Blowup algebras of square-free monomial ideals and some links to combinatorial optimization problems

Isidoro Gitler, Enrique Reyes[1],
and
Rafael H. Villarreal[2]
Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14–740
07000 México City, D.F.
e-mail: vila@math.cinvestav.mx

Abstract

Let $I=(x^{v_1},\ldots,x^{v_q})$ be a square-free monomial ideal of a polynomial ring $K[x_1,\ldots,x_n]$ over an arbitrary field K and let A be the incidence matrix with column vectors $v_1,\ldots,v_q$. We will establish some connections between algebraic properties of certain graded algebras associated to I and combinatorial optimization properties of certain polyhedra and clutters associated to A and I respectively. Some applications to Rees algebras and combinatorial optimization are presented.

1 Introduction

Let $R=K[x_1,\ldots,x_n]$ be a polynomial ring over a field K and let I be an ideal of R of height $g\geq 2$ minimally generated by a finite set of square-free monomials $F=\{x^{v_1},\ldots,x^{v_q}\}$ of degree at least two. As usual we use x^a as an abbreviation for $x_1^{a_1}\cdots x_n^{a_n}$, where $a=(a_1,\ldots,a_n)\in\mathbb{N}^n$. A *clutter* with vertex set X is a family of subsets of X, called edges, none of which is included in another. We associate to the ideal I a *clutter* $\mathcal{C}$ by taking the set of indeterminates $X=\{x_1,\ldots,x_n\}$ as vertex set and $E=\{S_1,\ldots,S_q\}$ as edge set, where

$$S_k=\mathrm{supp}(x^{v_k})=\{x_i|\,\langle e_i,v_k\rangle=1\}.$$

Here $\langle\ ,\ \rangle$ denotes the standard inner product and e_i is the ith unit vector. For this reason I is called the *edge ideal* of $\mathcal{C}$. To stress the relationship between I and $\mathcal{C}$ we will use the notation $I=I(\mathcal{C})$. A basic example of clutter is a graph. Algebraic

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and combinatorial properties of edge ideals and graded algebras associated to graphs have been studied in [10, 20, 32, 33, 38]. The related notion of facet ideal has been studied by Faridi [16, 17] and Zheng [44].

The *blowup algebras* studied here are the *Rees algebra*

$$R[It] = R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

where t is a new variable, and the *associated graded ring*

$$\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots \oplus I^i/I^{i+1} \oplus \cdots \simeq R[It] \otimes_R (R/I),$$

with multiplication $(a + I^{i+1})(b + I^{j+1}) = ab + I^{i+j+1}$, $a \in I^i$, $b \in I^j$.

In the sequel A will denote the *incidence matrix* of order $n \times q$ whose column vectors are $v_1, \ldots, v_q$. In order to link the properties of these algebras with combinatorial optimization problems we consider the *set covering polyhedron*

$$Q(A) = \{x \in \mathbb{R}^n \,|\, x \geq 0;\, xA \geq \mathbf{1}\},$$

and the related system of linear inequalities $x \geq 0;\ xA \geq \mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$. Recall that this system is called *totally dual integral* (TDI) if the maximum in the LP-duality equation

$$\min\{\langle \alpha, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq \alpha\} \qquad (1)$$

has an integral optimum solution y for each integral vector α with finite maximum. If the system is totally dual integral it is seen that $Q(A)$ has only integral vertices, this follows from [30, Theorem 22.1, Corollary 22.1.a, pp. 310-311].

We are able to express algebraic properties of blowup algebras in terms of TDI systems and combinatorial properties of clutters, such as the integrality of $Q(A)$ and the König property. An important goal here is to establish bridges between commutative algebra and combinatorial optimization, which could be beneficial to both areas. Necessary and/or sufficient conditions for the normality of $R[It]$ and the reducedness of $\mathrm{gr}_I(R)$ are shown. Some of our results give some support to a conjecture of Conforti and Cornuéjols (Conjecture 4.17). Applications to Rees algebras theory and combinatorial optimization are presented.

Along the paper we introduce some of the algebraic and combinatorial notions that are most relevant. For unexplained terminology and notation we refer to [26, 29, 30] and [28, 35]. See [11] for detailed information about clutters.

2 Vertex covers of clutters

The set of non-negative real numbers will be denoted by $\mathbb{R}_+$. To avoid repetitions throughout this article we shall use the notation and assumptions introduced in Section 1. For convenience we shall always assume that each variable x_i occurs in at least one monomial of F.

Definition 2.1 A subset $C\subset X$ is a *minimal vertex cover* of the clutter $\mathcal{C}$ if: (i) every edge of $\mathcal{C}$ contains at least one vertex of C, and (ii) there is no proper subset of C with the first property. If C satisfies condition (i) only, then C is called a *vertex cover* of $\mathcal{C}$.

The first aim is to characterize this notion in terms of the integral vertices of set covering polyhedrons and the minimal primes of edge ideals.

Notation The *support* of $x^a=x_1^{a_1}\cdots x_n^{a_n}$ is $\mathrm{supp}(x^a)=\{x_i\,|\,a_i>0\}$.

Proposition 2.2 *The following are equivalent*:

(a) $\mathfrak{p}=(x_1,\ldots,x_r)$ *is a minimal prime of* $I=I(\mathcal{C})$.

(b) $C=\{x_1,\ldots,x_r\}$ *is a minimal vertex cover of* $\mathcal{C}$.

(c) $\alpha=e_1+\cdots+e_r$ *is a vertex of* $Q(A)$.

Proof. (a) $\Leftrightarrow$ (b): It follows readily by noticing that the minimal primes of the square-free monomial ideal I are face ideals, that is, they are generated by subsets of the set of variables, see [39, Proposition 5.1.3].

(b) $\Rightarrow$ (c): Fix $1\leq i\leq r$. To make notation simpler fix $i=1$. We may assume that there is an s_1 such that $x^{v_j}=x_1m_j$ for $j=1,\ldots,s_1$ and $x_1\notin\mathrm{supp}(x^{v_j})$ for $j>s_1$. Notice that $\mathrm{supp}(m_{k_1})\cap(C\setminus\{x_1\})=\emptyset$ for some $1\leq k_1\leq s_1$, otherwise $C\setminus\{x_1\}$ is a vertex cover of $\mathcal{C}$ strictly contained in C, a contradiction. Thus $\mathrm{supp}(m_{k_1})\cap C=\emptyset$ because I is square-free. Hence for each $1\leq i\leq r$ there is v_{k_i} in $\{v_1,\ldots,v_q\}$ such that $x^{v_{k_i}}=x_im_{k_i}$ and $\mathrm{supp}(m_{k_i})\subset\{x_{r+1},\ldots,x_n\}$. The vector α is clearly in $Q(A)$, and since $\{e_i\}_{i=r+1}^n\cup\{v_{k_1},\ldots,v_{k_r}\}$ is linearly independent, and

$$\langle\alpha,e_i\rangle=0\quad(i=r+1,\ldots,n);\quad\langle\alpha,v_{k_i}\rangle=1\quad(i=1,\ldots,r),$$

we get that the vector α is a basic feasible solution. Therefore by [1, Theorem 2.3] α is a vertex of $Q(A)$.

(c) $\Rightarrow$ (b): It is clear that C intersects all the edges of the clutter $\mathcal{C}$ because $\alpha\in Q(A)$. If $C'\subsetneq C$ is a vertex cover of $\mathcal{C}$, then the vector $\alpha'=\sum_{x_i\in C'}e_i$ satisfies $\alpha'A\geq\mathbf{1}$ and $\alpha'\geq 0$. Using that α is a basic feasible solution in the sense of [1] it is not hard to verify that α' is also a vertex of $Q(A)$. By the finite basis theorem [41, Theorem 4.1.3] we can write

$$Q(A)=\mathbb{R}_+^n+\mathrm{conv}(V),$$

where V is the vertex set of $Q(A)$. As $\alpha=\beta+\alpha'$, for some $0\neq\beta\in\mathbb{R}_+^n$, we get

$$Q(A)=\mathbb{R}_+^n+\mathrm{conv}(V\setminus\{\alpha\}).$$

Hence the vertices of $Q(A)$ are contained in $V\setminus\{\alpha\}$ (see [4, Theorem 7.2]), a contradiction. Thus C is a minimal vertex cover. $\Box$

Corollary 2.3 *A vector $\alpha \in \mathbb{R}^n$ is an integral vertex of $Q(A)$ if and only if α is equal to $e_{i_1} + \cdots + e_{i_s}$ for some minimal vertex cover $\{x_{i_1}, \ldots, x_{i_s}\}$ of $\mathcal{C}$.*

Proof. By Proposition 2.2 it suffices to observe that any integral vertex of $Q(A)$ has entries in $\{0, 1\}$ because A has entries in $\{0, 1\}$. See [34, Lemma 4.6]. $\Box$

A set of edges of the clutter $\mathcal{C}$ is *independent* if no two of them have a common vertex. We denote the smallest number of vertices in any minimal vertex cover of $\mathcal{C}$ by $\alpha_0(\mathcal{C})$ and the maximum number of independent edges of $\mathcal{C}$ by $\beta_1(\mathcal{C})$. These numbers are related to min-max problems because they satisfy:

$$
\begin{aligned}
\alpha_0(\mathcal{C}) \geq \min\{\langle \mathbf{1}, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} \\
= \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq \mathbf{1}\} \geq \beta_1(\mathcal{C}).
\end{aligned}
$$

Notice that $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ if and only if both sides of the equality have integral optimum solutions.

These two numbers can be interpreted in terms of I. By Proposition 2.2 the height of the ideal I, denoted by $\mathrm{ht}(I)$, is equal to the *covering number* $\alpha_0(\mathcal{C})$. On the other hand the *independence number* $\beta_1(\mathcal{C})$ is equal to $\mathrm{mgrade}(I)$, the *monomial grade* of the ideal:

$$\beta_1(\mathcal{C}) = \max\{r \,|\, \exists \text{ a regular sequence of monomials } x^{\alpha_1}, \ldots, x^{\alpha_r} \in I\}.$$

The equality $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ is equivalent to require $x_1 \cdots x_n t^g \in R[It]$, where g is the covering number $\alpha_0(\mathcal{C})$.

