Explicit representations of the edge cone of a graph

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Abstract

Let G be an arbitrary graph. The main results are explicit representations of the edge cone of G as a finite intersection of closed halfspaces. If G is bipartite and connected we determine the facets of the edge cone and present a canonical irreducible representation.

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1 Edge cones

In the sequel we use standard terminology and notation from graph theory and adopt [1] as our main reference. For the reader's convenience, along the way, we recall a few notions about graphs and polyhedral geometry. We begin by fixing some notation that will be used throughout.

Let G be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{z_1, \ldots, z_q\}$. Thus every edge z_i is an unordered pair of distinct vertices $z_i = \{v_{i_j}, v_{i_k}\}$ and we set $\alpha_i = e_{i_j} + e_{i_k}$, where e_i is the *ith* unit vector in $\mathbb{R}^n$. The *incidence matrix* of G, denoted by $M_G = (a_{ij})$, is the $n \times q$ matrix whose columns are precisely the vectors $\alpha_1, \ldots, \alpha_q$. We set $\mathcal{A}_G$ (or simply $\mathcal{A}$ if G is understood) equal to the set $\{\alpha_1, \ldots, \alpha_q\}$ of column vectors of M_G. Since α_i represents an edge of G sometimes α_i is called an edge or an *edge vector*. The *edge cone* of G, denoted by $\mathbb{R}_+\mathcal{A}$, is defined as the cone generated by $\mathcal{A}$:

$$\mathbb{R}_+\mathcal{A} := \left\{ \sum_{i=1}^{q} a_i\alpha_i \,\middle|\, a_i \in \mathbb{R}_+ \text{ for all } i \right\} \subset \mathbb{R}^n,$$

where $\mathbb{R}_+$ is the set of non negative real numbers. Note $\mathbb{R}_+\mathcal{A} \neq (0)$ if G is not a discrete graph, i.e., a graph without edges. By [3] one has

$$n - c_0(G) = \text{rank}(M_G) = \dim \mathbb{R}_+\mathcal{A},$$

where $c_0(G)$ is the number of bipartite connected components of G. By the finite basis theorem [7, Chapter 4] $\mathbb{R}_+\mathcal{A}$ is a rational polyhedral cone, i.e., $\mathbb{R}_+\mathcal{A}$ is the intersection of finitely many *closed halfspaces* of the form:

$$H_a^+ := \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \geq 0\},$$

where $0 \neq a \in \mathbb{Z}^n$ and the nonzero entries of a are relatively prime. Here $\langle x, a\rangle$ denotes the standard inner product of x and a. Note that if $H_a^- := H_{-a}^+$, then the intersection $H_a^+ \cap H_a^-$ is the bounding hyperplane

$$H_a := \{x \,|\, \langle x, a\rangle = 0\}$$

with normal vector a. To simplify notation set $Q = \mathbb{R}_+\mathcal{A}$. Recall that a subset $F \subset \mathbb{R}^n$ is a *face* of Q if $F = Q \cap H_a$ for some hyperplane H_a such that $Q \subset H_a^+$ or $Q \subset H_a^-$. The hyperplane H_a is called a *supporting hyperplane* of Q. The *improper faces* of Q are Q and $\emptyset$, all the other faces are called *proper faces*. If a face of Q has dimension $\dim(Q) - 1$ it is called a *facet*. The dimension of Q is by definition the dimension of $\text{aff}(Q)$, the affine hull of Q. Note that a face of Q is again a finitely generated cone, see [7].

Definition 1.1 If $Q = \mathbb{R}_+\mathcal{A}$ is represented as

$$Q = \mathrm{aff}(Q) \cap \left(\bigcap_{i=1}^{r} H_{a_i}^{+} \right) \tag{1}$$

with $a_i \in \mathbb{R}^n \setminus \{0\}$ for all i and none of the closed halfspaces $H_{a_1}^+, \ldots, H_{a_r}^+$ can be omitted from the intersection, we say that Eq. (1) is an *irreducible representation* of Q.

Part of the importance of an irreducible representation can be seen in the following general fact.

Theorem 1.2 [7, Theorem 3.2.1] *If $Q = \mathrm{aff}(Q) \cap H_{a_1}^+ \cap \cdots \cap H_{a_r}^+$ is an irreducible representation of Q, then the facets of Q are precisely the sets $F_1, \ldots, F_r$, where $F_i = Q \cap H_{a_i}$. Moreover each proper face of Q is the intersection of those facets of Q that contain it.*

The main goal is to give an explicit combinatorial description of the edge cone of G, see Theorem 2.6 and Corollary 2.8. This description generalizes that of [5, Corollary 3.3]. In loc. cit. only the non bipartite case was studied. Another goal is to study in detail the facets of the edge cone of a connected bipartite graph and show a canonical irreducible representation of the edge cone, see Proposition 3.6 and Theorem 3.9. As an application the classical marriage theorem will follow.

