Admissions Examination for Master's / Doctorate
18 June 2018

Name: ____

Instructions: For each question encircle the correct answers. There may be several correct solutions for a single question (choose all, but points will be taken off for incorrectly chosen options). You may make calculations on the paper you are given, but you do not need to hand that in. The examination has 30 questions. We suggest reading all of the statements first. **Calculators or cell phones not allowed.**

Duration of the examination: 2 hours

1. ¿Which of the following sets is a vector space?

   (a) The set of solutions $\vec{x}$ of $A\vec{x} = 0$ where A is an $m \times n$ matrix;

   (b) The set of 2×2 matrices A such that $\det(A) = 0$;

   (c) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)dx = 0$;

   (d) The set of solutions of $y = y(t)$ of $y'' + 4y' + y = 0$.

2. Let V be the vector space of the real numbers over the field of the rational numbers. Which of the following statements are true?

   (a) $\dim(V)$ is countable;

   (b) $\dim(V)$ is not countable;

   (c) $\dim(V) = 1$;

   (d) V has no basis.

3. Let a, b, c be constants with $a \neq 0$. For which values of x is the matrix $\begin{pmatrix} 1 & 0 & c \\ 0 & a & -b \\ -1/a & x & x^2 \end{pmatrix}$ invertible?

   (a) $x = 0$;

   (b) $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$;

   (c) $x \neq \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x \neq \frac{-b-\sqrt{b^2-4ac}}{2a}$;

   (d) It is always invertible.

4. Let $A = \begin{pmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{pmatrix}$. For which values of a does the matrix A have $0, 3$ and -3 as eigenvalues?

   (a) $a = 0$;
   (b) $a = 1$;
   (c) No value of a;
   (d) $a = \sqrt{2}$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (2x + y, 0), S(x, y) = (x + y, xy).$$

   Which of the following are linear transformations?

   (a) T;
   (b) S;
   (c) $S \circ T$;
   (d) $T \circ S$.

6. Find all of the eigenvalues of the matrix

$$A = \begin{pmatrix} 10001 & 3 & 5 & 7 & 9 & 11 \\ 1 & 10003 & 5 & 7 & 9 & 11 \\ 1 & 3 & 10005 & 7 & 9 & 11 \\ 1 & 3 & 5 & 10007 & 9 & 11 \\ 1 & 3 & 5 & 7 & 10009 & 11 \\ 1 & 3 & 5 & 7 & 9 & 10011 \end{pmatrix}.$$

   (a) 0 and 1;
   (b) 1000 and 1036;
   (c) 10000 and 10036;
   (d) 1, 10, 100, 1000, 10000 and 100000.

7. Find the characteristic polynomial of the matrix

$$\begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 7 & 9 \end{pmatrix}$$

(a) $-\lambda^3 + 13\lambda^2 - 24\lambda - 6$;

(b) $\lambda^3 - 13\lambda^2 + 24\lambda + 6$;

(c) $-\lambda^3 - 3\lambda^2 - 12\lambda - 6$;

(d) $\lambda^3 + 13\lambda^2 - 24\lambda - 6$.

8. Let V be the vector space of continuous real-valued functions in the interval $[-\pi, \pi]$ with the inner product defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\mathrm{d}t$. Let $S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \dots\}$. Then

(a) S is orthogonal;

(b) S is orthonormal;

(c) S is a basis for V.

9. Consider $V = \mathbb{Z}_3^n$ as a vector space over $\mathbb{Z}_3$. How many subspaces of dimension 1 does V have?

(a) $(3^n - 1)$;

(b) $3n$;

(c) $(3^n - 1)/2$;

(d) 1.

10. A graph G is a pair (V, E) where $V = \{1, \dots, n\}$ is a set of vertices, and E is a set of pairs of vertices called *edges*. If $\{i, j\} \in E$ we say that i and j are adjacent. Let A be the $n \times n$ matrix where $A_{ij} = 1$ if i is adjacent to j and $A_{ij} = 0$ otherwise. Suppose that every vertex of G is adjacent to exactly d other vertices. Then

(a) A is always invertible;

(b) A is upper triangular;

(c) $(1, \dots, 1)$ is an eigenvector of A.

11. Let P_3 be the vector space of polynomials in R of degree at most 3. Let $D : P_3 \to P_3$ be the differential operator defined by $D(p(t)) = dp/dt$. Which of the following is the matrix of D with respect to the basis $\{1, t, t^2, t^3\}$?

(a) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(c) $\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

12. Let A be an $n \times n$ matrix and λ an eigenvalue of A with eigenvector v. Which of the following statements is true?

(a) $-v$ is an eigenvector of $-A$ with eigenvalue $-\lambda$;

(b) If B is an $n \times n$ matrix and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB;

(c) Let c be a scalar. Then $(\lambda + c)^2$ is an eigenvalue of $A^2 + 2cA + c^2 I$;

(d) If μ is an eigenvalue of an $n \times n$ matrix B, then $\lambda + \mu$ is an eigenvalue of $A + B$;

(e) $-\lambda$ is a root of the characteristic polynomial of A.

13. Let $f(x) = 1/(1 + x)$. What is the n-th derivative of f?

(a) $n!(1 + x)^{n+1}$;

(b) $-n!/(1 + x)^{n+1}$;

(c) $-n!/(1 + x)^n$;

(d) $(-1)^n n!/(1 + x)^{n+1}$.