Definition 2.4 If $\alpha_0(\mathcal{C}) = \beta_1(\mathcal{C})$ we say that the clutter $\mathcal{C}$ (or the ideal I) has the *König property*.

3 Rees algebras and polyhedral geometry

Let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of exponent vectors of $x^{v_1}, \ldots, x^{v_q}$ and let

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the *ith* unit vector. The *Rees cone* of $\mathcal{A}$ is the rational polyhedral cone, denoted by $\mathbb{R}_+\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative coefficients. Note $\dim(\mathbb{R}_+\mathcal{A}') = n + 1$. Thus according to [41] there is a unique irreducible representation

$$\mathbb{R}_+\mathcal{A}' = H^+_{e_1} \cap \cdots \cap H^+_{e_{n+1}} \cap H^+_{a_1} \cap \cdots \cap H^+_{a_r} \tag{2}$$

such that $0 \neq a_i \in \mathbb{Q}^{n+1}$ and $\langle a_i, e_{n+1}\rangle = -1$ for all i. As usual H^+_a denotes the closed halfspace

$$H^+_a = \{\alpha \in \mathbb{R}^{n+1} \,|\, \langle \alpha, a\rangle \geq 0\}$$

and H_a is the hyperplane through the origin with normal vector a.

Theorem 3.1 *The function $\varphi\colon \mathbb{Q}^n \rightarrow \mathbb{Q}^{n+1}$ given by $\varphi(\alpha) = (\alpha, -1)$ induces a bijective map*

$$\varphi\colon V \longrightarrow \{a_1, \ldots, a_r\}$$

between the set of vertices V of $Q(A)$ and the set $\{a_1, \ldots, a_r\}$ of normal vectors that occur in the irreducible representation of $\mathbb{R}_+\mathcal{A}'$.

Proof. First we show the containment $\varphi(V) \subset \{a_1, \ldots, a_r\}$. Take α in V. By [1, Theorem 2.3] α is a basic feasible solution. Hence $\langle \alpha, v_i \rangle \geq 1$ for $i = 1, \ldots, q$, $\alpha \geq 0$, and there exist n linearly independent vectors $v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_s}$ in $\mathcal{A} \cup \{e_1, \ldots, e_n\}$ such that $\langle \alpha, v_{i_h} \rangle = 1$ and $\langle \alpha, e_{j_m} \rangle = 0$ for all h, m. It follows that the set

$$F = H_{(\alpha,-1)} \cap \mathbb{R}_+\mathcal{A}'$$

has dimension n and $\mathbb{R}_+\mathcal{A}' \subset H^+_{(\alpha,-1)}$. Therefore F is a facet of $\mathbb{R}_+\mathcal{A}'$. Using [41, Theorem 3.2.1] we obtain that $F = \mathbb{R}_+\mathcal{A}' \cap H_{a_p}$ for some $1 \leq p \leq r$, and consequently $H_{(\alpha,-1)} = H_{a_p}$. Since the first n entries of a_p are non-negative and $\langle a_p, e_{n+1} \rangle = -1$ it follows that $\varphi(\alpha) = (\alpha, -1) = a_p$, as desired.

To show the reverse containment write $a_p = (\alpha, -1)$, with $1 \leq p \leq r$ and $\alpha \in \mathbb{R}^n$. We will prove that α is a vertex of $Q(A)$. Since Eq. (2) is an irreducible representation one has that the set

$$F = H_{(\alpha,-1)} \cap \mathbb{R}_+\mathcal{A}'$$

is a facet of the Rees cone $\mathbb{R}_+\mathcal{A}'$, see [41, Theorem 3.2.1]. Hence there is a linearly independent set

$$\{(v_{i_1}, 1), \ldots, (v_{i_k}, 1), e_{j_1}, \ldots, e_{j_s}\} \subset \mathcal{A}' \quad (k + s = n)$$

such that

$$\langle (v_{i_h}, 1), (\alpha, -1) \rangle = 0 \;\;\Rightarrow\;\; \langle v_{i_h}, \alpha \rangle = 1 \quad (h = 1, \ldots, k), \tag{3}$$

$$\langle e_{j_m}, (\alpha, -1) \rangle = 0 \;\;\Rightarrow\;\; \langle e_{j_m}, \alpha \rangle = 0 \quad (m = 1, \ldots, s). \tag{4}$$

It is not hard to see that $v_{i_1}, \ldots, v_{i_k}, e_{j_1}, \ldots, e_{j_s}$ are linearly independent vectors in $\mathbb{R}^n$. Indeed if

$$\lambda_1 v_{i_1} + \cdots + \lambda_k v_{i_k} + \mu_1 e_{j_1} + \cdots + \mu_s e_{j_s} = 0 \quad (\lambda_h, \mu_m \in \mathbb{R}),$$

then taking inner product with α and using Eqs. (3) and (4) we get

$$\lambda_1 + \cdots + \lambda_k = 0 \;\Rightarrow\; \lambda_1(v_{i_1}, 1) + \cdots + \lambda_k(v_{i_k}, 1) + \mu_1 e_{j_1} + \cdots + \mu_s e_{j_s} = 0.$$

Therefore $\lambda_h = 0$ and $\mu_m = 0$ for all h, m, as desired. From $\mathbb{R}_+\mathcal{A}' \subset H^+_{a_p}$ and $H^+_{a_p} = H^+_{(\alpha,-1)}$ we get $\alpha \geq 0$ and $\langle \alpha, v_i \rangle \geq 1$ for all i. Altogether we obtain that α is a basic feasible solution, that is, α is a vertex of $Q(A)$. $\square$

Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ be the minimal primes of the edge ideal $I=I(\mathcal{C})$ and let

$$C_k=\{x_i|\, x_i\in\mathfrak{p}_k\}\quad\ (k=1,\ldots,s)$$

be the corresponding minimal vertex covers of the clutter $\mathcal{C}$. By Proposition 2.2 and Theorem 3.1 in the sequel we may assume that

$$a_k=(\textstyle\sum_{x_i\in C_k}e_i,-1)\quad\ (k=1,\ldots,s).$$

Notation Let d_k be the unique positive integer such that d_ka_k has relatively prime integral entries. We set $\ell_k=d_ka_k$ for $k=1,\ldots,r$. If the first n rational entries of a_k are written in lowest terms, then d_k is the least common multiple of the denominators. For $1\leq k\leq r$, we have $d_k=-\langle\ell_k,e_{n+1}\rangle$.

Definition 3.2 The set covering polyhedron $Q(A)$ is *integral* if all its vertices have integral entries.

Corollary 3.3 *The irreducible representation of the Rees cone has the form*

$$\mathbb{R}_+\mathcal{A}'=H_{e_1}^+\cap\cdots\cap H_{e_{n+1}}^+\cap H_{\ell_1}^+\cap\cdots\cap H_{\ell_r}^+,\tag{5}$$

$d_k=1$ *if and only if* $1\leq k\leq s$, *and* $Q(A)$ *is integral if and only if* $r=s$.

Proof. It follows from Theorem 3.1 and Corollary 2.3. $\Box$

Notation In the sequel we shall always assume that $\ell_1,\ldots,\ell_r$ are the integral vectors of Eq. (5).

Recall that the *Simis cone* of $\mathcal{A}$ is the rational polyhedral cone

$$\mathrm{Cn}(\mathcal{A})=H_{e_1}^+\cap\cdots\cap H_{e_{n+1}}^+\cap H_{\ell_1}^+\cap\cdots\cap H_{\ell_s}^+,$$

and the *symbolic Rees algebra* of I is the K algebra:

$$R_s(I)=R+I^{(1)}t+I^{(2)}t^2+\cdots+I^{(i)}t^i+\cdots\subset R[t],$$

where $I^{(i)}=\mathfrak{p}_1^i\cap\cdots\cap\mathfrak{p}_s^i$ is the *ith* symbolic power of I. Symbolic Rees algebras have a combinatorial interpretation [22]. Notice the following description:

$$I^{(b)}=(\{x^a|\,\langle(a,b),\ell_i\rangle\geq 0\text{ for }i=1,\ldots,s\}).$$

A first use of the Simis cone is the following expression for the symbolic Rees algebra. In particular $R_s(I)$ is a finitely generated K-algebra [27] by Gordan's Lemma [6].

Theorem 3.4 ([15]) *If* $S=\mathbb{Z}^{n+1}\cap\mathrm{Cn}(\mathcal{A})$ *and* $K[S]=K[\{x^at^b|\,(a,b)\in S\}]$ *is its semigroup ring, then* $R_s(I)=K[S]$.

Let $\mathbb{N}\mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integer coefficients. The Rees algebra of I can be written as

$$\begin{aligned} R[It] &= K[\{x^a t^b \mid (a,b) \in \mathbb{N}\mathcal{A}'\}] && (6)\\ &= R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t]. && (7) \end{aligned}$$

According to [39, Theorem 7.2.28] and [36, p. 168] the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\begin{aligned} \overline{R[It]} &= K[\{x^a t^b \mid (a,b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'\}] && (8)\\ &= R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots, && (9) \end{aligned}$$

where $\overline{I^i} = (\{x^a \in R \mid \exists\, p \geq 1; (x^a)^p \in I^{pi}\})$ is the integral closure of I^i. Hence, by Eqs. (6) to (9), we get that $R[It]$ is a normal domain if and only if any of the following two equivalent conditions hold:

(a) $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$.

(b) $I^i = \overline{I^i}$ for all $i \geq 1$.

If the second condition holds we say that I is a *normal* ideal.