To show our results we use graph theory, linear algebra, and polyhedral geometry. The proofs require a careful analysis at the graph theoretical level. Our main references for graphs, algebra and geometry are [1, 2, 4, 6, 7].

Definition 1.3 A graph G is *bipartite* if there is a *bipartition* (V_1, V_2) of G, that is, V_1 and V_2 are vertex classes satisfying:

(a) $V(G) = V_1 \cup V_2$,

(b) $V_1 \cap V_2 = \emptyset$, and

(c) every edge of G joins a vertex of V_1 to a vertex of V_2.

Recall that a graph G is bipartite if and only if all its cycles are of even length. If G is bipartite, then its incidence matrix is totally unimodular, that is, all the $i \times i$ minors of M_G are equal to 0 or ± 1 for all $i \geq 1$, see [4].

2 An explicit representation of the edge cone

Lemma 2.1 *If v_i is not an isolated vertex of G, then the set $F = H_{e_i} \cap \mathbb{R}_+\mathcal{A}$ is a proper face of the edge cone.*

Proof. Note $F \neq \emptyset$ because $0 \in F$, and $\mathbb{R}_+\mathcal{A} \subset H_{e_i}^+$. Since v_i is not an isolated vertex $\mathbb{R}_+\mathcal{A} \not\subset H_{e_i}$. □

Given a subset $A \subset V(G)$, the *neighbor set* of A, denoted $N_G(A)$ or simply $N(A)$, is defined as

$$N(A) = \{v \in V(G) \mid v \text{ is adjacent to some vertex in } A\}.$$

Let A be an *independent set* of vertices of G, that is, no two vertices of A are adjacent. The supporting hyperplane of the edge cone of G defined by

$$\sum_{v_i \in A} x_i = \sum_{v_i \in N(A)} x_i$$

will be denoted by H_A.

Lemma 2.2 *If A is an independent set of vertices of G and $F = \mathbb{R}_+\mathcal{A} \cap H_A$, then either F is a proper face of the edge cone or $F = \mathbb{R}_+\mathcal{A}$.*

Proof. It suffices to prove the containment $\mathbb{R}_+\mathcal{A} \subset H_A^-$. Take an edge $\{v_j, v_\ell\}$ of G. If $\{v_j, v_\ell\} \cap A \neq \emptyset$, then $e_j + e_\ell$ is in H_A, else $e_j + e_\ell$ is in H_A^-. □

Definition 2.3 The *support* of a vector $\beta = (\beta_i) \in \mathbb{R}^n$ is defined as

$$\mathrm{supp}(\beta) = \{\beta_i \mid \beta_i \neq 0\}.$$

Lemma 2.4 ([5]) *Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of G and let $G_1, \ldots, G_r$ be the connected components of G. If G_1 is a tree with at least two vertices and $G_2, \ldots, G_r$ are unicyclic non bipartite graphs, then $\ker(M_G^t) = (\beta)$ for some β in $\mathbb{R}^n$ with $\mathrm{supp}(\beta) = \{1, -1\}$ such that $V(G_1) = \{v_i \in V \mid \beta_i = \pm 1\}$.*

For use below we recall the following form of Farkas's Lemma, which is called the fundamental theorem of linear inequalities, see [4, Theorem 7.1].

Theorem 2.5 *Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ be a set of vectors in $\mathbb{R}^n$ and let $\alpha \in \mathbb{R}^n$. If $\alpha \notin \mathbb{R}_+\mathcal{A}$ and $t = \mathrm{rank}\{\alpha_1, \ldots, \alpha_q, \alpha\}$, then there exists a hyperplane H_a containing $t-1$ linearly independent vectors from $\mathcal{A}$ such that $\langle a, \alpha\rangle > 0$ and $\langle a, \alpha_i\rangle \leq 0$ for $i = 1, \ldots, q$.*

Theorem 2.6 *If G is a connected graph with vertex set $V = \{v_1, \ldots, v_n\}$ and $\mathbb{R}_+\mathcal{A}$ is the edge cone of G, then*

$$\mathbb{R}_+\mathcal{A} = \left(\bigcap_{A\in\mathcal{F}} H_A^-\right) \bigcap \left(\bigcap_{i=1}^{n} H_{e_i}^+\right), \tag{2}$$

where $\mathcal{F}$ is the family of all the independent sets of vertices of G and $H_{e_i}^+$ is the closed halfspace $\{x \in \mathbb{R}^n \mid x_i \geq 0\}$.

Proof. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ be the set of column vectors of the incidence matrix of G. Since $\mathbb{R}_+\mathcal{A}$ is clearly contained in the right hand side of Eq. (2) it suffices to prove the other containment. Take $\alpha \in \mathbb{R}^n$ in the right hand side of Eq. (2). The proof is by contradiction, that is, assume that $\alpha \notin \mathbb{R}_+\mathcal{A}$. By [5, Corollary 3.3] we may assume that G bipartite with $n \geq 3$ vertices.

