\mathcal{P}_1 = \{x, x_1, \ldots, x_r, y\}, \qquad \mathcal{P}_2 = \{x, z_1, \ldots, z_t, y\},$$
$$\mathcal{P}_3 = \{x, y_1, \ldots, y_s, y\},$$

where r, s, t are greater or equal than 1. We may assume that the sum of the lengths of the $\mathcal{P}_i$'s is minimal. Consider the cycles

$$\begin{aligned} c_1 &= \{x, x_1, \ldots, x_r, y, z_t, \ldots, z_1, x\}, \\ c_2 &= \{x, z_1, \ldots, z_t, y, y_s, \ldots, y_1, x\}, \\ c_3 &= \{x, x_1, \ldots, x_r, y, y_s, \ldots, y_1, x\}. \end{aligned}$$

Thus we are in the following situation:



Observe that, by the choice of the $\mathcal{P}_i$'s, a chord of the cycle c_1 (resp. c_2, c_3) must join x_i and z_j (resp. z_i and y_j, x_i and y_j) for some i, j. If c_1 is not primitive, we can write

$$\mathbf{c}_1 = \mathbf{a}_1 + \cdots + \mathbf{a}_{n_1}$$

for some distinct cycle vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{n_1}$ of primitive cycles $\mathrm{a}_1, \ldots, \mathrm{a}_{n_1}$ such that each cycle a_i contains at least one edge of the form $\{x_j, z_k\}$. Similarly if c_2 (resp. c_3) is not primitive we can write:

$$\mathbf{c}_2 = \mathbf{b}_1 + \cdots + \mathbf{b}_{n_2} \quad (\text{resp.} \quad \mathbf{c}_3 = \mathbf{d}_1 + \cdots + \mathbf{d}_{n_3})$$

for some distinct cycle vectors $\mathbf{b}_1, \ldots, \mathbf{b}_{n_2}$ (resp. $\mathbf{d}_1, \ldots, \mathbf{d}_{n_3}$) of primitive cycles such that each cycle b_i (resp. d_i) contains at least one edge of the form $\{z_j, y_k\}$ (resp. $\{x_j, y_k\}$). Therefore we can write

$$\mathbf{c}_1 = \sum_{i=1}^{n_1} \mathbf{a}_i, \quad \mathbf{c}_2 = \sum_{i=1}^{n_2} \mathbf{b}_i, \quad \mathbf{c}_3 = \sum_{i=1}^{n_3} \mathbf{d}_i$$

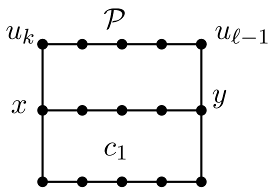
where $\mathbf{a}_1, \ldots, \mathbf{a}_{n_1}, \mathbf{b}_1, \ldots, \mathbf{b}_{n_2}, \mathbf{d}_1, \ldots, \mathbf{d}_{n_3}$ are distinct cycle vectors of primitive cycles of G. Thus from the equality $\mathbf{c}_3 = \mathbf{c}_1 + \mathbf{c}_2$ we get a non trivial linear relation of the set $\mathcal{B}$ of cycle vectors of primitive cycles, i.e., $\mathcal{B}$ is linearly dependent, a contradiction to Corollary 2.3. $\square$

Lemma 2.10 *Let G be a graph. If* $\operatorname{rank}(G) = \operatorname{frank}(G)$*, then G has the primitive cycle property.*

Proof. Let c_1, c_2 be two distinct primitive cycles. Assume that c_1 and c_2 intersect in at least two edges. Thus c_1 and c_2 must intersect in at least two non adjacent vertices u, v. The cycle c_2 can be written as:

$$c_2 = \{u = u_0, u_1, \ldots, u_s, v = u_{s+1}, v_1, \ldots, v_m, u\}.$$

At least one of the paths $\mathcal{P}_1 = \{u, u_1, \ldots, u_s, v\}$, $\mathcal{P}_2 = \{v, v_1, \ldots, v_m, u\}$ that form the cycle c_2 must contain a vertex not in c_1, otherwise $c_1 = c_2$. Assume that the path $\mathcal{P}_1$ has this property. Hence there is $u_k \notin c_1$ such that $u_i \in c_1$ for $i < k$, and there is $u_\ell \in c_1$, with $k < \ell$, such that $u_i \notin c_1$ for $k \le i < \ell$. Hence there are two non adjacent vertices $x = u_{k-1}, y = u_\ell$ in c_1 and a path $\mathcal{P} = \{x, u_k, \ldots, u_{\ell-1}, y\}$ of length at least two that intersect c_1 in exactly the vertices x, y:



This contradicts Lemma 2.9. □

Lemma 2.11 *Let G be a graph. If G satisfies* PCP *and G does not contain a subdivision of $\mathcal{K}_4$ as a subgraph, then for any two non adjacent vertices x, y of G there are at most two vertex disjoint paths joining x and y.*

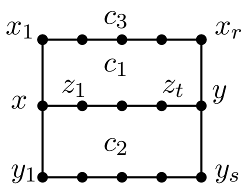
Proof. Assume that there are three vertex disjoint paths joining x and y:

$$\mathcal{P}_1 = \{x, x_1, \ldots, x_r, y\},\ \mathcal{P}_2 = \{x, z_1, \ldots, z_t, y\},\ \mathcal{P}_3 = \{x, y_1, \ldots, y_s, y\},$$

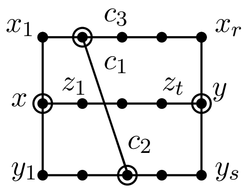
where r, s, t are greater or equal than 1. We may assume that the sum of the lengths of the $\mathcal{P}_i$'s is minimal. Consider the cycles