Proposition 3.5 *For $1 \leq i \leq r$ we write $a_i = (a_i', -1)$. Let B be the matrix with column vectors $a_1', \ldots, a_r'$ and let $Q = \mathbb{Q}_+^n + \mathrm{conv}(v_1, \ldots, v_q)$. Then*

(a) $\overline{I^i} = (\{x^a \in R \mid a \in iQ \cap \mathbb{Z}^n\})$.

(b) $Q = Q(B) = \{x \mid x \geq 0;\, xB \geq \mathbf{1}\}$. *In particular $Q(B)$ is integral.*

Proof. Part (a) follows from Eq. (9) and part (b) follows from Eq. (2). □

In the sequel $J_k^{(d_k)}$ will denote the ideal of $R[It]$ given by

$$J_k^{(d_k)} = (\{x^a t^b \in R[It] \mid \langle (a,b), \ell_k \rangle \geq d_k\}) \quad (k = 1, \ldots, r)$$

and J_k will denote the ideal of $R[It]$ given by

$$J_k = (\{x^a t^b \in R[It] \mid \langle (a,b), \ell_k \rangle > 0\}) \quad (k = 1, \ldots, r),$$

where $d_k = -\langle \ell_k, e_{n+1} \rangle$. If $d_k = 1$, we have $J_k^{(1)} = J_k$. In general $J_k^{(d_k)}$ might not be equal to the $d_k th$ symbolic power of J_k. The localization of $R[It]$ at $R \setminus \mathfrak{p}_k$ is denoted by $R[It]_{\mathfrak{p}_k}$.

Proposition 3.6 *$J_1, \ldots, J_r$ are height one prime ideals containing $IR[It]$ and J_k is equal to $\mathfrak{p}_k R[It]_{\mathfrak{p}_k} \cap R[It]$ for $k = 1, \ldots, s$. If $Q(A)$ is integral, then*

$$\mathrm{rad}(IR[It]) = J_1 \cap J_2 \cap \cdots \cap J_s.$$

Proof. $IR[It]$ is clearly contained in J_k for all k by construction. To show that J_k is a prime ideal of height one it suffices to notice that the right hand side of the isomorphism:

$$R[It]/J_k \simeq K[\{x^a t^b \in R[It] | \langle (a,b), \ell_k\rangle = 0\}]$$

is an n-dimensional integral domain, because $F_k = \mathbb{R}_+\mathcal{A}' \cap H_{\ell_k}$ is a facet of the Rees cone for all k. Set $P_k = \mathfrak{p}_k R[It]_{\mathfrak{p}_k} \cap R[It]$ for $1 \leq k \leq s$. This ideal is a minimal prime of $IR[It]$ (see [25]) and admits the following description

$$\begin{aligned} P_k &= \mathfrak{p}_k R_{\mathfrak{p}_k}[\mathfrak{p}_k R_{\mathfrak{p}_k} t] \cap R[It] \\ &= \mathfrak{p}_k + (\mathfrak{p}_k^2 \cap I)t + (\mathfrak{p}_k^3 \cap I^2)t^2 + \cdots + (\mathfrak{p}_k^{i+1} \cap I^i)t^i + \cdots \end{aligned}$$

Notice that $x^a \in \mathfrak{p}_k^{b+1}$ if and only if $\langle a, \sum_{x_i \in C_k} e_i\rangle \geq b+1$. Hence $J_k = P_k$.

Assume that $Q(A)$ is integral, i.e., $r = s$. Take $x^\alpha t^b \in J_k$ for all k. Using Eq. (5) it is not hard to see that $(\alpha, b+1) \in \mathbb{R}_+\mathcal{A}'$, that is $x^\alpha t^{b+1}$ is in $\overline{R[It]}$ and $x^\alpha t^{b+1} \in \overline{I^{b+1}} t^{b+1}$. It follows that $x^\alpha t^b$ is a monomial in the radical of $IR[It]$. This proves the asserted equality. $\Box$

For use below recall that the *analytic spread* of I is given by

$$\ell(I) = \dim R[It]/\mathfrak{m}R[It]; \quad \mathfrak{m} = (x_1, \ldots, x_n).$$

Corollary 3.7 *If $Q(A)$ is integral, then $\ell(I) < n$.*

Proof. Since $Q(A)$ is integral, we have $r = s$. If $\ell(I) = n$, then the height of $\mathfrak{m}R[It]$ is equal to 1. Hence there is a height one prime ideal P of $R[It]$ such that $IR[It] \subset \mathfrak{m}R[It] \subset P$. By Proposition 3.6 the ideal P has the form $\mathfrak{p}_k R[It]_{\mathfrak{p}_k} \cap R[It]$, this readily yields a contradiction. $\Box$

Theorem 3.8 ([9, 13]) $\inf\{\mathrm{depth}(R/I^i) | i \geq 1\} \leq \dim(R) - \ell(I)$. *If* $\mathrm{gr}_I(R)$ *is Cohen-Macaulay, then the equality holds.*

By a result of Brodmann [3], the depth of R/I^k is constant for k sufficiently large. Broadmann improved this inequality by showing that the constant value is bounded by $\dim(R) - \ell(I)$. For a study of the initial and limit behavior of the numerical function $f(k) = \mathrm{depth}\, R/I^k$ see [21].

Lemma 3.9 *Let $x_1 \in C_k$ for some $1 \leq k \leq s$. If $x^{v_i} = x_1 x^{v_i'}$ for $1 \leq i \leq p$ and $x_1 \notin$* $\mathrm{supp}(x^{v_i})$ *for $i > p$, then there is $x^{v_j'}$ such that* $\mathrm{supp}(x^{v_j'}) \cap C_k = \emptyset$.

Proof. If $\mathrm{supp}(x^{v_j'}) \cap C_k \neq \emptyset$ for all j, then $C_k \setminus \{x_1\}$ is a vertex cover of $\mathcal{C}$, a contradiction because C_k is a minimal vertex cover. $\Box$

Proposition 3.10 *If $P \in \{J_1, \ldots, J_s\}$, then $R[It] \cap IR[It]_P = P$.*

Proof. Set $P = J_k$. We may assume that $x_1, \ldots, x_m$ (resp. $x^{v_1}t, \ldots, x^{v_p}t$) is the set of all x_i (resp. $x^{v_i}t$) such that $x_i \in P$ (resp. $x^{v_i}t \in P$). Notice that $\mathfrak{p}_k$ is equal to $(x_1, \ldots, x_m)$ and set $C = \{x_1, \ldots, x_m\}$. In general the left hand side is contained in P. To show the reverse inclusion we first prove the equality

$$P = (x_1, \ldots, x_m, x^{v_1}t, \ldots, x^{v_p}t)R[It]. \tag{10}$$

Let $x^a t^b \in P$. Thus $x^a t^b = x_1^{\mu_1} \cdots x_n^{\mu_n} (x^{v_1}t)^{\lambda_1} \cdots (x^{v_q}t)^{\lambda_q}$ and $\langle (a, b), \ell_k \rangle > 0$. Hence $\langle e_i, \ell_k \rangle > 0$ for some i or $\langle (v_j, 1), \ell_k \rangle > 0$ for some j. Therefore $x^a t^b$ belongs to the right hand side of Eq. (10), as required.

Case (I): Consider x_ℓ with $1 \leq \ell \leq m$. By Lemma 3.9 there is j such that $x^{v_j} = x_\ell x^\alpha$ and $\mathrm{supp}(x^\alpha) \cap C = \emptyset$. Thus since x^α is not in P (because of the second condition) we obtain $x_\ell \in R[It] \cap IR[It]_P$.

Case (II): Consider $x^{v_\ell}t$ with $1 \leq \ell \leq p$. Since

$$\langle (v_\ell, 1), e_1 + \cdots + e_m - e_{n+1} \rangle \geq 1,$$

the monomial x^{v_ℓ} contains at least two variables in C. Thus we may assume that x_1, x_2 are in the support of x^{v_ℓ}. Again by Lemma 3.9 there are j, j_1 such that $x^{v_j} = x_1 x^\alpha$, $x^{v_{j_1}} = x_2 x^\gamma$, and the support of x^α and x^γ disjoint from C. Hence the monomial $x^{v_\ell} x^{\alpha+\gamma} t$ belongs to $I^2 t$ and $x^{\alpha+\gamma}$ is not in P. Writing

$$x^{v_\ell}t = (x^{v_\ell} x^{\alpha+\gamma} t)/x^{\alpha+\gamma},$$

we get $x^{v_\ell}t \in R[It] \cap IR[It]_P$. $\Box$

Lemma 3.11 $\mathrm{rad}(J_k^{(d_k)}) = J_k$ *for* $1 \leq k \leq r$.

Proof. By construction one has $\mathrm{rad}(J_k^{(d_k)}) \subset J_k$. The reverse inclusion follows by noticing that if $x^a t^b \in J_k$, then $(x^a t^b)^{d_k} \in J_k^{(d_k)}$. $\Box$

Proposition 3.12 *If* $R[It]$ *is normal, then* $IR[It] = J_1^{(d_1)} \cap \cdots \cap J_r^{(d_r)}$.

Proof. "$\subset$": Let $x^a t^b \in IR[It]$. Since $x^a \in I^{b+1}$, we obtain $(a, b+1) \in \mathbb{N}\mathcal{A}'$. In particular we get $(a, b+1) \in \mathbb{R}_+\mathcal{A}'$. Therefore

$$0 \leq \langle (a, b+1), \ell_k \rangle = \langle (a, b), \ell_k \rangle - d_k$$

and consequently $x^a t^b \in J^{(d_k)}$ for $1 \leq k \leq r$.

"$\supset$": Let $x^a t^b \in J^{(d_k)}$ for all k. Since $(a, b+1) \in \mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{n+1}$, using that $R[It]$ is normal yields $(a, b+1) \in \mathbb{N}\mathcal{A}'$. It follows that $x^a t^b \in I^{b+1}t^b \subset IR[It]$. $\Box$

A similar formula is shown in [8]. The normality of $R[It]$ can be described in terms of primary decompositions of $IR[It]$, see [24, Proposition 2.1.3].