Note that if (V_1, V_2) is the bipartition of G, then $\mathrm{aff}(\mathbb{R}_+\mathcal{A})$ is the hyperplane

$$\sum_{v_i\in V_1} x_i = \sum_{v_i\in V_2} x_i,$$

because $\dim(\mathbb{R}_+\mathcal{A}) = n-1$. As $H_{V_1}^- \cap H_{V_2}^- = H_{V_1}$, the vector α is in $\mathrm{aff}(\mathbb{R}_+\mathcal{A})$. As a consequence $\mathrm{rank}(\mathcal{A} \cup \{\alpha\}) = n-1$.

By Theorem 2.5 there is $a \in \mathbb{R}^n$ and there are linearly independent vectors $\alpha_1, \ldots, \alpha_{n-2}$ in $\mathcal{A}$ such that

(i) $\langle a, \alpha_i\rangle = 0$ for $i = 1, 2, \ldots, n-2$,

(ii) $\langle a, \alpha_i\rangle \leq 0$ for $i = 1, 2, \ldots, q$, and

(iii) $\langle a, \alpha\rangle > 0$.

Observe that $\mathbb{R}_+\mathcal{A} \not\subset H_a$ because $\mathrm{aff}(\mathbb{R}_+\mathcal{A}) \neq H_a$. There exists α_j in $\mathcal{A}$ such that $\alpha_1, \ldots, \alpha_{n-2}, \alpha_j$ is a basis of $\mathrm{aff}(\mathbb{R}_+\mathcal{A})$ as a real vector space. In particular we can write

$$\alpha = \lambda_1\alpha_1 + \cdots + \lambda_{n-2}\alpha_{n-2} + \lambda_j\alpha_j \quad (\lambda_i \in \mathbb{R}) \tag{3}$$

It follows that $\langle \alpha, a\rangle = \lambda_j\langle \alpha_j, a\rangle > 0$. Thus $\lambda_j < 0$.

Consider the subgraph D of G whose edges correspond to $\alpha_1, \ldots, \alpha_{n-2}$ and its vertex set is the union of the vertices in the edges of D. Set $k = |V(D)|$. By [3] one has:

$$n-2 = \mathrm{rank}(M_D) = k - c_0(D),$$

where M_D is the incidence matrix of D and $c_0(D)$ is the number of bipartite components of D. Thus $0 \leq n-k = 2 - c_0(D)$. This shows that either $c_0(D) = 1$ and $k = n-1$ or $c_0(D) = 2$ and $k = n$.

Case (I): Assume that $C_0(D) = 1$ and $k = n-1$. Set $V(D) = \{v_1, \ldots, v_{n-1}\}$. As D is a tree with $n-2$ edges and $\langle \alpha_i, a\rangle = 0$ for $i = 1, \ldots, n-2$, applying Lemma 2.4, one may assume that $a = (a_1, \ldots, a_{n-1}, a_n)$, where $a_i = \pm 1$ for $1 \leq i \leq n-1$.

Set $a' = (0, \ldots, 0, -1) = -e_n$. Next we prove the following

(a) $\langle \alpha_i, a'\rangle = 0$ for $i = 1, \ldots, n-2$.

(b) $\langle \alpha_j, a'\rangle = -1$ and $\langle \alpha_j, a\rangle < 0$.

Condition (a) is clear. To prove (b) first note $\alpha_j \notin \mathbb{R}(\alpha_1, \ldots, \alpha_{n-2})$. Then $\alpha_j = e_k + e_n$, because otherwise the "edge" α_j added to the tree D form a graph with a unique even cycle, to derive a contradiction recall that a set of edge vectors forming an even cycle are linearly dependent. Thus $\langle \alpha_j, a'\rangle = -1$.

On the other hand $\langle \alpha_j, a\rangle < 0$, because if $\langle \alpha_j, a\rangle = 0$, then the hyperplane H_a would contain the linearly independent vectors $\alpha_1, \ldots, \alpha_{n-2}, \alpha_j$ and consequently $\mathrm{aff}(\mathbb{R}_+\mathcal{A})$ would be equal to H_a, a contradiction.

To finish the proof of this case we use the inequality

$$\langle \alpha, a'\rangle = \lambda_j \langle \alpha_j, a'\rangle > 0$$

to conclude $\langle \alpha, a'\rangle > 0$, a contradiction because $\alpha \in H_{e_n}^+$.

Case (II): Assume that $C_0(D) = 2$ and $k = n$. Let D_1 and D_2 be the components of D and set $U_1 = V(D_1)$ and $U_2 = V(D_2)$.