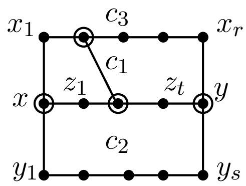
$$c_1 = \{x, x_1, \ldots, x_r, y, z_t, \ldots, z_1, x\}, \quad c_2 = \{x, z_1, \ldots, z_t, y, y_s, \ldots, y_1, x\},$$
$$c_3 = \{x, x_1, \ldots, x_r, y, y_s, \ldots, y_1, x\}.$$

Thus we are in the following situation:



Observe that, by the choice of the $\mathcal{P}_i$'s, a chord of the cycle c_1 (resp. c_2, c_3) must join x_i and z_j (resp. z_i and y_j, x_i and y_j) for some i, j. Notice that the cycles c_1 and c_3 are primitive. Indeed if c_1 or c_3 have a chord, then one of the following





is a subgraph of G, which is impossible because both subgraphs are subdivisions of $\mathcal{K}_4$. Since c_1 and c_3 are primitive and have at least two edges in common we obtain that G does not satisfy PCP, a contradiction. □

Lemma 2.12 *Let G be a graph. If* $\operatorname{rank}(G) = \operatorname{frank}(G)$*, then G does not contain a subdivision of $\mathcal{K}_4$ as a subgraph.*

Proof. Assume there is a subgraph $H \subset G$ which is a subdivision of $\mathcal{K}_4$. If $\mathcal{K}_4$ is a subgraph of G, then G has four distinct triangles whose cycle vectors are linearly dependent, a contradiction to Corollary 2.3. If $\mathcal{K}_4$ is not a subgraph of G, then H is a strict subdivision of $\mathcal{K}_4$, i.e., H has more than four vertices. It follows that there are two vertices x, y in $V(H)$ which are non adjacent in G. Notice that x, y can be chosen in $\mathcal{K}_4$ before subdivision. Therefore there are at least three non adjacent paths joining x and y, a contradiction to Lemma 2.9. □

The main result of this section is:

Theorem 2.13 *Let G be a graph. Then the following conditions are equivalent*:

(a) *G is a ring graph.*

(b) $\mathrm{rank}(G) = \mathrm{frank}(G)$.

(c) *G satisfies* PCP *and G does not contain a subdivision of $\mathcal{K}_4$ as a subgraph.*

Proof. (a) $\Rightarrow$ (b): By induction on the number of vertices it is not hard to see that any ring graph G satisfies the equality $\mathrm{rank}(G) = \mathrm{frank}(G)$.

(b) $\Rightarrow$ (c): It follows at once from Lemmas 2.10 and 2.12.

(c) $\Rightarrow$ (a): Let $G_1, \ldots, G_r$ be the blocks of G. The proof is by induction on the number of vertices of G. If each G_i is either a bridge or an isolated vertex, then G is a forest and consequently a ring graph. Hence by Lemma 2.4 we may assume that G is 2-connected and that G is not a cycle. We claim that G has at least one vertex of degree 2. If $\deg(v) \geq 3$ for all $v \in V(G)$, then by Lemma 2.8 there is a subgraph $H \subset G$ which is a subdivision of $\mathcal{K}_4$, which is impossible. Let $v_0 \in V(G)$ be a vertex of degree 2 as claimed. By the primitive cycle property there is a unique primitive cycle $c = \{v_0, v_1, \ldots, v_s = v_0\}$ of G containing v_0. The graph $H = G \setminus \{v_0\}$ satisfies PCP and does not has a subdivision of $\mathcal{K}_4$ as a subgraph. Consequently H is a ring graph. Thus we may assume that c is not a triangle, otherwise G is a ring graph because it can be obtained by adding the H-path $\{v_2, v_0, v_1\}$ to H.

Next we claim that if $1 \leq i < j < k \leq s-1$, then v_i and v_k cannot be in the same connected component of $H \setminus \{v_j\}$. Otherwise there is a path of $H \setminus \{v_j\}$ than joins v_i with v_k. It follows that there is a path $\mathcal{P}$ of $H \setminus \{v_j\}$ with at least three vertices that joins a vertex of $\{v_{j+1}, \ldots, v_{s-1}\}$ with a vertex of $\{v_1, \ldots, v_{j-1}\}$ and such that $\mathcal{P}$ intersects c exactly in its ends, but this contradicts Lemma 2.11. This proves the claim. In particular v_i is a cutvertex of H for $i = 2, \ldots, s-2$ and v_{i-1}, v_{i+1} are in different connected components of $H \setminus \{v_i\}$. For each $1 \leq i \leq s-2$ there is a block K_i of H such that $\{v_i, v_{i+1}\}$ is an edge of K_i. Notice that if $1 \leq i < j < k \leq s-1$, then v_i, v_j, v_k cannot lie in some K_ℓ. Indeed if the three vertices lie in some K_ℓ, then there is a path $\mathcal{P}'$ in $K_\ell \setminus \{v_j\}$ that joins v_i and v_k. Since $\mathcal{P}'$ is also a path in $H \setminus \{v_j\}$, we get that v_i and v_k are in the same connected component of $H \setminus \{v_j\}$, but this contradicts the last claim. In particular $V(K_\ell)$ intersects the cycle c in exactly the vertices $v_\ell, v_{\ell+1}$ for $1 \leq \ell \leq s-2$.

Observe that at least one of the edges of c not containing v_0 is not a bridge of H. To show this pick $x \notin c$ such that $\{x, v_k\}$ is an edge of H. We may assume that $v_{k+1} \neq v_0$ (or $v_{k-1} \neq v_0$). Since $G' = G \setminus \{v_k\}$ is connected, there is a path $\mathcal{P}$ of G' joining x and v_{k+1} (or v_{k-1}). This readily yields a cycle of H containing an edge of c which is not a bridge of H. Hence at least one of the blocks $K_1, \ldots, K_{s-2}$, say K_i, contains vertices outside c.