The following two nice formulas, pointed out to us by Vasconcelos, describe the difference between the symbolic Rees algebra of I and the normalization of its Rees algebra. If $\mathfrak{q}_k = J_k \cap R$ for $k = 1, \dots, r$, then

$$R_s(I) = \bigcap_{k=1}^{s} R[It]_{\mathfrak{q}_k} \cap R[t]; \qquad \overline{R[It]} = \bigcap_{k=1}^{r} \overline{R[It]}_{\mathfrak{q}_k} \cap R[t].$$

These representations are linked to the so called Rees valuations of the ideal I, see [37, Chapter 8] for further details.

Proposition 3.13 *The following conditions are equivalent*

(a) $Q(A)$ *is integral.*

(b) $\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+$, *i.e.,* $r = s$.

(c) $R_s(I) = \overline{R[It]}$.

(d) *The minimal primes of* $IR[It]$ *are of the form* $\mathfrak{p}_k R[It]_{\mathfrak{p}_k} \cap R[It]$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): These implications follow from Theorems 3.1 and 3.4. The other implications follow readily using Proposition 3.6. $\Box$

Definition 3.14 Let $x^{u_k} = \prod_{x_i \in C_k} x_i$ for $1 \leq k \leq s$. The ideal of *vertex covers* of $\mathcal{C}$ is the ideal

$$I_c(\mathcal{C}) = (x^{u_1}, \dots, x^{u_s}) \subset R.$$

The clutter of minimal vertex covers, denoted by $\mathcal{D}$ or $b(\mathcal{C})$, is the *blocker* of $\mathcal{C}$.

In the literature $I_c(\mathcal{C})$ is also called the *Alexander dual* of I because if Δ is the Stanley-Reisner complex of I, then $I_c(\mathcal{C})$ is the Stanley-Reisner ideal of the Alexander dual of Δ. The survey article [20] explains the role of Alexander duality to prove combinatorial and algebraic theorems.

Example 3.15 Let $I = (x_1x_2x_5, x_1x_3x_4, x_2x_3x_6, x_4x_5x_6)$. The clutter of I is denoted by $\mathcal{Q}_6$. Using *Normaliz* [7] and Proposition 3.13 we obtain:

$$R[It] \subsetneq R_s(I) = \overline{R[It]} = R[It][x_1 \cdots x_6 t^2] \;\text{ and }\; R[I_c(\mathcal{Q}_6)t] = \overline{R[I_c(\mathcal{Q}_6)t]}.$$

Proposition 3.16 ([18]) *If* $\overline{R[It]} = R_s(I)$ *and* $J = I_c(\mathcal{C})$, *then* $\overline{R[Jt]} = R_s(J)$.

Corollary 3.17 [11, Theorem 1.17] *If* $Q(A)$ *is integral and* A' *is the incidence matrix of the clutter of minimal vertex covers of* $\mathcal{C}$, *then* $Q(A')$ *is integral.*

Proof. It follows at once from Propositions 3.13 and 3.16. $\Box$

Definition 3.18 Let $X' = \{x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s}\}$ be a subset of X. A *minor* of I is a proper ideal I' of $R' = K[X \setminus X']$ obtained from I by making $x_{i_k} = 0$ and $x_{j_\ell} = 1$ for all k, ℓ. The ideal I is considered itself a minor. A *minor* of $\mathcal{C}$ is a clutter $\mathcal{C}'$ that corresponds to a minor $(0) \subsetneq I' \subsetneq R'$.

Notice that $\mathcal{C}'$ is obtained from I' by considering the unique set of square-free monomials of R' that minimally generate I'.

Proposition 3.19 *If $\overline{I^i} = I^{(i)}$ for some $i \geq 2$ and $J = I'$ is a minor of I, then $\overline{J^i} = J^{(i)}$.*

Proof. Assume that J is the minor obtained from I by making $x_1 = 0$. Take $x^a \in J^{(i)}$. Then $x^a \in I^{(i)} = \overline{I^i}$ because $J \subset I$. Thus $x^a \in \overline{I^i}$. Since $x_1 \notin \operatorname{supp}(x^a)$ it follows that $x^a \in \overline{J^i}$. This proves $J^{(i)} \subset \overline{J^i}$. The other inclusion is clear because $J^{(i)}$ is integrally closed.

Assume that J is the minor obtained from I by making $x_1 = 1$. Take $x^a \in J^{(i)}$. Notice that $x_1^i x^a \in I^{(i)} = \overline{I^i}$. Indeed if $x_1 \in \mathfrak{p}_k$, then $x_1^i \in \mathfrak{p}_k^i$, and if $x_1 \notin \mathfrak{p}_k$, then $J \subset \mathfrak{p}_k$ and $x^a \in \mathfrak{p}_k^i$. Since $x_1 \notin \operatorname{supp}(x^a)$ it follows that $x^a \in \overline{J^i}$. $\Box$

Corollary 3.20 *If $R_s(I) = \overline{R[It]}$, then $R_s(I') = \overline{R'[I't]}$ for any minor I' of I.*

Proposition 3.21 *Let $\mathcal{D}$ be the clutter of minimal vertex covers of $\mathcal{C}$. If $\overline{R[It]}$ is equal to $R_s(I)$ and $|A \cap B| \leq 2$ for $A \in \mathcal{C}$ and $B \in \mathcal{D}$, then $R[It]$ is normal.*

Proof. Let $x^a t^b = x_1^{a_1} \cdots x_n^{a_n} t^b \in \overline{R[It]}$ be a minimal generator, that is (a, b) cannot be written as a sum of two non-zero integral vectors in the Rees cone $\mathbb{R}_+\mathcal{A}'$. We may assume $a_i \geq 1$ for $1 \leq i \leq m$, $a_i = 0$ for $i > m$, and $b \geq 1$.

Case (I): $\langle (a, b), \ell_i \rangle > 0$ for all i. The vector $\gamma = (a, b) - e_1$ satisfies $\langle \gamma, \ell_i \rangle \geq 0$ for all i, that is $\gamma \in \mathbb{R}_+\mathcal{A}'$. Thus since $(a, b) = e_1 + \gamma$ we derive a contradiction.

Case (II): $\langle (a, b), \ell_i \rangle = 0$ for some i. We may assume

$$\{\ell_i | \langle (a, b), \ell_i \rangle = 0\} = \{\ell_1, \ldots, \ell_p\}.$$

Subcase (II.a): $e_i \in H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for some $1 \leq i \leq m$. It is not hard to verify that the vector $\gamma = (a, b) - e_i$ satisfies $\langle \gamma, \ell_k \rangle \geq 0$ for all $1 \leq k \leq s$. Thus $\gamma \in \mathbb{R}_+\mathcal{A}'$, a contradiction because $(a, b) = e_i + \gamma$.

Subcase (II.b): $e_i \notin H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for all $1 \leq i \leq m$. Since the vector (a, b) belongs to $\mathbb{R}_+\mathcal{A}'$, it follows (see the proof of Theorem 4.1) that we can write

$$(a, b) = \lambda_1(v_1, 1) + \cdots + \lambda_q(v_q, 1) \quad (\lambda_i \geq 0). \tag{11}$$

By the choice of $x^a t^b$ we may assume $0 < \lambda_1 < 1$. Set $\gamma = (a, b) - (v_1, 1)$ and notice that by Eq. (11) this vector has non-negative entries. We claim that γ is in the Rees cone. Since by hypothesis one has $0 \leq \langle (v_1, 1), \ell_j \rangle \leq 1$ for all j we readily obtain

$$\langle \gamma, \ell_k \rangle = \left\{ \begin{array}{lcll} \langle (a,b), \ell_k \rangle - \langle (v_1, 1), \ell_k \rangle & = & 0 & \text{if } 1 \leq k \leq p, \\ \langle (a,b), \ell_k \rangle - \langle (v_1, 1), \ell_k \rangle & \geq & 0 & \text{otherwise.} \end{array} \right.$$

Thus $\gamma \in \mathbb{R}_+\mathcal{A}'$ and $(a,b) = (v_1, 1) + \gamma$. As a result $\gamma = 0$ and $x^a t^b \in R[It]$, as desired. $\Box$

4 König property of clutters and normality

Let us introduce a little bit more notation and definitions. Recall that the *Ehrhart ring* of the lattice polytope $P = \text{conv}(v_1, \ldots, v_q)$ is the subring

$$A(P) = K[\{x^a t^i |\, a \in \mathbb{Z}^n \cap iP; i \in \mathbb{N}\}] \subset R[t],$$

and the homogeneous *monomial subring* generated by $Ft = \{x^{v_1}t, \ldots, x^{v_q}t\}$ over the field K is the subring $K[Ft] \subset R[t]$.

Theorem 4.1 *If $\overline{R[It]} = R_s(I)$ and $K[Ft] = A(P)$, then $R[It]$ is normal.*

Proof. Let $x^a t^b = x_1^{a_1} \cdots x_n^{a_n} t^b \in \overline{R[It]}$ be a minimal generator, that is $x^a t^b$ cannot be written as a product of two non-constant monomials of $\overline{R[It]}$. We may assume $a_i \geq 1$ for $1 \leq i \leq m$, $a_i = 0$ for $i > m$, and $b \geq 1$.

Case (I): $\langle (a,b), \ell_i \rangle > 0$ for all i. The vector $\gamma = (a,b) - e_1$ satisfies $\langle \gamma, \ell_i \rangle \geq 0$ for all i, that is $\gamma \in \mathbb{R}_+\mathcal{A}'$. Thus since x_1 and $x_1^{a_1 - 1} x_2^{a_2} \cdots x_n^{a_n} t^b$ are in $\overline{R[It]}$ we get a contradiction. In conclusion this case cannot occur.