Using Lemma 2.4 we can relabel the vertices of the graph D and write $a = rb + sc$, where $0 \neq r \geq s \geq 0$ are rational numbers,

$$b = (b_1, \ldots, b_m, 0, \ldots, 0), \qquad c = (0, \ldots, 0, c_{m+1}, \ldots, c_n),$$

$U_1 = \{v_1, \ldots, v_m\}$, $b_i = \pm 1$ for $i \leq m$, and $c_i = \pm 1$ for $i > m$. Set $a' = b$. Note the following:

(a) $\langle \alpha_i, a'\rangle = 0$ for $i = 1, \ldots, n-2$.

(b) $\langle \alpha_j, a'\rangle = -1$ and $\langle \alpha_j, a\rangle < 0$; this holds for any $\alpha_j \notin \mathbb{R}(\alpha_1, \ldots, \alpha_{n-2})$.

Condition (a) is clear. To prove (b) first note that the inequality $\langle \alpha_j, a\rangle < 0$ can be shown as in case (I). Observe that if an "edge" α_k has vertices in U_1 (resp. U_2), then $\langle \alpha_k, a\rangle = 0$. Indeed if we add the edge α_k to the tree D_1 (resp. D_2) we get a graph with a unique even cycle and this implies that $\alpha_1, \ldots, \alpha_{n-2}, \alpha_k$ are linearly dependent, that is, $\langle \alpha_k, a\rangle = 0$. Thus $\alpha_j = e_i + e_\ell$ for some $v_i \in U_1$ and $v_\ell \in U_2$. From the inequality

$$\langle \alpha_j, a\rangle = r\langle \alpha_j, b\rangle + s\langle \alpha_j, c\rangle = rb_i + sc_\ell < 0$$

we obtain $b_i = -1 = \langle \alpha_j, a'\rangle$, as required.

Next we set

$$A = \{v_i \in V \,|\, b_i = 1\} \quad \text{and} \quad B = \{v_i \in V \,|\, b_i = -1\}.$$

Note that $\emptyset \neq A \subset U_1$ and $\emptyset \neq B \subset U_1$, because D_1, D_2 are trees with at least two vertices. We will show that A is an independent set of G and $B = N_G(A)$.

If A is not an independent set of G, there is an edge $\{v_i, v_\ell\}$ of G for some v_i, v_ℓ in A. Thus $\alpha_k = e_i + e_\ell$, by (a) and (b) we get $\langle a', \alpha_k\rangle \leq 0$, which is impossible because $\langle a', \alpha_k\rangle = 2$. This proves that A is an independent set of G.

Next we show $N_G(A) = B$. If $v_i \in N_G(A)$, then $\alpha_k = e_i + e_\ell$ for some v_ℓ in A, using (a) and (b) we obtain $\langle a', \alpha_k\rangle = b_i + 1 \leq 0$ and $b_i = -1$, hence $v_i \in B$. Conversely if $v_i \in B$, since D_1 has no isolated vertices, there is $1 \leq k \leq n-2$ so that $\alpha_k = e_i + e_\ell$, for some ℓ, by (b) we obtain $\langle a', \alpha_k\rangle = -1 + b_\ell = 0$, which shows that $v_\ell \in A$ and $v_i \in N_G(A)$.

Therefore $H_{a'} = H_A$. Since $\mathbb{R}_+\mathcal{A} \not\subset H_{a'}$ (this follows from (b)), there is $\alpha_\ell \notin H_A$, thus $H^-_{a'} \cap H^-_A \neq \emptyset$ and consequently $H^-_A = H^-_{a'}$. By hypothesis $\alpha \in H^-_A$, hence $\langle \alpha, a'\rangle \leq 0$. From Eq. (3) together with and (a) and (b) one has

$$\langle \alpha, a'\rangle = \lambda_j \langle \alpha_j, a'\rangle = -\lambda_j > 0,$$

a contradiction. $\Box$

The next two results give an explicit representation by closed halfspaces of the edge cone of an arbitrary graph. Those representations were known for connected non bipartite graphs only [5].

Corollary 2.7 *If G is a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and $\mathbb{R}_+\mathcal{A}$ is the edge cone of G, then*

$$\mathbb{R}_+\mathcal{A} = \left(\bigcap_A H^-_A\right) \bigcap \left(\bigcap_{i=1}^n H^+_{e_i}\right), \tag{4}$$

where the intersection is taken over all the independent sets of vertices A of G and $H^+_{e_i} = \{x \in \mathbb{R}^n \,|\, x_i \geq 0\}$.