Next we show that two distinct blocks B_1, B_2 of H cannot intersect outside c. We proceed by contradiction assuming that $V(B_1) \cap V(B_2) = \{z\}$ for some z not in c. Let $H_1, \ldots, H_t$ be the connected components of $H \setminus \{z\}$. Notice that $t \geq 2$ because $\{z\}$ is the intersection of two different blocks of H. We may assume that $\{v_1, \ldots, v_{s-1}\}$ are contained in H_1. Consider the subgraph H_1' of $G \setminus \{z\}$ obtained from H_1 by adding the vertex v_0 and the edges $\{v_0, v_1\}$, $\{v_0, v_{s-1}\}$. It follows that the connected components of $G \setminus \{z\}$ are $H_1', H_2, \ldots, H_t$, which is impossible because G is 2-connected.

Let K_i be a block of H that contains vertices outside c for some $1 \leq i \leq s-2$. By induction hypothesis K_i is a ring graph. Thus by Remark 2.7 we can construct K_i starting with a primitive cycle c_1 that contains the edge $\{v_i, v_{i+1}\}$, and then adding appropriate paths. Suppose that $\mathcal{P}_1, \ldots, \mathcal{P}_m$ is the sequence of paths added to c_1 to obtain K_i. If we remove the path $\mathcal{P}_m$ from G and use the fact that distinct blocks of H cannot intersect outside c, then again by induction hypothesis we obtain a ring graph. It follows that G is a ring graph as well. □

An immediate consequence of Theorem 2.13 is:

Corollary 2.14 *Let G be a graph. If* $\mathrm{rank}(G) = \mathrm{frank}(G)$*, then G is planar.*

Corollary 2.15 *If G is a ring graph and H is an induced subgraph of G, then H is a ring graph.*

Proof. It follows from part (c) of Theorem 2.13. □

Two graphs H_1 and H_2 are called *homeomorphic* if there exists a graph G such that both H_1 and H_2 are subdivisions of G. A graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on a common face; it is usual to choose this face to be the exterior face. The complete bipartite graph with bipartition (V_1, V_2) is denoted by $\mathcal{K}_{t,s}$, where $|V_1| = t$ and $|V_2| = s$.

Theorem 2.16 [10, Theorem 11.10] *A graph is outerplanar if and only if it has no subgraph homeomorphic to $\mathcal{K}_4$ or $\mathcal{K}_{2,3}$ except $\mathcal{K}_4 \setminus \{e\}$, where e is an edge.*

Proposition 2.17 *If G is an outerplanar graph, then* $\mathrm{rank}(G) = \mathrm{frank}(G)$.

Proof. By Theorem 2.13(c) it suffices to prove that G satisfies PCP and G does not contain a subdivision of $\mathcal{K}_4$ as a subgraph. If G contains a subdivision H of $\mathcal{K}_4$ as a subgraph, then G contains a subgraph, namely H, homeomorphic to $\mathcal{K}_4$, but this is impossible by Theorem 2.16. To finish the proof we now show that G has the PCP property. Let $c_1 = \{x_1, x_2, \ldots, x_m = x_1\}$ and $c_2 = \{y_1, y_2, \ldots, y_n = y_1\}$ be two distinct primitive cycles having at least one common edge. We may assume that $x_i = y_i$ for $i = 1, 2$ and $x_3 \neq y_3$. Notice that $y_3 \notin c_1$ because otherwise $\{y_2, y_3\} = \{x_2, y_3\}$ is a chord of c_1. We need only show that $\{x_1, x_2\} = c_1 \cap c_2$, because this implies that c_1 and c_2 cannot have more than one edge in common. Assume that $\{x_1, x_2\} \subsetneq c_1 \cap c_2$. Let r be the minimum integer such that y_r belong to $(c_1 \cap c_2) \setminus \{x_1, x_2\}$. Notice that $y_r \neq x_3$ because otherwise $\{x_2, x_3\}$ is a chord of c_2. Hence c_1 together with the path $\{x_2 = y_2, y_3, \ldots, y_r\}$ give a subgraph H of G which is a subdivision of $\mathcal{K}_{2,3}$, a contradiction to Theorem 2.16. $\Box$

3 Toric ideals of graphs

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$. The *edge subring* of the graph G, denoted by $k[G]$, is the k-subalgebra of R generated by the monomials corresponding to the edges of G:

$$k[G] = k[\{x_i x_j \,|\, x_i \text{ is adjacent to } x_j\}] \subset R.$$

There is a graded epimorphism of k-algebras

$$\varphi\colon B = k[t_1, \ldots, t_q] \longrightarrow k[G], \quad \{x, y\} \longmapsto xy,$$

where B is a polynomial ring graded by $\deg(t_i) = 1$ for all i and $k[G]$ has the normalized grading $\deg(f_i) = 1$ for all i. The kernel of φ, denoted by $P(G)$, is a graded prime ideal of B called the *toric ideal* of G. The graded structure of $P(G)$ will not play a role in what follows. Later we will emphasize the fact that toric ideals of oriented graphs may not have a graded structure. Having a grading is useful if one studies the projective toric variety defined by $P(G)$ or systems of generators of $P(G)$.

The Krull dimension of $k[G]$ equals the rank of the incidence matrix of G [11]. If G is a connected graph, then by [23, Corollary 6.3] one has:

$$\dim(k[G]) = \begin{cases} n & \text{if } G \text{ is not bipartite, and} \\ n-1 & \text{otherwise.} \end{cases}$$

Since $B/P(G) \simeq k[G]$, we obtain that height of $P(G)$ is $q - n + 1$ if G is a connected bipartite graph and that height of $P(G)$ is $q - n$ if G is a connected non-bipartite graph.