Case (II): $\langle (a,b), \ell_i \rangle = 0$ for some i. We may assume

$$\{\ell_i |\, \langle (a,b), \ell_i \rangle = 0\} = \{\ell_1, \ldots, \ell_p\}.$$

Subcase (II.a): $e_i \in H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for some $1 \leq i \leq m$. For simplicity of notation assume $i = 1$. The vector $\gamma = (a,b) - e_1$ satisfies

$$\langle \gamma, \ell_k \rangle = \left\{ \begin{array}{lcll} \langle (a,b), \ell_k \rangle - \langle e_1, \ell_k \rangle & = & 0 & \text{if } 1 \leq k \leq p, \\ \langle (a,b), \ell_k \rangle - \langle e_1, \ell_k \rangle & \geq & 0 & \text{otherwise.} \end{array} \right.$$

Thus $\gamma \in \mathbb{R}_+\mathcal{A}'$. Proceeding as in Case (I) we derive a contradiction.

Subcase (II.b): $e_i \notin H_{\ell_1} \cap \cdots \cap H_{\ell_p}$ for all $1 \leq i \leq m$. The vector (a,b) belongs to the polyhedral cone

$$C = H_{\ell_1} \cap \cdots \cap H_{\ell_p} \cap \mathbb{R}_+\mathcal{A}'.$$

Hence we can write

$$
\begin{aligned}
(a,b) &= \lambda_1(v_1,1)+\cdots+\lambda_q(v_q,1)+\mu_1e_1+\cdots+\mu_ne_n \qquad (\lambda_i;\mu_j\geq 0),\\
\langle (a,b),\ell_k\rangle &= \lambda_1\langle (v_1,1),\ell_k\rangle+\cdots+\lambda_q\langle (v_q,1),\ell_k\rangle+\\
&\quad \mu_1\langle e_1,\ell_k\rangle+\cdots+\mu_n\langle e_n,\ell_k\rangle=0
\end{aligned}
$$

for $k=1,\ldots,p$. From the first equality we get $\mu_i=0$ for $i>m$ because $a_i=0$ for $i>m$. If $\mu_i>0$ for some $1\leq i\leq m$, then $\langle e_i,\ell_k\rangle=0$ for $1\leq k\leq p$, a contradiction. Hence $\mu_i=0$ for all i. Therefore $a/b\in P$ and $a\in\mathbb{Z}^n\cap bP$. This proves $x^at^b\in A(P)=K[Ft]\subset R[It]$, as desired. $\Box$

Proposition 4.2 ([40]) *If $\ell_k=d_ka_k$ has the form*

$$\ell_k=e_{i_1}+\cdots+e_{i_k}-d_ke_{n+1} \qquad (1\leq i_1<\cdots<i_k\leq n)$$

for $k=1,\ldots,r$, then $A(P)[x_1,\ldots,x_n]=\overline{R[It]}$.

Proof. The Ehrhart ring is contained in $\overline{R[It]}$. Thus the equality follows using the proof of Theorem 4.1. $\Box$

Proposition 4.3 *If $x^{v_1},\ldots,x^{v_q}$ have degree $d\geq 2$ and $\overline{I^b}=I^{(b)}$ for all b, then $\overline{I^b}$ is generated by monomials of degree bd for $b\geq 1$.*

Proof. The monomial ideal $\overline{I^b}$ has a unique minimal set of generators consisting of monomials. Take x^a in this minimal set. Notice that $(a,b)\in\mathbb{R}_+\mathcal{A}'$. Thus we may proceed as in the proof of Theorem 4.1 to obtain that (a,b) is in the cone generated by $\{(v_1,1),\ldots,(v_q,1)\}$. This yields $\deg(x^a)=bd$. $\Box$

Proposition 4.4 *If $x^{v_1},\ldots,x^{v_q}$ have degree $d\geq 2$, then $I^i=I^{(i)}$ for all $i\geq 1$ if and only if $Q(A)$ is integral and $K[Ft]=A(P)$.*

Proof. $\Rightarrow$) By Proposition 3.13 the polyhedron $Q(A)$ is integral. Since $I^{(i)}$ is integrally closed [39, Corollary 7.3.15], we get that $R[It]$ is normal. Therefore applying [14, Theorem 3.15] we obtain $K[Ft]=A(P)$, here the hypothesis on the degrees of x^{v_i} is essential.

$\Leftarrow$) By Proposition 3.13 $\overline{I^i}=I^{(i)}$ for all i, thus applying Theorem 4.1 gives $R[It]$ normal and we get the required equality. Here the hypothesis on the degrees of x^{v_i} is not needed. $\Box$

Definition 4.5 The clutter $\mathcal{C}$ satisfies the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle\alpha,x\rangle\,|\,x\geq 0; xA\geq\mathbf{1}\}=\max\{\langle y,\mathbf{1}\rangle\,|\,y\geq 0; Ay\leq\alpha\} \tag{12}$$

have integral optimum solutions x and y for each non-negative integral vector α.

It follows from [30, pp. 311-312] that $\mathcal{C}$ has the MFMC property if and only if the maximum in Eq. (12) has an optimal integral solution y for each non-negative integral vector α. Thus the system $x \geq 0;\ xA \geq \mathbf{1}$ is TDI if and only if $\mathcal{C}$ has the max-flow min-cut property.

A ring is called *reduced* if 0 is its only nilpotent element. For convenience let us state some known characterizations of the reducedness of the associated graded ring.

Theorem 4.6 ([15, 19, 25]) *The following conditions are equivalent*

(i) $\mathrm{gr}_I(R)$ *is reduced.*

(ii) $R[It]$ *is normal and* $Q(A)$ *is an integral polyhedron.*

(iii) I *is normally torsion free, that is,* $I^i = I^{(i)}$ *for all* $i \geq 1$.

(iv) $x \geq 0;\ xA \geq \mathbf{1}$ *is a* TDI *system.*

(iv) $\mathcal{C}$ *has the max-flow min-cut property.*

Corollary 4.7 [12, Theorem 1.3] *Let* $\mathcal{D}$ *be the clutter of minimal vertex covers of* $\mathcal{C}$. *If* $Q(A)$ *is integral and* $|A \cap B| \leq 2$ *for* $A \in \mathcal{C}$ *and* $B \in \mathcal{D}$, *then* $x \geq 0;\ xA \geq \mathbf{1}$ *is a* TDI *system.*

Proof. By Proposition 3.21 the Rees algebra $R[It]$ is normal. To complete the proof apply Theorem 4.6. $\Box$

Lemma 4.8 *If* I *is a monomial ideal of* R, *then the nilradical of the associated graded ring of* I *is given by*

$$\mathrm{nil}(\mathrm{gr}_I(R)) = (\{\overline{x^\alpha} \in I^i/I^{i+1} |\, x^{s\alpha} \in I^{si+1}; i \geq 0; s \geq 1\}).$$

Proof. The nilradical of $\mathrm{gr}_I(R)$ is graded with respect to the fine grading, and thus it is generated by homogeneous elements. $\Box$

Definition 4.9 The matrix A is *balanced* if A has no square submatrix of odd order with exactly two 1's in each row and column. A is *totally unimodular* if each $i \times i$ minor of A is 0 or ± 1 for all $i \geq 1$.

Proposition 4.10 *If* A *is balanced, then* $\mathrm{gr}_I(R)$ *is reduced.*

Proof. Let $\overline{x^\alpha} \in I^i/I^{i+1}$ be in $\mathrm{nil}(\mathrm{gr}_I(R))$, that is $x^{s\alpha} \in I^{is+1}$ for some $0 \neq s \in \mathbb{N}$. By Lemma 4.8 we need only show $\overline{x^\alpha} = \overline{0}$. It follows rapidly that the maximum in Eq. (1) is greater or equal than $i + 1/s$. By [30, Theorem 21.8, p. 305] the maximum in Eq. (1) has an integral optimum solution $y = (y_1, \ldots, y_q)$. Thus $y_1 + \cdots + y_q \geq i + 1$. Since y satisfies $y \geq 0$ and $Ay \leq \alpha$ we obtain $x^\alpha \in I^{i+1}$. This proves $\overline{x^\alpha} = \overline{0}$, as required. $\Box$

Proposition 4.11 *If A is balanced and $J = I_c(\mathcal{C})$, then $R[Jt] = R_s(J)$.*

Proof. Let $\mathcal{D}$ be the blocker of $\mathcal{C}$. By [31, Corollary 83.1a(v), p. 1441], we get that $\mathcal{D}$ satisfy the max-flow min-cut property. Thus the equality follows at once from Theorem 4.6. $\square$

Proposition 4.12 *If $\mathrm{gr}_I(R)$ is reduced (resp. $R[It]$ is normal) and I' is a minor of I, then $\mathrm{gr}_{I'}(R')$ is reduced (resp. $R'[I't]$ is normal).*

Proof. Notice that we need only show the result when I' is a minor obtained from I by making $x_1 = 0$ or $x_1 = 1$. Using Lemma 4.8 both cases are quite easy to prove. $\square$

Definition 4.13 A clutter $\mathcal{C}$ satisfies the *packing property* (PP) if all its minors satisfy the König property, that is, $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for any minor $\mathcal{C}'$ of $\mathcal{C}$.

Corollary 4.14 *If the ring $\mathrm{gr}_I(R)$ is reduced, then $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for any minor $\mathcal{C}'$ of $\mathcal{C}$.*

Proof. Let $\mathcal{C}'$ be any minor of $\mathcal{C}$ and let I' be its clutter ideal. We denote the incidence matrix of $\mathcal{C}'$ by A'. By Proposition 4.12 the associated graded ring $\mathrm{gr}_{I'}(R')$ is reduced. Hence by Theorem 4.6 the clutter $\mathcal{C}'$ has the max-flow min-cut property. In particular the LP-duality equation

$$\min\{\langle \mathbf{1}, x\rangle \,|\, x \geq 0; xA' \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; A'y \leq \mathbf{1}\}$$

has optimum integral solutions x, y. To complete the proof notice that the left hand side of this equality is $\alpha_0(\mathcal{C}')$ and the right hand side is $\beta_1(\mathcal{C}')$. $\square$

Remark 4.15 If I is the facet ideal of a simplicial tree, then $\mathrm{gr}_I(R)$ is reduced. This follows from [16, p. 174] using the proof of [32, Corollary 3.2, p. 399]. In particular $\mathcal{C}$ has the König property, this was shown in [17, Theorem 5.3].