Proof. Let $G_1, \ldots, G_r$ be the connected components of G. For simplicity of notation we assume that $r = 2$ and $V(G_1) = \{v_1, \ldots, v_m\}$. There is a decomposition

$$\mathbb{R}_+\mathcal{A} = \mathbb{R}_+\mathcal{A}_{G_1} \oplus \mathbb{R}_+\mathcal{A}_{G_2}.$$

Let δ be a vector in the right hand side of Eq. (4). One can write

$$\delta = (\delta_i) = \beta + \gamma = (\delta_1, \ldots, \delta_m, 0, \ldots, 0) + (0, \ldots, 0, \delta_{m+1}, \ldots, \delta_n).$$

Let A be an independent set of G_1. Note $N_G(A) = N_{G_1}(A)$, hence

$$\sum_{v_i \in A} \delta_i \leq \sum_{v_i \in N_G(A)} \delta_i = \sum_{v_i \in N_{G_1}(A)} \delta_i.$$

Applying Theorem 2.6 yields $\beta \in \mathbb{R}_+ \mathcal{A}_{G_1}$. Similarly one has $\gamma \in \mathbb{R}_+ \mathcal{A}_{G_2}$. Hence $\delta \in \mathbb{R}_+ \mathcal{A}_G$, as required. $\Box$

Corollary 2.8 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$. Then a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is in $\mathbb{R}_+ \mathcal{A}$ if and only if x is a solution of the system of linear inequalities*

$$\begin{array}{rcl} -x_i & \leq & 0, \quad i = 1, \ldots, n \\ \sum_{v_i \in A} x_i - \sum_{v_i \in N(A)} x_i & \leq & 0, \quad \textit{for all independent sets } A \subset V. \end{array}$$

Proof. It follows at once from Corollary 2.7. $\Box$

Theorem 2.9 *If G is a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and F is a facet of the edge cone of G, then either*

(a) *$F = \mathbb{R}_+ \mathcal{A} \cap \{x \in \mathbb{R}^n \mid x_i = 0\}$ for some $1 \leq i \leq n$, or*

(b) *$F = \mathbb{R}_+ \mathcal{A} \cap H_A$ for some independent set A of G.*

Proof. By Corollary 2.7 we can write

$$\mathbb{R}_+ \mathcal{A} = \operatorname{aff}(\mathbb{R}_+ \mathcal{A}) \cap H_1^- \cap \cdots \cap H_r^-$$

for some hyperplanes $H_1, \ldots, H_r$ such that none of the halfspaces H_j^- can be omitted in the intersection and each H_j is either of the form H_{-e_i} or $H_j = H_A$ for some independent set A. By Theorem 1.2 the facets of $\mathbb{R}_+ \mathcal{A}$ are precisely the sets $F_1, \ldots, F_r$, where $F_i = H_i \cap \mathbb{R}_+ \mathcal{A}$. $\Box$

3 Studying the bipartite case

For connected bipartite graphs we will present sharper results on the irreducible representations of edge cones and give a characterization of their facets.

Proposition 3.1 *Let G be a connected bipartite graph with bipartition (V_1, V_2). If A is an independent set of G such that $A \neq V_i$ for $i = 1, 2$, then $F = \mathbb{R}_+ \mathcal{A} \cap H_A$ is a proper face of the edge cone.*

Proof. Assume that $N(A) = V_2$. Take any $v_i \in V_1 \setminus A$ and any $v_j \in V_2$ adjacent to v_i, then $e_i + e_j \notin H_A$. Thus we may assume that $N(A) \neq V_i$ for $i = 1, 2$.

Case (I): $N(A) \cap V_i \neq \emptyset$ for $i = 1, 2$. If the vertices in $N(A) \cap V_i$ for $i = 1, 2$ are only adjacent to vertices in A, then pick vertices $v_i \in N(A) \cap V_i$ and note that there is no path between v_1 and v_2, a contradiction. Thus there must be a vector in the edge cone which is not in H_A.

Case (II): $A \subsetneq V_1$. If the vertices in $N(A)$ are only adjacent to vertices in A. Then a vertex in A cannot be joined by a path to a vertex in $V_2 \setminus N(A)$, a contradiction. As before we obtain $\mathbb{R}_+\mathcal{A} \not\subset H_A$. $\square$

Proposition 3.2 *Let G be a connected bipartite graph with bipartition (V_1, V_2) and $\mathcal{F}$ the family of independent sets A of G such that $H_A \cap \mathbb{R}_+\mathcal{A}_G$ is a facet. If A is in $\mathcal{F}$ and $V_i \cap A \neq \emptyset$ for $i = 1, 2$, then the halfspace H_A^- is redundant in the following expression of the edge cone*

$$\mathbb{R}_+\mathcal{A} = \mathrm{aff}(\mathbb{R}_+\mathcal{A}) \cap \left(\bigcap_{A \in \mathcal{F}} H_A^- \right) \bigcap \left(\bigcap_{i=1}^{n} H_{e_i}^+ \right).$$