Definition 3.1 The toric ideal $P(G)$ is called a *complete intersection* if it can be generated by g polynomials, where g is the height of $P(G)$. The graph G is called a *complete intersection* if $P(G)$ is a complete intersection.

The complete intersection property is independent of k [14, Theorem 3.9]. In the area of complete intersection toric ideals there are some recent papers, see [3, 4] and the introduction of [14], where one can find additional properties and references on this active area.

Next we describe a generating set for $P(G)$ that shows how the cycle structure of G determine $P(G)$. Let

$$c = \{x_0, x_1, \ldots, x_r = x_0\}$$

be an even cycle of G such that $f_i = x_{i-1}x_i$. Notice that the binomial

$$t_c = t_1 t_3 \cdots t_{r-1} - t_2 t_4 \cdots t_r$$

is in $P(G)$. If G is bipartite, then $P(G)$ is minimally generated by the set of all t_c such that c is a primitive cycle of G, see [21].

The next result can be extended to non connected bipartite graphs.

Theorem 3.2 [19, Theorem 2.5] *If G is a bipartite connected graph, then G is a complete intersection if and only if* $\mathrm{rank}(G) = \mathrm{frank}(G)$.

This was the first characterization of complete intersection bipartite graphs. For these graphs the equality $\mathrm{rank}(G) = \mathrm{frank}(G)$ can also be interpreted in homological terms [19]. Another characterization is the following:

Theorem 3.3 ([13]) *If G is a bipartite graph, then G is a complete intersection if and only if G is planar and satisfies* PCP.

The next result is interesting because it shows how to construct all the complete intersection bipartite graphs.

Corollary 3.4 *If G is a bipartite graph, then G is a complete intersection if and only if G is a ring graph.*

Proof. By Theorem 3.2 G is a complete intersection if and only if $\mathrm{rank}(G) = \mathrm{frank}(G)$ and the result follows from Theorem 2.13. □

Notation For $a = (a_1, \ldots, a_q) \in \mathbb{N}^q$ and $f_1, \ldots, f_q$ in a commutative ring we set $f^a = f_1^{a_1} \cdots f_q^{a_q}$. The *support* of f^a is the set $\mathrm{supp}(f^a) = \{f_i \mid a_i \neq 0\}$.

Definition 3.5 Let $g_1 = t^{\alpha_1} - t^{\beta_1}, \ldots, g_r = t^{\alpha_r} - t^{\beta_r}$ be a sequence of homogeneous binomials of degree at least 2 in the polynomial ring $B = k[t_1, \ldots, t_q]$. We say that $\mathcal{B} = \{g_1, \ldots, g_r\}$ is a *foliation* if the following conditions are satisfied:

(a) t^{α_i} and t^{β_i} are square-free monomials for all i,

(b) $\mathrm{supp}(t^{\alpha_i}) \cap \mathrm{supp}(t^{\beta_i}) = \emptyset$ for all i, and

(c) $|(\cup_{i=1}^{j} C_i) \cap C_{j+1}| = 1$ for $1 \leq j < r$, where $C_i = \mathrm{supp}(t^{\alpha_i}) \cup \mathrm{supp}(t^{\beta_i})$.

Proposition 3.6 *If $\mathcal{B} = \{g_1, \ldots, g_r\}$ is a foliation, then the ideal $I = (\mathcal{B})$ generated by $\mathcal{B}$ is a complete intersection and $\mathcal{B}$ is a Gröbner basis of I.*

Proof. By the constructive nature of $\mathcal{B}$ we can order the variables $t_1, \ldots, t_q$ such that the leading terms of $g_1, \ldots, g_r$, with respect to the lexicographical order, are relatively prime. Let $\mathrm{in}(g_i)$ be the leading term of g_i. Then $\mathcal{B}$ is a Gröbner basis by a result of Buchberger [22, Theorem 2.4.15]. Since B/I and $B/(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_r))$ have the same Krull dimension by a result of Macaulay [22, Corollary 2.4.13], we obtain that the height of I is equal to r, as required. □

Corollary 3.7 *If G is a 2-connected bipartite graph with at least four vertices, then the toric ideal $P(G)$ is a complete intersection if and only if it is generated by a foliation.*

Proof. It follows from Corollary 3.4 and the definition of a ring graph. □

4 Toric ideals of oriented graphs

Let G be a connected graph with n vertices and q edges and let $\mathcal{O}$ be an orientation of the edges of G, i.e., an assignment of a direction to each edge of G. Thus $\mathcal{D} = (G, \mathcal{O})$ is an *oriented graph.* To each oriented edge $e = (x_i, x_j)$ of $\mathcal{D}$, we associate the vector v_e defined as follows: the *ith* entry is -1, the *jth* entry is 1, and the remaining entries are zero. The *incidence matrix* $A_{\mathcal{D}}$ of $\mathcal{D}$ is the $n \times q$ matrix with entries in $\{0, \pm 1\}$ whose columns are the vectors of the form v_e, with e an edge of $\mathcal{D}$. For simplicity of notation we set $A = A_{\mathcal{D}}$. The set of column vectors of A will be denoted by $\mathcal{A} = \{v_1, \ldots, v_q\}$. It is well known [15] that A defines a matroid $M[A]$ on $\mathcal{A} = \{v_1, \ldots, v_q\}$ over the field $\mathbb{Q}$ of rational numbers, which is called the *vector matroid* of A, whose independent sets are the independent subsets of $\mathcal{A}$. A *minimal dependent set* or *circuit* of $M[A]$ is a dependent set all of whose proper subsets are independent. A subset B of $\mathcal{A}$ is called a *basis* of $M[A]$ if B is a maximal independent set. Recall that an integer matrix is called *totally unimodular* if each $i \times i$ minor (subdeterminant) of the matrix is 0 or ± 1 for all $i \geq 1$.