Corollary 4.16 *If $\mathcal{C}$ has the max-flow min-cut property, then $\mathcal{C}$ has the packing property.*

Proof. It follows at once from Theorem 4.6 and Corollary 4.14. $\square$

Conforti and Cornuéjols conjecture that the converse is also true:

Conjecture 4.17 [11, Conjecture 1.6] If the clutter $\mathcal{C}$ has the packing property, then $\mathcal{C}$ has the max-flow min-cut property.

Next we state the converse of Corollary 4.14 as an algebraic version of this interesting conjecture which to our best knowledge is still open:

Conjecture 4.18 If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then the ring $\mathrm{gr}_I(R)$ is reduced.

It is known [11, Theorem 1.8] that clutters with the packing property have integral set covering polyhedrons. As a consequence, using Theorem 4.6, this conjecture reduces to the following:

Conjecture 4.19 If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then $R[It]$ is normal.

In this paper we will give some support for this conjecture using an algebraic approach.

Proposition 4.20 *Let J_i be the ideal obtained from I by making $x_i = 1$. If $Q(A)$ is an integral polyhedron, then the ideal I is normal if and only if J_i is normal for all i and* $\mathrm{depth}(R/I^k) \geq 1$ *for all $k \geq 1$.*

Proof. $\Rightarrow$) The normality of an ideal is closed under minors [15, Proposition 4.3], hence J_i is normal for all i. Using Theorem 3.8 and Corollary 3.7 we get that $\mathrm{depth}(R/I^i) \geq 1$ for all i.

$\Leftarrow$) It follows readily by adapting the arguments given in the proof of the normality criterion [15, Theorem 4.4]. $\Box$

By Proposition 4.20 we obtain that Conjecture 4.18 also reduces to:

Conjecture 4.21 If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for any minor $\mathcal{C}'$ of $\mathcal{C}$, then

$$\mathrm{depth}(R/I^i) \geq 1 \ \text{ for all } i \geq 1.$$

Notation For an integral matrix $B \neq (0)$, the greatest common divisor of all the nonzero $r \times r$ subdeterminants of B will be denoted by $\Delta_r(B)$.

Corollary 4.22 *If $x^{v_1}, \ldots, x^{v_q}$ are monomials of degree $d \geq 2$ such that* $\mathrm{gr}_I(R)$ *is reduced and the matrix*

$$B = \begin{pmatrix} v_1 & \cdots & v_q \\ 1 & \cdots & 1 \end{pmatrix}$$

has rank r, then $\Delta_r(B) = 1$ and B diagonalizes over $\mathbb{Z}$ to an identity matrix.

Proof. By Proposition 4.4 we obtain $A(P) = K[Ft]$. Hence a direct application of [14, Theorem 3.9] gives $\Delta_r(B) = 1$. $\Box$

This result suggest the following weaker conjecture of Villarreal:

Conjecture 4.23 *If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$ and $x^{v_1}, \ldots, x^{v_q}$ have degree $d \geq 2$, then $\Delta_r(B) = 1$, where $r = \mathrm{rank}(B)$.*

Let G be a matroid on X of rank d and let $\mathcal{B}$ be the collection of bases of G. The set of all square-free monomials $x_{i_1}\cdots x_{i_d}\in R$ such that $\{x_{i_1},\ldots,x_{i_d}\}\in\mathcal{B}$ will be denoted by F_G and the subsemigroup (of the multiplicative semigroup of monomials of R) generated by F_G will be denoted by $\mathbb{M}_G$. The *basis monomial ring* of G is the monomial subring $K[F_G]=K[\mathbb{M}_G]$. The ideal $I(\mathcal{B})=(F_G)$ is called the *basis monomial ideal* of G. An open problem in the area is whether the toric ideal of $K[F_G]$ is generated by quadrics, see [43, Conjecture 12]. This has been shown for graphic matroids [2].

The next result implies the normality of the basis monomial ring of G.

Proposition 4.24 ([42]) *If x^a is a monomial of degree ℓd for some $\ell\in\mathbb{N}$ such that $(x^a)^p\in\mathbb{M}_G$ for some $0\neq p\in\mathbb{N}$, then $x^a\in\mathbb{M}_G$.*

Proposition 4.25 *If $I=I(\mathcal{B})$ and $\mathcal{B}$ satisfies the packing property, then $\mathrm{gr}_I(R)$ is reduced.*

Proof. First we show the equality $A(P)=K[F_Gt]$. It suffices to prove the inclusion $A(P)\subset K[F_Gt]$. Take $x^at^b\in A(P)$, i.e., $x^a\in\mathbb{Z}^n\cap bP$. Hence x^a has degree bd and $(x^a)^p\in\mathbb{M}_G$ for some positive integer p. By Proposition 4.24 we get $x^a\in\mathbb{M}_G$. It is seen that x^at^b is in $K[F_Gt]$. Since $Q(A)$ is integral [11], using Theorem 4.1 we get that $R[It]$ is normal. Thus both conditions yield that $\mathrm{gr}_I(R)$ is reduced according to Theorem 4.6. $\Box$

This proof can be simplified using that the basis monomial ideal of a matroid is normal [40].

Corollary 4.26 *Let $X_1,\ldots,X_d$ be a family of disjoint sets of variables and let M be the transversal matroid whose collection of basis is*

$$\mathcal{C}=\{\{y_1,\ldots,y_d\}|\,y_i\in X_i\,\forall\,i\}.$$

If $I=I(\mathcal{C})$, then $\mathrm{gr}_I(R)$ is reduced.

The combinatorial equivalencies in the next result are well known [11, 12]. Our contribution here is to link the reducedness of the associated graded ring with the integrality of $Q(A)$.

Proposition 4.27 *If $\mathcal{C}$ is a simple graph, then the following are equivalent*:

(a) $\mathrm{gr}_I(R)$ *is reduced.*

(b) $\mathcal{C}$ *is bipartite.*

(c) $Q(A)$ *is integral.*

(d) $\mathcal{C}$ *has the packing property.*

Proof. By [32, Theorem 5.9] (a) and (b) are equivalent. Applying Theorem 4.6 and Proposition 3.21 we obtain that (a) is equivalent to (c). By Corollary 4.14 condition (a) implies (d). Finally using [11, Theorem 1.8] we obtain that (d) implies (c). □

Corollary 4.28 ([18]) *If $\mathcal{C}$ is a bipartite graph and $J = I_c(\mathcal{C})$, then $\mathrm{gr}_J(R)$ is reduced.*

Proof. The matrix A is totally unimodular [30, p. 273], hence $Q(A)$ is integral. By Proposition 3.16 we get $\overline{R[Jt]} = R_s(J)$. On the other hand $R[Jt]$ is normal by Proposition 3.21. Thus by Theorem 4.6 the ring $\mathrm{gr}_J(R)$ is reduced. □

Proposition 4.29 *If $\mathcal{C}$ has the packing property and $I = I(\mathcal{C})$, then $I^2 = \overline{I^2}$.*

Proof. By induction on n. Assume $\overline{I^2} \neq I^2$ and consider $M = \overline{I^2}/I^2$. If $\mathfrak{p} \neq \mathfrak{m} = (x_1, \ldots, x_n)$ is a prime ideal of R, then by induction $M_{\mathfrak{p}} = (0)$. Thus $\mathfrak{m}$ is the only associated prime of M and there is an embedding $R/\mathfrak{m} \hookrightarrow M$, $\overline{1} \mapsto \overline{x^a}$, where $\overline{x^a} \in \overline{I^2} \setminus I^2$ and $x_i x^a \in I^2$ for all i. Notice that by induction all the entries a_i of a are positive. We consider two cases. Assume $a_i \geq 2$ for some i, say $i = 1$. Given a monomial x^α, the monomial obtained from x^α by making $x_1 = 1$ is denoted by $x^{\alpha'}$. Then making $x_1 = 1$ and using that $x_1 x^a \in I^2$ gives $x^{a'} = x^{v_1'} x^{v_2'} x^\delta$, hence $x^a = x_1^{a_1} x^{a'} = x^{v_1} x^{v_2} x^\gamma \in I^2$, a contradiction. On the other hand if $a_i = 1$ for all i, then $x^a = x_1 \cdots x_n \in I^g \subset I^2$, where $g = \mathrm{ht}(I)$, a contradiction. Therefore $I^2 = \overline{I^2}$. □

Recall that I is said to be *unmixed* if all the minimal vertex covers of $\mathcal{C}$ have the same cardinality.

Lemma 4.30 *If I is an unmixed ideal and $\mathcal{C}$ satisfies the König property, then $x^{\mathbf{1}} = x_1 x_2 \cdots x_n$ belongs to the subring $K[x^{v_1}, \ldots, x^{v_q}]$.*

Proof. We may assume $x^{\mathbf{1}} = x^{v_1} \cdots x^{v_g} x^\delta$, where g is the height of I. If $\delta \neq 0$, pick $x_n \in \mathrm{supp}(x^\delta)$. Since the variable x_n occurs in some monomial of I, there is a minimal prime $\mathfrak{p}$ containing x_n. Thus using that $x^{v_1}, \ldots, x^{v_g}$ have disjoint supports we conclude that $\mathfrak{p}$ contains at least $g + 1$ variables, a contradiction. □

Proposition 4.31 *Let $I_i = I \cap K[X \setminus \{x_i\}]$. If I is an unmixed ideal such that the following conditions hold*

(a$_1$) *$Q(A)$ is integral,*

(a$_2$) *I_i is normal for $i = 1, \ldots, n$, and*

(a$_3$) *$\mathcal{C}$ has the König property,*

then $R[It]$ is normal.