Proof. Set $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$. One can write $A = A_1 \cup A_2$ with $A_i \subset V_i$ for $i = 1, 2$. There are $\alpha_1, \ldots, \alpha_{n-2}$ linearly independent vectors in $H_A \cap \mathbb{R}_+\mathcal{A}$, where n is the number of vertices of G. Consider the subgraph D of G whose edges correspond to $\alpha_1, \ldots, \alpha_{n-2}$ and its vertex set is the union of the vertices in those edges. Note that D cannot be connected. Indeed there is no edge of D connecting a vertex in $N_G(A_1)$ with a vertex in $N_G(A_2)$ because all the vectors $\alpha_1, \ldots, \alpha_{n-2}$ satisfy the equation

$$\sum_{v_i \in A} x_i = \sum_{v_i \in N_G(A)} x_i.$$

Hence by the proof of Theorem 2.6 it follows that D is a spanning subgraph of G with two connected components D_1 and D_2 (which are trees) such that $V(D_i) = A_i \cup N_G(A_i)$, $i = 1, 2$. Therefore H_{A_i} is a proper support hyperplane defining a facet $F_i = H_{A_i} \cap \mathbb{R}_+\mathcal{A}$, that is A_1, A_2 are in $\mathcal{F}$. Since $H_{A_1}^- \cap H_{A_2}^-$ is contained in H_A^- the proof is complete. $\square$

Proposition 3.3 *Let G be a connected bipartite graph with bipartition (V_1, V_2). If $A_2 \subsetneq V_2$ and $F = H_{A_2} \cap \mathbb{R}_+\mathcal{A}$ is a facet of the edge cone of G, then*

$$H_{A_2}^- \cap \mathrm{aff}(\mathcal{A}') = \begin{cases} H_{A_1}^- \cap \mathrm{aff}(\mathcal{A}') & \textit{where } A_1 = V_1 \setminus N(A_2) \neq \emptyset, \textit{ or} \\ H_{e_i}^+ \cap \mathrm{aff}(\mathcal{A}') & \textit{for some vertex } v_i \textit{ with } G \setminus \{v_i\} \textit{ connected,} \end{cases}$$

where $\mathcal{A}' = \mathcal{A} \cup \{0\}$.

Proof. Let us assume that G has p vertices $v_1,\ldots,v_p$ and V_1 is the set of the first m vertices of G. Set $\mathcal{A}=\{\alpha_1,\ldots,\alpha_q\}$. There are $\alpha_1,\ldots,\alpha_{p-2}$ linearly independent vectors in the hyperplane H_{A_2}. Consider the subgraph D of G whose edges correspond to $\alpha_1,\ldots,\alpha_{p-2}$ and its vertex set is the union of the vertices in those edges. As G is connected either D is a tree with $p-1$ vertices or D is a spanning subgraph of G with two connected components.

If D is a tree, write $V(D)=V(G)\setminus\{v_i\}$ for some i. Note

$$\langle \alpha_j,\alpha_{A_2}\rangle=-\langle\alpha_j,e_i\rangle \quad (j=1,\ldots,q),$$

where

$$\alpha_{A_2}=\sum_{v_i\in A_2}e_i-\sum_{v_i\in N(A_2)}e_i.$$

Indeed if the "edge" α_j has vertices in $V(D)$, then both sides of the equality are zero, otherwise write $\alpha_j=e_i+e_\ell$. Observe $v_i\notin A_2$ and $v_\ell\in N(A_2)$ because H_{A_2} being a facet cannot contain α_j, thus both sides of the equality are equal to -1. As a consequence since $\mathrm{aff}(\mathcal{A}')=\mathbb{R}(\alpha_1,\ldots,\alpha_{p-2},\alpha_j)$ for some $\alpha_j=e_i+e_\ell$ we rapidly obtain

$$\langle\alpha,\alpha_{A_2}\rangle=-\langle\alpha,e_i\rangle \quad (\forall\,\alpha\in\mathrm{aff}(\mathcal{A}')).$$

Therefore

$$H_{A_2}^-\cap\mathrm{aff}(\mathcal{A}')=H_{e_i}^+\cap\mathrm{aff}(\mathcal{A}'),$$

as required.

We may now assume that D is not a tree. We claim that $A_1=V_1\setminus N(A_2)\neq\emptyset$. If $V_1=N(A_2)$. Take $v_i\in V_2\setminus A_2$ and $\{v_i,v_j\}$ and edge of D containing v_i. Hence since $v_j\in N(A_2)$ we get $\langle e_i+e_j,\alpha_{A_2}\rangle=-1$, a contradiction because e_i+e_j is in H_{A_2}. Thus $A_1\neq\emptyset$. Since all the vectors in $\mathrm{aff}(\mathcal{A}')$ satisfy the linear equation

$$\sum_{i=1}^m x_i=\sum_{i=m+1}^p x_i,$$

we obtain

$$H_{A_2}^-\cap\mathrm{aff}(\mathcal{A}')=\left\{x\in\mathrm{aff}(\mathcal{A}')\;\middle|\;\sum_{v_i\in V_1\setminus N(A_2)}x_i\ \leq\ \sum_{v_i\in V_2\setminus A_2}x_i\right\}.$$