Lemma 4.1 *The circuits of $M[A]$ are precisely the cycles of G, A is totally unimodular, and* $\mathrm{rank}(A) = n - 1$

Proof. It follows from [9, pp. 343-344] and [18, p. 274]. □

Let $\alpha \in \mathbb{R}^q$. The *support* of α is defined as $\text{supp}(\alpha) = \{i \,|\, \alpha_i \neq 0\}$. An *elementary vector* of $\ker(A)$ is a vector $0 \neq \alpha$ in $\ker(A)$ whose support is minimal with respect to inclusion, i.e., $\text{supp}(\alpha)$ does not properly contain the support of any other nonzero vector in $\ker(A)$. A *circuit* of $\ker(A)$ is an elementary vector of $\ker(A)$ with relatively prime integral entries (see [24, Section 2]). There is a one to one correspondence

$$\text{Circuits of } \ker(A) \quad \longrightarrow \quad \text{Circuits of } M[A] = \text{cycles of } G$$

given by $\alpha = (\alpha_1, \ldots, \alpha_q) \to C(\alpha) = \{v_i | \, i \in \text{supp}(\alpha)\}$. Thus the set of circuits of the kernel of A is the algebraic realization of the set of circuits of the vector matroid $M[A]$.

Consider the *edge subring* $k[\mathcal{D}] := k[x^{v_1}, \ldots, x^{v_q}] \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of the oriented graph $\mathcal{D}$. There is an epimorphism of k-algebras

$$\varphi \colon B = k[t_1, \ldots, t_q] \longrightarrow k[\mathcal{D}], \quad t_i \longmapsto x^{v_i},$$

where B is a polynomial ring. The kernel of φ, denoted by $P_{\mathcal{D}}$, is called the *toric ideal* of $\mathcal{D}$. Notice that $P_{\mathcal{D}}$ is no longer a graded ideal, see Proposition 4.7. The toric ideal $P_{\mathcal{D}}$ is a prime ideal of height $q - n + 1$ generated by binomials and $k[\mathcal{D}]$ is a normal domain. Thus any minimal generating set of $P_{\mathcal{D}}$ must have at least $q - n + 1$ elements, by the principal ideal theorem.

Let $\alpha \in \mathbb{R}^q$. Note that $\alpha = \alpha_+ - \alpha_-$, where α_+ and α_- are two non negative vectors with disjoint support. If $0 \neq \alpha \in \ker(A) \cap \mathbb{Z}^n$ we associate the binomial $t_\alpha = t^{\alpha_+} - t^{\alpha_-}$. Notice that $t_\alpha \in P_{\mathcal{D}}$. Given a cycle c of $\mathcal{D}$, we split c in two disjoint sets of edges c_+ and c_-, where c_+ is oriented clockwise and $c_- = c \setminus c_+$. The binomial

$$t_c = \prod_{v_i \in c_+} t_i - \prod_{v_i \in c_-} t_i$$

belongs to $P_{\mathcal{D}}$. If $c_+ = \emptyset$ or $c_- = \emptyset$ we set $\prod_{v_i \in c_+} t_i = 1$ or $\prod_{v_i \in c_-} t_i = 1$.

Definition 4.2 The toric ideal $P_{\mathcal{D}}$ is called a *binomial complete intersection* if $P_{\mathcal{D}}$ can be generated by $q - n + 1$ binomials.

If $P_{\mathcal{D}}$ is homogeneous and is generated by $q - n + 1$ polynomials, then $P_{\mathcal{D}}$ is a binomial complete intersection.

Proposition 4.3 *$P_{\mathcal{D}}$ is generated by the set of all binomials t_c such that c is a cycle of $\mathcal{D}$ and this set is a universal Gröbner basis.*

Proof. Let $\mathcal{U}_{\mathcal{D}}$ be the set of all binomials of the form t_α such that α is a circuit of $\ker(A)$. Since A is totally unimodular, by [20, Proposition 8.11], the set $\mathcal{U}_{\mathcal{D}}$ form a universal Gröbner basis of $P_{\mathcal{D}}$. Notice that the circuits of $\ker(A)$ are in one to one correspondence with the circuits of the vector matroid $M[A]$. To complete the proof it suffices to observe that the circuits of $M[A]$ are precisely the cycles of G, see Lemma 4.1. $\Box$

Proposition 4.4 *Let $c = \{x_1, x_2, \ldots, x_r, x_1\}$ be a circuit of $\mathcal{D}$. Suppose that (x_i, x_j) or (x_j, x_i) is an edge of $\mathcal{D}$, with $i + 1 < j$. Then t_c is a linear combination of t_{c_1} and t_{c_2}, where $c_1 = \{x_1, x_2, \ldots, x_i, x_j, x_{j+1}, \ldots, x_r, x_1\}$ and $c_2 = \{x_i, x_{i+1}, \ldots, x_j, x_i\}$.*

Proof. Suppose without loss of generality that $v_k = (x_i, x_j)$ is the edge of $\mathcal{D}$ with $i + 1 < j$. Then we can write $t_{c_1} = t^{\alpha_+} - t^{\alpha_-}$ and $t_{c_2} = t^{\beta_+} - t^{\beta_-}$ for some α, β. We may assume that $v_k \in c_{1_+} \cap c_{2_+}$, because otherwise we may multiply t_{c_1} or t_{c_2} by -1. As t_k divides t^{α_+} and t_k divides t^{β_+}, we get

$$\begin{aligned}\left(\frac{t^{\beta_+}}{t_k}\right) t_{c_1} - \left(\frac{t^{\alpha_+}}{t_k}\right) t_{c_2} &= \left(\frac{t^{\beta_+}}{t_k}\right)(t^{\alpha_+} - t^{\alpha_-}) - \left(\frac{t^{\alpha_+}}{t_k}\right)(t^{\beta_+} - t^{\beta_-}) \\ &= \left(\frac{t^{\alpha_+}}{t_k}\right) t^{\beta_-} - \left(\frac{t^{\beta_+}}{t_k}\right) t^{\alpha_-} = t^{\gamma_1} - t^{\gamma_2}.\end{aligned}$$