Proof. Take $x^at^b = x_1^{a_1}\cdots x_n^{a_n}t^b \in \overline{R[It]}$ a minimal generator. By the second condition we may assume $a_i \geq 1$ for all i. Set $g = \mathrm{ht}(I)$. Notice that $x_1\cdots x_nt^g$ is in $\overline{R[It]}$ because $Q(A)$ is integral, this follows from Corollary 3.3 and Eq. (8). We claim that $b \leq g$. If $b > g$, consider the decomposition

$$x^at^b = (x_1\cdots x_nt^g)(x_1^{a_1-1}\cdots x_n^{a_n-1}t^{b-g}).$$

To derive a contradiction consider the irreducible representation of the Rees cone $\mathbb{R}_+\mathcal{A}'$ given in Eq. (5). Observe that

$$\textstyle\sum_{x_i\in C_k} a_i \geq b \qquad (k = 1,\ldots,s)$$

because $(a,b) \in \mathbb{R}_+\mathcal{A}'$. Now since I is unmixed we get

$$\textstyle\sum_{x_i\in C_k}(a_i - 1) \geq b - g \qquad (k = 1,\ldots,s),$$

and consequently $x_1^{a_1-1}\cdots x_n^{a_n-1}t^{b-g} \in \overline{R[It]}$, a contradiction to the choice of x^at^b. Thus $b \leq g$. Using the third condition we get $x_1\cdots x_n \in I^g \subset I^b$, which readily implies $x^at^b \in R[It]$. $\Box$

According to Corollary 5.9 condition (a$_3$) is redundant when I is generated by monomials of the same degree.

Proposition 4.32 *Let $Y \subset X$ and let $I_Y = I \cap K[Y]$. If I_Y has the König property for all Y and $\overline{R[It]}$ is generated as a K-algebra by monomials of the form x^at^b, with x^a square-free, then $R[It]$ is normal.*

Proof. Take x^at^b a generator of $\overline{R[It]}$, with x^a square-free. By induction we may assume $x^at^b = x_1\cdots x_nt^b$. Hence, since $(1,\ldots,1,b)$ is in $\mathbb{R}_+\mathcal{A}'$, we get that $|C_k| \geq b$ for $k = 1,\ldots,s$. In particular $g = \mathrm{ht}(I) \geq b$. As I has the König property, we get $x_1\cdots x_n \in I^g$ and consequently $x^at^b \in R[It]$. $\Box$

Proposition 4.33 *Let $I_i = I \cap K[X \setminus \{x_i\}]$. If I_i is normal for $i = 1,\ldots,n$ and*

$$C = H_{\ell_1} \cap H_{\ell_2} \cap \cdots \cap H_{\ell_r} \cap \mathbb{R}_+^{n+1} \neq (0), \tag{13}$$

then $R[It]$ is normal.

Proof. Let $x^at^b = x_1^{a_1}\cdots x_n^{a_n}t^b \in \overline{R[It]}$ be a minimal generator, that is (a,b), cannot be written as a sum of two non-zero integral vectors in $\mathbb{R}_+\mathcal{A}'$. It suffices to prove that $0 \leq b \leq 1$ because this readily implies that x^a is either a variable or a monomial in F. Assume $b \geq 2$. Since I_i is normal we may assume that $a_i \geq 1$ for all i. As each variable occurs in at least one monomial of F, using that C is contained in $\mathbb{R}_+\mathcal{A}'$ together with Eq. (13), it follows that there is $(v_k,1)$ such that $\langle (v_k,1),\ell_i\rangle = 0$ for $i = 1,\ldots,r$. Therefore

$$\langle (a - v_k, b-1),\ell_i\rangle \geq 0 \qquad (i = 1,\ldots,r).$$

Thus $(a,b) - (v_k,1) \in \mathbb{R}_+\mathcal{A}'$, a contradiction to the choice of x^at^b. $\Box$

5 Some applications to Rees algebras and clutters

Throughout this section we assume $\deg(x^{v_i}) = d \geq 2$ for all i. By assigning $\deg(x_i) = 1$ and $\deg(t) = -(d-1)$, the Rees algebra $R[It]$ becomes a standard graded K-algebra, i.e., it is generated by elements of degree 1. The a-invariant of $R[It]$, with respect to this grading, is denoted by $a(R[It])$. If $R[It]$ is a normal domain, then according to a formula of Danilov-Stanley [6, Theorem 6.3.5] its *canonical module* is the ideal of $R[It]$ given by

$$\omega_{R[It]} = (\{x_1^{a_1}\cdots x_n^{a_n}t^{a_{n+1}} \,|\, a = (a_i) \in (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}} \cap \mathbb{Z}^{n+1}\}),$$

where $(\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$ is the topological interior of the Rees cone.

Theorem 5.1 *If* $\mathrm{gr}_I(R)$ *is reduced, then*

$$a(R[It]) \geq -\left[n - (d-1)(\alpha_0(\mathcal{C}) - 1)\right],$$

with equality if I is unmixed.

Proof. It is well known (see [6]) that the a-invariant can be expressed as

$$a(R[It]) = -\min\{\, i \,|\, (\omega_{R[It]})_i \neq 0\}.$$

Set $\alpha_0 = \alpha_0(\mathcal{C})$. Using Eq. (5) it is seen that the vector $(1,\ldots,1,\alpha_0 - 1)$ is in the interior of the Rees cone. Thus the inequality follows by computing the degree of $x_1\cdots x_n t^{\alpha_0 - 1}$.

Assume that I is unmixed. Take an arbitrary monomial $x^a t^b = x_1^{a_1}\cdots x_n^{a_n}t^b$ in the ideal $\omega_{R[It]}$, that is, $(a,b) \in (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$. By Proposition 3.13 the vector (a,b) has positive entries and satisfies

$$-b + \textstyle\sum_{x_i \in C_k} a_i \geq 1 \qquad (k = 1,\ldots,s). \tag{14}$$

If $\alpha_0 \geq b+1$, we obtain the inequality

$$\deg(x^a t^b) = a_1 + \cdots + a_n - b(d-1) \geq n - (d-1)(\alpha_0 - 1). \tag{15}$$

Now assume $\alpha_0 \leq b$. Using the normality of $R[It]$ and Eqs. (5) and (14) it follows that the monomial

$$m = x_1^{a_1-1}\cdots x_n^{a_n-1}t^{b-\alpha_0+1}$$

belongs to $R[It]$. Since $x^a t^b = m x_1\cdots x_n t^{\alpha_0-1}$, the inequality (15) also holds in this case. Altogether we conclude the desired equality. □

Corollary 5.2 ([18]) *If I is unmixed with $\alpha_0(\mathcal{C}) = 2$ and $\mathrm{gr}_I(R)$ is reduced, then $R[It]$ is a Gorenstein ring and*

$$a(R[It]) = -(n - d + 1).$$

Proof. From the proof of Theorem 5.1 it follows that $x_1 \cdots x_n t$ generates the canonical module. $\Box$

Notice that if $\alpha_0(\mathcal{C}) \geq 3$, then $R[It]$ is not Gorenstein because the monomials $x_1 \cdots x_n t^{\alpha_0 - 1}$ and $x_1 \cdots x_n t$ are distinct minimal generators of $\omega_{R[It]}$. This holds in a more general setting (see Proposition 5.5 below).

Corollary 5.3 *Let $J = I_c(\mathcal{C})$ be the ideal of vertex covers of $\mathcal{C}$. If $\mathcal{C}$ is a bipartite graph and $I = I(\mathcal{C})$ is unmixed, then $R[Jt]$ is a Gorenstein ring and*

$$a(R[Jt]) = -(n - \alpha_0(\mathcal{C}) + 1).$$

Proof. Notice that $R[Jt]$ has the grading induced by assigning $\deg(x_i) = 1$ and $\deg(t) = 1 - \alpha_0(\mathcal{C})$. Thus the formula follows from Corollary 5.2 once we recall that $\mathrm{gr}_J(R)$ is a reduced ring according to Corollary 4.28. $\Box$

Lemma 5.4 ([5], [35, p. 142]) *If S is a regular local ring and J is an ideal of S generated by a regular sequence $h_1, ..., h_g$, then $S[Jt]$ is determinantal:*

$$S[Jt] \simeq S[z_1, \ldots, z_g]/I_2 \begin{pmatrix} z_1 & \cdots & z_g \\ h_1 & \cdots & h_g \end{pmatrix}$$

and its canonical module is $\omega_S(1, t)^{g-2}$.