14. Find the derivative with respect to x of $\sqrt{x+\sqrt{x+\sqrt{x}}}$.

   (a) $\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{\sqrt{x}}\right)\right]$;
   (b) $\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$;
   (c) $\frac{1}{\sqrt{x+\sqrt{x}}}\left[1+\frac{1}{\sqrt{x}}\left(1+\frac{1}{\sqrt{x}}\right)\right]$;
   (d) $\frac{2}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{2}{\sqrt{x+\sqrt{x}}}\left(1+\frac{2}{\sqrt{x}}\right)\right]$.

15. The following functions are defined for all values of x except $x = 0$. Which functions can be defined at $x = 0$ so that the function will be continuous at $x = 0$?

   (a) $(1/x)$;
   (b) $\cos(1/x)$;
   (c) $\frac{\tan(x)}{x}$;
   (d) x/x^2.

16. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ a function such that its Taylor series converges to $f(x)$ for every real number x. If $f(0) = 2$, $f'(0) = 2$ and $f^{(n)}(0) = 3$ for every $n \geq 2$, then $f(x)$ is equal to

   (a) $3e^x + 2x - 1$;
   (b) $e^{3x} + 2x + 1$;
   (c) $3e^x - x - 1$;
   (d) $e^{3x} - x + 1$.

17. Let
$$f(x) = \int_{\frac{\pi}{2}}^{x} \sqrt[3]{\sin t}\,dt.$$
At which value of x in the interval $[0, 2\pi]$ does $f(x)$ attain its maximum?

   (a) 0;
   (b) $\pi/2$;
   (c) π;
   (d) $3\pi/2$.

18. A particle moves in a straight line so that its velocity at time t is equal to $v(t) = 3t^3$. At what time t in the interval $t = 0$ to $t = 9$ is its velocity equal to its average velocity during the whole interval?

(a) 3;
(b) 4;
(c) $3\sqrt[3]{3}$;
(d) $\frac{9}{2}\sqrt[3]{2}$.

19. Let

$$a_n = n\sin\left(\frac{1}{n}\right) + (-1)^n\frac{\cos(n)}{n}.$$

What statements are true about the sequence a_n?

(a) It converges to 0;
(b) It is bounded but does not converge;
(c) It converges to a positive number.
(d) It diverges.

20. Which of the following conditions is necessary for a function f to be Riemann integrable in the closed interval $[a, b]$?

(a) f is bounded in $[a, b]$;
(b) f is continuous in $[a, b]$;
(c) f is differentiable in $[a, b]$.

21. Calculate the improper integral $\int_1^\infty \frac{1}{e^x+1}dx$.

(a) $\ln(1+e^{-1})$;
(b) $-\ln(1+e^{-1})$;
(c) It does not exist;
(d) $\ln(1+e)$.

22. Which of the following is the equation of the tangent plane to the surface $x^2+y^2-3z = 2$ at the point $(-2, -4, 6)$?

(a) $x+y+z-2 = 0$;
(b) $-2x-4y+6z = 0$;
(c) $-2x-4y+6z-2 = 0$;
(d) $4x+8y+3z+22 = 0$.

23. Let $u = xy^z$. Calculate

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

(a) $3u$;

(b) $u(1+z+z^2/y)$;

(c) $u(1+z^2/y+and\ln z)$;

(d) $u(1+z+z\ln y)$.

24. Calculate

$$\int_0^2 \int_0^1 x^3 and e^{x^2y^2} dxdy.$$

(a) $\frac{e^4-5}{16}$;

(b) $\frac{e^4}{8} - \frac{1}{72}$;

(c) $\frac{e^4}{4} - 1$;

(d) $\frac{e^4}{2} - \frac{1}{18}$.

25. How many of its elements generate $\mathbb{Z}_{12}$?

(a) 1;

(b) 4;

(c) 3;

(d) 6.

26. Let G be a multiplicative group such that $(ab)^{-1} = a^{-1}b^{-1}$ for every a, b in G. Then

(a) G is abelian;

(b) G is not abelian;

(c) G is simple;

(d) G is cyclic.

27. Let S_n be the group of permutationes of $\{1, \dots, n\}$. Let H be the group generated by the permutations $(1,2)$ and $(1,2,3,\dots,n)$. Which of the following statements is valid?

(a) H is abelian;

(b) H the dihedral group D_n;

(c) H is the alternating group A_n;

(d) H is all of S_n.

28. Which of the following functions is **not** uniformly continuous in $(0, 1)$?

(a) x^2;

(b) $1/x^2$;

(c) $f(x) = 1$ for $x \in (0, 1)$ and $f(0) = f(1) = 0$;

(d) $\frac{(x)}{x}$.

29. Which of the following functions define a métric in $\mathbb{R}$?

(a) $d(x, y) = xy$;

(b) $d(x, y) = 0$ si $x = y$ and $d(x, y) = 1$ si $x \neq y$;

(c) $d(x, y) = \max\{|x|, |y|\}$;

(d) $d(x, y) = (x - y)^2$.

30. Which of the following are compact subsets of $\mathbb{R}$?

(a) $[0, 1] \cup [5, 6]$;

(b) $\{x \in \mathbb{R} : x \geq 0\}$;

(c) $\{x \in \mathbb{R} : 0 \leq x \leq 1$ x is irracional $\}$;

(d) $\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$.