Hence we need only show $V_2\setminus A_2=N(A_1)$. The containment $N(A_1)\subset V_2\setminus A_2$ holds in general. For the reverse containment take $v_i\in V_2\setminus A_2$. There is v_j such that $\{v_i,v_j\}$ is an edge of D. If $v_j\in N(A_2)$, then $\langle e_i+e_j,\alpha_{A_2}\rangle=-1$, a contradiction because $e_i+e_j\in H_{A_2}$. Hence $v_j\notin N(A_2)$ and $v_i\in N(A_1)$. $\Box$

For use below we state the following duality of facets which follows from the proof of Proposition 3.3.

Lemma 3.4 *Let G be a connected bipartite graph with bipartition (V_1, V_2) and let $F = H_A \cap \mathbb{R}_+\mathcal{A}$ be a facet of $\mathbb{R}_+\mathcal{A}$ with $A \subsetneq V_1$. Then*

(a) *If $N(A) = V_2$, then $A = V_1 \setminus \{v_i\}$ for some $v_i \in V_1$ and $F = H_{e_i} \cap \mathbb{R}_+\mathcal{A}$.*

(b) *If $N(A) \subsetneq V_2$, then $F = H_{V_2 \setminus N(A)} \cap \mathbb{R}_+\mathcal{A}$ and $N(V_2 \setminus N(A)) = V_1 \setminus A$.*

Definition 3.5 For any set of vertices S of a graph G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S.

Proposition 3.6 *Let G be a connected bipartite graph with bipartition (V_1, V_2) and let $A \subsetneq V_1$. Then $F = H_A \cap \mathbb{R}_+\mathcal{A}$ is a facet of $\mathbb{R}_+\mathcal{A}$ if and only if*

(a) *$\langle A \cup N(A) \rangle$ is connected with vertex set $V(G) \setminus \{v\}$ for some $v \in V_1$, or*

(b) *$\langle A \cup N(A) \rangle$ and $\langle (V_2 \setminus N(A)) \cup (V_1 \setminus A) \rangle$ are connected and their union is a spanning subgraph of G.*

Moreover any facet has the form $F = H_A \cap \mathbb{R}_+\mathcal{A}$ for some $A \subsetneq V_i$, $i = 1$ or $i = 2$.

Proof. The first statement follows readily from Lemma 3.4 and using part of the proof of Theorem 2.6. The last statement follows combining Theorem 2.6 with Proposition 3.2. □

Remark 3.7 (a) In Proposition 3.6 the case (a) is included in case (b). To see this make $N(A) = V_2$ and note that $\langle (V_2 \setminus N(A)) \cup (V_1 \setminus A) \rangle$ must consist of a point. (b) The facets of the edge cone of G for G non bipartite were characterized in [5, Theorem 3.2].

Lemma 3.8 *Let G be a connected bipartite graph with bipartition (V_1, V_2) and let F be a facet of $\mathbb{R}_+\mathcal{A}$. If $F = H_A \cap \mathbb{R}_+\mathcal{A} = H_B \cap \mathbb{R}_+\mathcal{A}$ with $A \subsetneq V_1$ and $B \subsetneq V_1$, then $A = B$.*

Proof. Set $V_1 = \{v_1, \ldots, v_m\}$ and $V_2 = \{v_{m+1}, \ldots, v_{m+n}\}$. Recall that the equality

$$x_1 + \cdots + x_m = x_{m+1} + \cdots + x_{m+n}$$

defines $\text{aff}(\mathbb{R}_+\mathcal{A})$.

Case (I): $N(A) = V_2$. Then by Lemma 3.4 (after permutation of vertices) $A = \{v_1, \ldots, v_{m-1}\}$. Hence any $x \in F$ satisfies

$$\sum_{v_i \in A} x_i - \sum_{v_i \in N(A)} x_i = -x_m$$

and thus $F = H_{e_m} \cap \mathbb{R}_+\mathcal{A}$. If $v_m \in B$, then $\{v_m, v_j\} \in E(G)$ for some v_j in $N(B)$, thus $e_m + e_j \in H_B$ and consequently $e_m + e_j \in H_{e_m}$, a contradiction.

Hence $v_m \notin B$, that is, $B \subset A$. If $N(B) = V_2$, then by Lemma 3.4 $A = B$. Assume that $V_2 \setminus N(B) \neq \emptyset$, to complete the proof for this case we will show that this assumption leads to a contradiction. First note that v_m is not adjacent to any $v_j \in V_2 \setminus N(B)$. Indeed if $\{v_m, v_j\} \in E(G)$, then $e_m + e_j \in H_B$. Thus $e_m + e_j \in H_{e_m}$, a contradiction. Therefore by the connectivity of G at least one vertex $v_i \in V_1 \setminus B$ must be adjacent to both a vertex $v_j \in V_2 \setminus N(B)$ and a vertex $v_k \in N(B)$, which is impossible because $e_i + e_k \in H_{e_m}$ and $e_i + e_k \notin H_B$.