Hence $t^{\gamma_1} - t^{\gamma_2}$ is in $P_{\mathcal{D}}$, where $\gamma_1 = (\alpha_+ - e_k) + \beta_-$ and $\gamma_2 = (\beta_+ - e_k) + \alpha_-$. Then t^{γ_1} is the product of the edges of $(c_{1_+} \setminus \{t_k\}) \cup c_{2_-}$, but these are the edges of c_+. By the same reason t_{γ_2} is the product of the edges of c_-. Thus $t_c = t^{\gamma_1} - t^{\gamma_2}$. From the equality above we get that t_c is a linear combination of t_{c_1} and t_{c_2}. $\Box$

As an immediate consequence of Propositions 4.3 and 4.4 we get:

Corollary 4.5 *$P_{\mathcal{D}}$ is generated by the set of binomials corresponding to primitive cycles.*

We say that a cycle c of $\mathcal{D}$ is *oriented* if all the arrows of c are oriented in the same direction. If $\mathcal{D}$ does not have oriented cycles, we say that $\mathcal{D}$ is acyclic.

Proposition 4.6 ([10]) *$\mathcal{D}$ is acyclic if and only if there is a linear ordering of the vertices such that every edge of $\mathcal{D}$ has the form (x_i, x_j) with $i < j$.*

The ordering of the last proposition is called a *topological ordering*. The next result is not hard to prove.

Proposition 4.7 *If $\mathcal{D}$ has a topological ordering, then $P_{\mathcal{D}}$ is generated by homogeneous binomials with respect to the grading induced by degree$(t_k) = j - i$, where t_k maps to $x_i^{-1} x_j$ and (x_i, x_j) is an edge.*

Corollary 4.8 *If $\mathcal{D}$ is acyclic, then $P_{\mathcal{D}}$ is a complete intersection if and only if $P_{\mathcal{D}}$ is generated by $q - n + 1$ binomials corresponding to primitive cycles.*

Proof. Since $P_{\mathcal{D}}$ is a graded ideal, it suffices to recall that all the homogeneous minimal sets of generators of $P_{\mathcal{D}}$ have the same number of elements. $\Box$

In general the binomial complete intersection property of $P_{\mathcal{D}}$ depends on the orientation of G. However we have:

Corollary 4.9 *If G is a ring graph, then $P_{\mathcal{D}}$ is a complete intersection for any orientation of G. The converse holds if G is bipartite.*

Proof. By Corollary 4.5, $P_{\mathcal{D}}$ is generated by $q-n+1$ binomials. To show the converse assume that G is bipartite. Let (V_1, V_2) be a bipartition of G. Consider the oriented graph $\mathcal{D}$ obtained from G by orienting all the edges of G from V_1 to V_2, i.e., all the arrows of G have tail at V_1 and head at V_2. Since every vertex of $\mathcal{D}$ is either a source or a sink it follows that $P(G) = P_{\mathcal{D}}$. Hence $P(G)$ is a complete intersection and G is a ring graph by Corollary 3.4. $\Box$

An interesting problem that remains unsolved is to characterize the graphs with the property that $P_{\mathcal{D}}$ is a binomial complete intersection for all orientations of G. Apart from ring graphs it has been shown that complete graphs have this property [17, 16].

A special orientation

Let G be a connected graph. Here we show that there is always an orientation of G such that $P_{\mathcal{D}}$ is a complete intersection generated by the binomials that correspond to a cycle basis of a certain spanning tree of G.

Definition 4.10 Let S be a set of vertices of a graph G. The *neighbor set* of S, denoted by $N_G(S)$ or simply by $N(S)$ if G is understood, is the set of vertices of G that are adjacent with at least one vertex of S.

Lemma 4.11 *If H is a subgraph of a connected graph G and $N_G(V(H)) \subset V(H)$, then $V(G) = V(H)$.*

Proof. Fix a vertex $x \in V(H)$. Let $y \in V(G)$. Since G is connected, there is a path $\mathcal{P} = \{b_1 = x, b_2, \ldots, b_\ell = y\}$ from x to y. Using that $\{b_j, b_{j+1}\} \in E(G)$ for $1 \leq j < \ell - 1$ and that $b_1 \in V(H)$, by induction we get that $b_j \in V(H)$ for all j. Thus $y \in V(H)$. $\Box$

We begin by constructing a proper nested sequence $A_1, \ldots, A_m$ of subtrees of G labeled by $V(A_j) = \{y_1^j, \ldots, y_{r_j}^j\}$ such that A_m is a spanning tree of G and $V(A_i) \subsetneq V(A_{i+1})$ for $i < m$. First we construct the sequence $A_1, \ldots, A_m$ and then we show that it has the required properties. Let A_1 be a path of G maximal with respect to inclusion. Set $V(A_1) = \{y_1^1, y_2^1, \ldots, y_{r_1}^1\}$. We define

$$i_1 = \max\{u \in \mathbb{N} | N_G(y_1^1, \ldots, y_u^1) \subset V(A_1)\},$$

where $N_G(B)$ is the neighbor set of B. If $i_1 = r_1$, then $N_G(V(A_1)) \subset V(A_1)$ and by Lemma 4.11 we get $V(A_1) = V(G)$, in this case A_1 is the required spanning tree and we set $m = 1$. If $i_1 < r_1$, we define $a_1 = y^1_{i_1+1}$. By induction we define the sequence of subgraphs $A_1, \ldots, A_m$. Suppose that A_j has been defined, where $V(A_j) = \{y^j_1, \ldots, y^j_{r_j}\}$. We define

