Admission Exam

Department of Mathematics, Cinvestav. June 30, 2017

Nombre: ____

Instructions: For each question circle the correct answers. For each question there is at least one and possible more than one correct answer; fill all answers that apply.

Exam duration: 2 hours.

1. Which of the following sets is a subspace of $\mathbb{Q}^n$?

   (a) $\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : \text{ where every } x_i \text{ is an integer }\}$;

   (b) $\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : \text{ where } x_1 \text{ or } x_2 \text{ is zero}\}$;

   (c) $\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : \text{ where } x_1 = 0\}$;

   (d) $\{(x_1, \ldots, x_n) \in \mathbb{Q}^n : \text{ where } 3x_1 + 4x_2 = 1\}$.

2. Let V be the vector space of all continuous real valued functions on the interval $[0, 1]$. The dimension of V is:

   (a) finite;

   (b) infinite.

3. If $T : U \to V$ is a linear transformation from U to V, then

   (a) the kernel of T is a subspace of U;

   (b) the kernel of T is a subspace of V;

   (c) the image of T is a subspace of U;

   (d) the image of T is a subspace of V;

   (e) V is equal to the image of T if and only if $\ker T = \{0\}$.

4. Let P_3 be the vector space of polynomials on R of degree at most 3. Let $D : P_3 \to P_3$ be the differential operator defined by $D(p(t)) = dp/dt$. Which of the following is the matrix of D with respect to the basis $\{1, t, t^2, t^3\}$.?

(a) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(c) $\begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

5. Let A and B be two real $n \times n$ matrices. Let $\text{tr}(A)$ be the trace of A. Which of the following statements always holds?

(a) $\text{tr}(AB) = \text{tr}(BA)$;

(b) $\text{tr}(AB) = \text{tr}(A)\,\text{tr}(B)$;

(c) $\text{tr}(ABC) = \text{tr}(ACB)$;

(d) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

6. The matrix $\begin{pmatrix} 1/9 & 8/9 & -4/9 \\ 4/9 & -4/9 & -7/9 \\ 8/9 & 1/9 & 4/9 \end{pmatrix}$ is

(a) upper triangular;

(b) orthogonal;

(c) symmetric;

(d) invertible.

7. The vectors $(1+i, 2i)$, $w = (1, 1+i)$ are

   (a) linearly dependent;

   (b) linearly independent;

   (c) linearly dependent over $\mathbb{R}$;

   (d) linearly dependent over $\mathbb{C}$.

8. Let u, v and w be linearly independent vectors. Then $u+v$, $u-v$ y $u-2v+w$:

   (a) are always linearly independent;

   (b) are always linearly dependent;

   (c) they may be linearly dependent or linearly independent.

9. Let V be vector space of all polynomials in t of degree at most n. The following are bases of V:

   (a) $\{1, t, \ldots, t^n\}$;

   (b) $\{1, t-1, (t-1)^2, \ldots, (t-1)^n\}$;

   (c) $\{1+t, t+t^2, t^2+t^3, \ldots, t^{n-1}+t^n\}$.

10. Which of the following sets is a basis for the subspace of $\mathbb{R}^4$ of all vectors orthogonal to $(0, 1, 1, 1)$ and $(1, 1, 1, 0)$?

    (a) $\{(0, -1, 1, 0)\}$;

    (b) $\{(1, 0, 0, 0), (0, 0, 0, 1)\}$;

    (c) $\{(-2, 1, 1, -2), (0, 1, -1, 0)\}$

    (d) $\{(1, -1, 0, 1), (-1, 1, 0, -1), (0, 1, -1, 0)\}$;

    (e) $\{(0, 0, 0, 0), (-1, 1, 0, -1), (0, 1, -1, 0)\}$.

11. Let V be the vector space of all continuous real valued functions on the interval $[-\pi, \pi]$ with interior product defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\mathrm{d}t$. Let $S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots\}$. Then

    (a) S is orthogonal;

    (b) S is orthonormal;

    (c) S is a basis of V.

12. Let $V = \mathbb{Z}_3^n$ be a vector space over $\mathbb{Z}_3$. How many subspaces of dimension equal to 1 does V have?

    (a) $(3^n - 1)$;

    (b) $3n$;

    (c) $(3^n - 1)/2$;

    (d) None, V is not a vector space.

13. Find the characteristic polynomial of $\begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & 1 \\ 4 & 0 & -2 \end{pmatrix}$.

    (a) $t^3 - t^2 - 2t + 4$;

    (b) $t^3 - t^2 + 2t$;

    (c) $t^3 + t^2 + 2t + 4$;

    (d) $t^3 - t^2 + 2t + 4$.

14. A graph G is a pair (V, E) where $V = \{1, \ldots, n\}$ is a finite set of *vertices*, and E is a set of paris of these vertices; these are called *edges*. If $\{i, j\} \in E$, we say that i and j are adjacent. Let A be the $n \times n$ matrix where $A_{ij} = 1$ if i is adjacent to j and $A_{ij} = 0$ otherwise. Suppose that every vertex of G is adjacent to exactly d other vertices of G. Then:

    (a) A is always invertible;

    (b) A is always upper triangular;

    (c) $(1, \ldots, 1)$ is an eigenvector of A.