Case (II): $N(A) \subsetneq V_2$ and $N(B) \subsetneq V_2$. We begin by considering the subcase $A \cap B \neq \emptyset$. Take $v_0 \in A \cap B$ and $v_0 \neq v \in B$. By Proposition 3.6 the subgraph $\langle B \cup N(B) \rangle$ is connected, hence there is a path of even length

$$\mathcal{P} = \{v_0, v_1, v_2, \ldots, v_{2r-1}, v_{2r} = v\}$$

such that $v_{2i} \in B$ for all i. Note that $v_2 \in A$. If $v_2 \notin A$, then $e_1 + e_2 \in H_B$ and $e_1 + e_2 \notin H_A$, a contradiction. By induction we get $v_{2i} \in A \cap B$ for all i. Hence $v \in A$. This proves $B \subset A$, a similar argument proves $A = B$.

Assume now that $A \cap B = \emptyset$. We claim $N(A) \cap N(B) = \emptyset$, for otherwise if $\{v_j, v_k\}$ is an edge with $v_j \in B$ and $v_k \in N(A) \cap N(B)$, then $e_j + e_k \notin H_A$ because $v_j \notin A$ and $e_j + e_k \in H_B$, a contradiction.

We may now assume that $A \cap B = N(A) \cap N(B) = \emptyset$. Observe $A \cup B \neq V_2$ because if $V_2 = A \cup B$, then G would be disconnected with components $\langle A \cup N(A) \rangle$ and $\langle B \cup N(B) \rangle$. Take $v_j \in V_1 \setminus (A \cup B)$ such that v_j is adjacent to some v_k in $N(A) \cup N(B)$, this choice is possible because G is connected. Say $v_k \in N(A)$. Note $v_k \notin N(B)$. Then $e_j + e_k \in H_B$ and $e_j + e_k \notin H_A$, a contradiction. $\Box$

Putting together the previous results we obtain the following canonical way of representing the edge cone. The uniqueness follows from Lemma 3.4 and Lemma 3.8.

Theorem 3.9 *If G is a connected bipartite graph with bipartition (V_1, V_2), then there is a unique irreducible representation*

$$\mathbb{R}_+\mathcal{A} = \operatorname{aff}(\mathbb{R}_+\mathcal{A}) \cap (\cap_{i=1}^r H_{A_i}^-) \cap (\cap_{i \in \mathcal{I}} H_{e_i}^+)$$

such that $A_i \subsetneq V_1$ for all i and $v_i \in V_2$ for $i \in \mathcal{I}$.

Lemma 3.10 *If G is a bipartite graph, then*

$$\mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\mathcal{A}.$$

In particular if $(\beta_1, \ldots, \beta_n)$ is an integral vector in the edge cone, then $\sum_{i=1}^n \beta_i$ is an even integer.

Proof. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ be the set of column vectors of the incidence matrix M of G. Take $\alpha \in \mathbb{Z}^n \cap \mathbb{R}_+\mathcal{A}$, then by Carathéodory's Theorem [2, Theorem 2.3] and after an appropriate permutation of the α_i's we can write

$$\alpha = \eta_1\alpha_1 + \cdots + \eta_r\alpha_r \quad (\eta_i \geq 0),$$

where r is the rank of M and $\alpha_1, \ldots, \alpha_r$ are linearly independent. On the other hand the submatrix $M' = (\alpha_1 \cdots \alpha_r)$ is totally unimodular because G is bipartite (see [4]), hence by Kronecker's lemma [4, p. 51] the system of equations $M'x = \alpha$ has an integral solution. Hence α is a linear combination of $\alpha_1, \ldots, \alpha_r$ with coefficients in $\mathbb{Z}$. It follows that $\eta_i \in \mathbb{N}$ for all i, that is, $\alpha \in \mathbb{N}\mathcal{A}$. The other containment is clear. $\Box$

As an application we recover the following version of the marriage problem for bipartite graphs, see [1]. Recall that a pairing of all the vertices of a graph G is called a *perfect matching*.

Theorem 3.11 (Marriage Theorem) *If G is a bipartite graph, then G has a perfect matching if and only if*

$$|A| \leq |N(A)|$$

for every independent set of vertices A of G.

Proof. Note that G has a perfect matching if and only if the vector $\beta = (1, 1, \ldots, 1)$ is in $\mathbb{N}\mathcal{A}$. By Lemma 3.10 β is in $\mathbb{N}\mathcal{A}$ if and only if $\beta \in \mathbb{R}_+\mathcal{A}$. Thus the result follows from Corollary 2.8. $\Box$

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