Proposition 5.5 *If I has height $g \geq 2$ and $S = R[It]$ is Gorenstein, then $g = 2$.*

Proof. Since $I_\mathfrak{p}$ is a complete intersection for all associated prime ideals $\mathfrak{p}$ of I and S is Gorenstein one has $\omega_S \simeq \omega_R(1, t)^{g-2}$ [23]. Then

$$S \simeq \omega_S \simeq \omega_R(1, t)^{g-2} = R \oplus Rt \oplus \cdots \oplus Rt^{g-2} \oplus It^{g-1} \oplus \cdots \tag{16}$$

Take a minimal prime $\mathfrak{p}$ of I of height g. Then $S_\mathfrak{p} = R_\mathfrak{p}[I_\mathfrak{p} t]$ is the Rees algebra of the ideal $I_\mathfrak{p}$, which is generated by a regular sequence. Thus localizing the extremes of Eq. (16) at $\mathfrak{p}$ and using Lemma 5.4 we obtain

$$S_\mathfrak{p} = R_\mathfrak{p}[I_\mathfrak{p} t] \simeq \omega_{R_\mathfrak{p}}(1, t)^{g-2} \simeq \omega_{S_\mathfrak{p}}.$$

Note that it is important to know a priori that the canonical module of $S_\mathfrak{p}$ is $\omega_{R_\mathfrak{p}}(1, t)^{g-2}$. Hence $S_\mathfrak{p}$ is Gorenstein. To finish the proof note that the only Gorenstein determinantal rings that occur in Lemma 5.4 are those with $g = 2$. Here the hypothesis on the degrees of x^{v_i} is not needed. $\Box$

Lemma 5.6 *If $\overline{R[It]} = R_s(I)$, then there is a minimal vertex cover C_k of $\mathcal{C}$ such that $|\mathrm{supp}(x^{v_i}) \cap C_k| = 1$ for $i = 1, \ldots, q$.*

Proof. We claim that $J_k = \mathfrak{p}_k R[It]$ for some $1 \leq k \leq s$. If not, using Eq. (10), we can pick $x^{v_k}t \in J_k$ for $k = 1, \ldots, s$. Then by Proposition 3.6 the product of these monomials is in the radical of $IR[It]$. Therefore

$$[(x^{v_1}t)\cdots(x^{v_s}t)]^p \in IR[It]$$

for some $0 \neq p \in \mathbb{N}$. Thus $(x^{v_1}\cdots x^{v_s})^p \in I^{sp+1}$. By degree considerations, using that $\deg(x^{v_i}) = d$ for all i, one readily derives a contradiction. This proves the claim. Hence $\langle (v_i, 1), \ell_k \rangle = 0$ for all i and $v_1, \ldots, v_q$ lie on the hyperplane

$$\textstyle\sum_{x_i \in C_k} x_i = 1.$$

Therefore $|\mathrm{supp}(x^{v_i}) \cap C_k| = 1$ for all i, as desired. $\Box$

Proposition 5.7 *If $\overline{R[It]} = R_s(I)$ and I is unmixed, then*

$$H_{\ell_1} \cap H_{\ell_2} \cap \cdots \cap H_{\ell_r} \cap \mathbb{R}_+^{n+1} \neq (0).$$

Proof. Let $J = I_c(\mathcal{C})$ be the Alexander dual of I. Using Proposition 3.16 one has $\overline{R[Jt]} = R_s(J)$. Thus by Lemma 5.6 there is v_k such that $|\mathrm{supp}(x^{v_k}) \cap C_i| = 1$ for $i = 1, \ldots, r$. This means that $(v_k, 1)$ is in the intersection of $H_{\ell_1}, \ldots, H_{\ell_r}$. $\Box$

Proposition 5.8 *If $\overline{R[It]} = R_s(I)$, then there are $C_1, \ldots, C_d$ mutually disjoint minimal vertex covers of $\mathcal{C}$ such that $\cup_{i=1}^q \mathrm{supp}(x^{v_i}) = \cup_{i=1}^d C_i$ and*

$$|\mathrm{supp}(x^{v_i}) \cap C_k| = 1 \qquad \forall\ i, k.$$

Proof. By induction on d. By Lemma 5.6 there is a minimal vertex cover C_1 of $\mathcal{C}$ such that $|\mathrm{supp}(x^{v_i}) \cap C_1| = 1$ for all i. Consider the ideal I' obtained from I by making $x_i = 1$ for all $x_i \in C_1$. Then I' is an ideal generated by monomials of degree $d-1$ and $\overline{R[I't]} = R_s(I')$ by Corollary 3.20. Thus we can apply induction to get the required assertion. $\Box$

Corollary 5.9 *If I is unmixed and $\overline{R[It]} = R_s(I)$, then both $\mathcal{C}$ and the clutter $\mathcal{D}$ of minimal vertex covers of $\mathcal{C}$ have the König property.*

Proof. That $\mathcal{D}$ has the König property follows from Proposition 5.8, because $\alpha_0(\mathcal{D}) = d$ and $C_1, \ldots, C_d$ are independent edges of $\mathcal{D}$. Now $I_c(\mathcal{C})$ is unmixed, is generated by monomials of degree $\alpha_0(\mathcal{C})$, and according to Proposition 3.16 one has $\overline{R[I_c(\mathcal{C})]} = R_s(I_c(\mathcal{C}))$. Thus, using again Proposition 5.8, we conclude that $\mathcal{C}$ has the König property. $\Box$

Combining Corollary 5.9 with Proposition 4.31 we obtain:

Theorem 5.10 *Let $I_i = I \cap K[X \setminus \{x_i\}]$. If I is unmixed and $Q(A)$ is integral, then $\mathrm{gr}_I(R)$ is reduced if and only if I_i is normal for $i = 1, \ldots, n$.*

References

[1] D. Bertsimas and J. N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, Massachusetts, 1997.

[2] J. Blasiak, The toric ideal of a graphic matroid is generated by quadrics, arXiv:math.CO/0511223.

[3] M. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. **86** (1979), 35–39.

[4] A. Brøndsted, *Introduction to Convex Polytopes*, Graduate Texts in Mathematics **90**, Springer-Verlag, 1983.

[5] W. Bruns, The canonical module of a determinantal ring, in Commutative Algebra: Durham 1981 (R. Sharp, Ed.), London Math. Soc. Lecture Notes Series **72**, Cambridge University Press, Cambridge, 1982, 109–120.

[6] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, Revised Edition, 1997.

[7] W. Bruns and R. Koch, *Normaliz*, a program to compute normalizations of affine semigroups. Available via anonymous ftp from: ftp.mathematik.uni-osnabrueck.de/pub/osm/kommalg/software/.

[8] W. Bruns and G. Restuccia, Canonical modules of Rees algebras, J. Pure Appl. Algebra **201** (2005), 189–203.

[9] L. Burch, Codimension and analytic spread, Proc. Camb. Phil. Soc. **72** (1972), 369-373.

[10] J. Chen, S. Morey and A. Sung, The stable set of associated primes of the ideal of a graph, Rocky Mountain J. Math. **32** (2002), 71–89.

[11] G. Cornuéjols, *Combinatorial optimization*: *Packing and covering*, CBMS-NSF Regional Conference Series in Applied Mathematics **74**, SIAM (2001).

[12] G. Cornuéjols, B. Guenin and F. Margot, The packing property, Math. Program. **89** (2000), no. 1, Ser. A, 113–126.

[13] D. Eisenbud and C. Huneke, Cohen-Macaulay Rees algebras and their specializations, J. Algebra **81** (1983), 202–224.

[14] C. Escobar, J. Martínez-Bernal and R. H. Villarreal, Relative volumes and minors in monomial subrings, Linear Algebra Appl. **374** (2003), 275–290.

[15] C. Escobar, R. H. Villarreal and Y. Yoshino, Torsion freeness and normality of blowup rings of monomial ideals, *Commutative Algebra*, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 69-84.

[16] S. Faridi, The facet ideal of a simplicial complex, Manuscripta Math. **109** (2002), 159-174.

[17] S. Faridi, Cohen-Macaulay properties of square-free monomial ideals, J. Combin. Theory Ser. A **109(2)** (2005), 299-329.

[18] I. Gitler, E. Reyes and R. H. Villarreal, Blowup algebras of ideals of vertex covers of bipartite graphs, Contemp. Math. **376** (2005), 273–279.

[19] I. Gitler, C. Valencia and R. H. Villarreal, A note on Rees algebras and the MFMC property, arXiv:math.AC/0511307.

[20] J. Herzog, Alexander duality in commutative algebra and combinatorics, Algebra Colloq. **11** (2004), no. 1, 21–30.

[21] J. Herzog and T. Hibi, The depth of powers of an ideal, J, Algebra **291** (2005), 534–550.

[22] J. Herzog, T. Hibi and N. V. Trung, Symbolic powers of monomial ideals and vertex cover algebras, arXiv:math.AC/0512423.

[23] J. Herzog, A. Simis and W. V. Vasconcelos, On the canonical module of the Rees algebra and the associated graded ring of an ideal, J. Algebra **105** (1987), 285–302.

[24] J. Herzog, A. Simis and W. V. Vasconcelos, Arithmetic of normal Rees algebras, J. Algebra **143** (1991), 269–294.

[25] C. Huneke, A. Simis and W. V. Vasconcelos, Reduced normal cones are domains, Contemp. Math. **88** (1989), 95–101.

[26] B. Korte and J. Vygen, *Combinatorial Optimization Theory and Algorithms*, Springer-Verlag, 2000.

[27] G. Lyubeznik, On the arithmetical rank of monomial ideals, J. Algebra **112** (1988), 86–89.

[28] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986.

[29] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[30] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1986.

[31] A. Schrijver,*Combinatorial Optimization*, Algorithms and Combinatorics **24**, Springer-Verlag, Berlin, 2003.

[32] A. Simis, W. V. Vasconcelos and R. H. Villarreal, On the ideal theory of graphs, J. Algebra, **167** (1994), 389–416.

[33] C. Valencia, *Studies on graded rings associated to graphs*, PhD thesis, Cinvestav–IPN, 2003.

[34] C. Valencia and R. H. Villarreal, Canonical modules of certain edge subrings, European J. Combin. **24(5)** (2003), 471–487.

[35] W. V. Vasconcelos, *Arithmetic of Blowup Algebras*, London Math. Soc., Lecture Note Series **195**, Cambridge University Press, Cambridge, 1994.

[36] W. V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer-Verlag, 1998.

[37] W. V. Vasconcelos, *Integral Closure*, Springer Monographs in Mathematics, Springer, New York, 2005.

[38] R. H. Villarreal, Cohen-Macaulay graphs, Manuscripta Math. **66** (1990), 277–293.

[39] R. H. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001.

[40] R. H. Villarreal, Rees cones of matroids, arXiv:math.AC/0601520.

[41] R. Webster, *Convexity*, Oxford University Press, Oxford, 1994.

[42] N. L. White, The basis monomial ring of a matroid, Adv. in Math. **24** (1977), 292–297.

[43] N. L. White, A unique exchange property for bases, Linear Algebra Appl. **31** (1980), 81–91.

[44] X. Zheng, Resolutions of facet ideals, Comm. Algebra **32(6)** (2004), 2301–2324.