$$i_j = \max\{u \in \mathbb{N} | N_G(y^j_1, \ldots, y^j_u) \subset V(A_j)\}.$$

If $i_j = r_j$, then by Lemma 4.11 we get $V(A_j) = V(G)$, in this case we set $m = j$ and $A_1, \ldots, A_j$ is the desired sequence. If $i_j < r_j = |V(A_j)|$, we define $a_j = y^j_{i_j+1}$. Let $\mathcal{L}_j$ be a maximal path with respect to inclusion such that $V(\mathcal{L}_j) \cap V(A_j) = \{a_j\}$ and $V(\mathcal{L}_j) = \{z^j_1, z^j_2, \ldots, z^j_{s_j} = a_j\}$, the final vertex of $\mathcal{L}_j$ is a_j. We define A_{j+1} as follows: $V(A_{j+1}) = V(A_j) \cup V(\mathcal{L}_j) = \{y^{j+1}_1, \ldots, y^{j+1}_{r_j+s_j-1}\}$, where

$$y^{j+1}_i = \begin{cases} y^j_i & \text{if } i \leq i_j, \\ z^j_{i-i_j} & \text{if } i_j + 1 \leq i \leq i_j + s_j, \\ y^j_{i-s_j+1} & \text{if } i_j + s_j + 1 \leq i \leq r_j + s_j - 1, \end{cases} \quad (1)$$

$E(A_{j+1}) = E(A_j) \cup E(\mathcal{L}_j)$, and $r_{j+1} = r_j + s_j - 1$.

Lemma 4.12 *$i_{k+1} > i_k$ for $1 \leq k \leq m - 1$.*

Proof. By construction $y^{k+1}_i = y^k_i$ for $1 \leq i \leq i_k$ and $y^{k+1}_{i_k+1} = z^k_1$ (see Eq.(1)). By the maximality of $\mathcal{L}_j$ we have

$$N_G(y^{k+1}_1, y^{k+1}_2, \ldots, y^{k+1}_{i_k}, y^{k+1}_{i_k+1}) \subset V(A_{k+1}),$$

thus $i_{k+1} > i_k$ by definition of i_{k+1}. □

Suppose that the process finish at step m, i.e., $i_m = r_m$. We now prove that $A_1, \ldots, A_m$ has the required properties:

Lemma 4.13 *A_i is a tree for $1 \leq i \leq m$ and A_m is a spanning tree of G.*

Proof. By induction on i. For $i = 1$ the assertion is clear. Suppose that A_i is a tree. Recall that $\mathcal{L}_i$ is a tree and $V(\mathcal{L}_i) \cap V(A_i) = \{a_i\}$. On the other hand $V(A_{i+1}) = V(A_i) \cup V(\mathcal{L}_i)$ and $E(A_{i+1}) = E(A_i) \cup E(\mathcal{L}_i)$, then A_{i+1} is connected and does not has cycles. By Lemma 4.11 we get that $V(A_m) = V(G)$ and A_m is a spanning tree. □

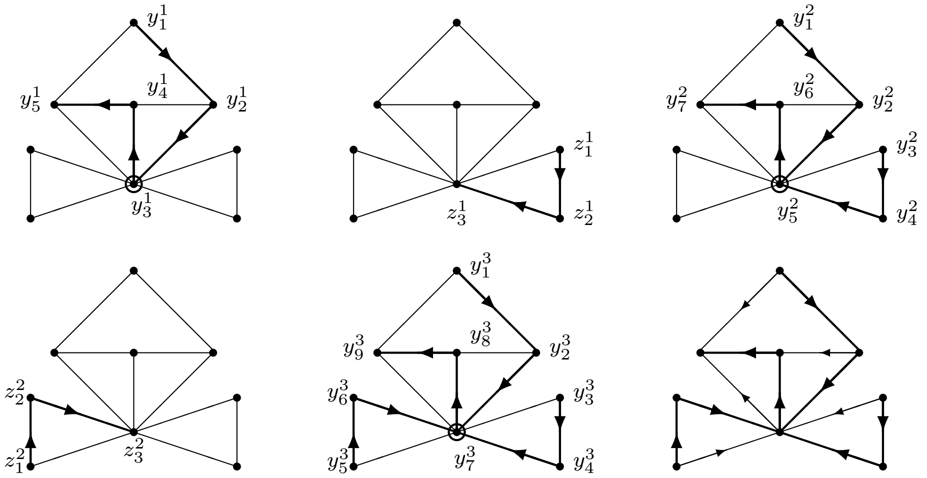
Orientation of the tree A_m and the graph G. Let $\tau = (A_m, \mathcal{O})$ be the oriented tree obtained from A_m using the following orientation:

$$(y^m_i, y^m_j) \in E(\tau) \text{ if and only if } \{y^m_i, y^m_j\} \in E(A_m) \text{ and } j > i.$$

By Lemma 4.13 we have $V(G) = V(A_m) = \{y_1^m, y_2^m, \ldots, y_{r_m}^m\}$ and we orient G to obtain the oriented graph $\mathcal{D} = (G, \mathcal{O})$ in the following way:

$$(y_i^m, y_j^m) \in \mathcal{D} \text{ if and only if } \{y_i^m, y_j^m\} \in E(G) \text{ and } j > i.$$

Example 4.14 *The construction of the spanning tree A_m and the orientation $\mathcal{O}$ of G is illustrated below.*



Notation For each $f_i \in E(\mathcal{D}) \setminus E(\tau)$ the unique cycle of the subgraph $\tau \cup \{f_i\}$ is denoted by $c(\tau, f_i)$.

Proposition 4.15 *For each $f_i \in E(\mathcal{D}) \setminus E(\tau)$ all the edges of $c(\tau, f_i) \setminus \{f_i\}$ are oriented in the same direction and f_i is oriented in the opposite direction.*

Proof. By induction on m, the number of subtrees $A_1, \ldots, A_m$. If $m = 1$ the result is easy to verify because A_1 is a spanning path of G. Assume $m > 1$. Consider the subgraphs

$$\overline{G} = G \setminus \{y_1^1, \ldots, y_{i_1}^1\}, \quad \overline{A}_i = A_i \setminus \{y_1^1, \ldots, y_{i_1}^1\}, \ i \geq 2.$$

We set $\overline{\mathcal{D}} = (\overline{G}, \overline{\mathcal{O}})$ and $\overline{\tau} = (\overline{A}_m, \overline{\mathcal{O}})$, where $\overline{\mathcal{O}}$ is the orientation induced from $\mathcal{O}$. Notice that $\overline{G}$ is connected because $\overline{A}_m$ is a spanning tree of $\overline{G}$. Using the equality

$$V(\overline{A}_2) = \{y_{i_1+1}^2, \ldots, y_{r_1+s_1-1}^2\} = \{z_1^1, \ldots, z_{s_1}^1, y_{i_1+2}^1, \ldots, y_{r_1}^1\}$$

and $z_{s_1} = y_{i_1+1}^1$ it is not hard to see that $\overline{A}_2$ is a maximal path of $\overline{G}$ and the result follows by induction. Indeed a fundamental cycle of τ is equal to $c(\tau, f_i) = c(\overline{\tau}, f_i)$

with $f_i \in E(\overline{\mathcal{D}}) \setminus E(\overline{\tau})$ or $c(\tau, f_i) = c(\tau', f_i)$ with $f_i \in E(H) \setminus E(\tau')$ where H is the induced subgraph on $\{y_1^1, \ldots, y_{i_1}^1\}$ and τ' is the spanning path of H given by $y_1^1, \ldots, y_{i_1}^1$. In the first case we apply induction to obtain that the edges of $c(\overline{\tau}, f_i)$ are properly oriented, in the second case it is easy to verify that $c(\tau', f_i)$ has the required orientation. $\Box$

Theorem 4.16 $P_{\mathcal{D}} = \big(\{t_{c(\tau,f_i)} | f_i \in E(\mathcal{D}) \setminus E(\tau)\}\big)$.

Proof. Set $E(\mathcal{D}) \setminus E(\tau) = \{f_1, \ldots, f_{q-n+1}\}$. Suppose without loss of generality that $t_1, \ldots, t_{q-n+1}$ are the variables associated to $f_1, \ldots, f_{q-n+1}$ respectively. By Proposition 4.15 $t_{c(\tau,f_i)} = t_i - t^{\beta_i}$, where t^{β_i} is a product of variables associated to edges in τ. Let I be the ideal generated by the set $\{t_{c(\tau,f_i)} | f_i \in E(\mathcal{D}) \setminus E(\tau)\}$ in $B = k[t_1, \ldots, t_q]$. Let $h = t^\alpha - t^\beta$ be a binomial in $P_{\mathcal{D}}$. Thus $\overline{t_i} = \overline{t^{\beta_i}}$ in B/I for $i = 1, \ldots, q-n+1$. Then $\overline{h} = \overline{t^\gamma} - \overline{t^\omega}$, where t^γ and t^ω are products of variables associated to edges of τ. As $I \subset P_{\mathcal{D}}$, then $t^\gamma - t^\omega \in P_{\mathcal{D}} = \ker(\varphi)$. But τ is a tree, thus $t^\gamma = t^\omega$, and $\overline{h} = \overline{0}$ in B/I. Since $P_{\mathcal{D}}$ is generated by binomials, $P_{\mathcal{D}} = I$. $\Box$

Corollary 4.17 *Assume that $\mathcal{D}$ is the oriented graph constructed above. Then $P_{\mathcal{D}}$ is a homogeneous ideal generated by $q - n + 1$ binomials corresponding to primitive cycles.*

Proof. By Theorem 4.16 it follows that $P_{\mathcal{D}}$ does not contains binomials of the form $1 - t^a$, i.e., $\mathcal{D}$ is acyclic. Thus we may apply Corollary 4.8. $\Box$

A tournament $\mathcal{D}$ is a complete graph $\mathcal{K}_n$ with a given orientation.

Proposition 4.18 ([10]) *If $\mathcal{D}$ is a tournament, then $\mathcal{D}$ has a spanning oriented path.*

Proposition 4.19 ([12]) *If $\mathcal{D}$ is an acyclic tournament, then $P_{\mathcal{D}}$ is a complete intersection minimally generated by a Gröbner basis.*

Proof. Let τ be a spanning oriented path of $\mathcal{D}$, i.e., $\tau = \{x_1, x_2, \ldots, x_n\}$ and (x_i, x_{i+1}) is an edge of $\mathcal{D}$ for all $i < n$. Since $\mathcal{D}$ is acyclic, using the proof of Theorem 4.16, it follows that

$$P_{\mathcal{D}} = (\{t_{c(\tau,f_i)} | f_i \in E(\mathcal{D}) \setminus E(\tau)\}),$$

where for each $f_i \in E(\mathcal{D}) \setminus E(\tau)$, the unique cycle of the subgraph $\tau \cup \{f_i\}$ is denoted by $c(\tau, f_i)$. $\Box$

Similarly we can prove the following generalization:

Proposition 4.20 *If $\mathcal{D}$ is an acyclic oriented graph with a spanning oriented path, then $P_{\mathcal{D}}$ is a complete intersection minimally generated by a Gröbner basis.*

Acknowledgments. We thank the referees for their careful reading of the paper and for the improvements that they suggested.

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