15. Let $B = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$; then

    (a) B is diagonalizable with $P^{-1}BP = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}$, and $P = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$;

    (b) B is diagonalizable with $P^{-1}BP = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$, and $P = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$;

    (c) B is diagonalizable with $P^{-1}BP = \begin{pmatrix} -5 & 0 \\ 0 & -2 \end{pmatrix}$, and $P = \begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix}$;

    (d) is not diagonalizable.

16. The series

$$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n} \text{ y } \sum_{n=1}^{\infty} \frac{3^n n!}{n^n}:$$

(a) are both convergent;

(b) are both divergent;

(c) the first diverges and the second converges;

(d) the first converges the second diverges.

17. Let $\{a_n\}$ be the sequence defined recursively as follows. $a_1 = \sqrt{2}$ y $a_n = \sqrt{2 + a_{n-1}}$. Then the sequence $\{a_n\}$

(a) diverges;

(b) converges to 2;

(c) converge to $\frac{2}{\sqrt{2}}$;

(d) converge to e.

18. Compute

$$\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right).$$

(a) ∞;

(b) 1;

(c) 0;

(d) π.

19. Compute

$$\lim_{x\to y} \frac{x^n - y^n}{x - y}.$$

(a) 0;

(b) ∞;

(c) ny^{n-1};

(d) nx^{n-1}.

20. The function

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0 \end{cases}$$

(a) is discontinuous at $x = 0$;

(b) is continuous at all values of x.

21. Find the derivative with respect to x of $\sqrt{x+\sqrt{x+\sqrt{x}}}$.

(a) $\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{\sqrt{x}}\right)\right]$;

(b) $\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$;

(c) $\frac{1}{\sqrt{x+\sqrt{x}}}\left[1+\frac{1}{\sqrt{x}}\left(1+\frac{1}{\sqrt{x}}\right)\right]$;

(d) $\frac{2}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{2}{\sqrt{x+\sqrt{x}}}\left(1+\frac{2}{\sqrt{x}}\right)\right]$.

22. Find the maxima of the function $f(x)=3\sqrt[3]{x^2}-x^2$.

(a) $x=2$ y $x=-2$;

(b) $x=0$;

(c) $x=\sqrt{2}$ y $x=-\sqrt{2}$;

(d) $x=1$ y $x=-1$.

23. How many inflection points are on the curve defined by $y=\frac{x+1}{x^2+1}$?

(a) 3;

(b) 0;

(c) 2;

(d) 1.

24. Which of the following are asymptotes of the curve defined by $y=\frac{3x}{2}\ln\left(e-\frac{1}{3x}\right)$?

(a) $x=0$;

(b) $x=1/(3e)$;

(c) $y=\frac{3x}{2}-\frac{1}{2e}$;

(d) $y=-\frac{3x}{2}-\frac{1}{2}$.

25. Compute the indefinite integral $\int\frac{1}{1+e^x}\mathrm{d}x$.

(a) $\ln(1+e^x)+C$;

(b) $x+\ln(1+e^x)+C$;

(c) $x-\ln(1+e^x)+C$;

(d) $x-\ln(1-e^x)+C$.

26. Compute the indefinite integral $\int x \ln\left(1+\frac{1}{x}\right) \mathrm{d}x$.

(a) $\frac{1}{2}(x^2-1)\ln(x+1) - \frac{x^2}{2}\ln x + \frac{x}{2} + C$;

(b) $\frac{1}{2}(x^2+1)\ln(x-1) - \frac{x^2}{2}\ln x + \frac{x}{2} + C$;

(c) $\frac{1}{2}(x^2-1)\ln(x+1) + \frac{x^2}{2}\ln x + \frac{x}{2} + C$;

(d) $\frac{1}{2}(x^2+1)\ln(x-1) + \frac{x^2}{2}\ln x - \frac{x}{2} + C$.

27. Compute the area bounded by the parabolas $x = -2y^2$ and $x = 1 - 3y^2$.

(a) $\sqrt{2}$;

(b) $\frac{4}{3}$;

(c) $\frac{3}{4}$;

(d) $\frac{\sqrt{2}}{2}$.

28. Compute the tangent plane to the surface $z = (\cos x)(\cos y)$ at the point $(0, \pi/2, 0)$.

(a) $z + y = \pi/2$;

(b) $x + y = \pi/2$;

(c) $z - y = \pi/2$;

(d) $x - y = \pi/2$.

29. Compute the volume of the region bounded by the surface $z = x^2 + y$, and the planes $x = 0$, $x = 1$, $y = 1$, $y = 2$ and $z = 0$.

(a) $\frac{11}{6}$;

(b) 2;

(c) $\frac{13}{6}$;

(d) $\sqrt{2}$.

30. Compute the matrix of partial derivatives of $f(x, y) = (xe^y + \cos y, x, x + e^y)$.

(a) $\begin{pmatrix} e^y & xe^y - \sin y \\ x & e^y - \cos y \\ 1 & e^y \end{pmatrix}$;

(b) $\begin{pmatrix} xe^y & xe^y - \sin y \\ x & 0 \\ 1 & e^y \end{pmatrix}$;

(c) $\begin{pmatrix} e^y & e^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{pmatrix}$;

(d) $\begin{pmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{pmatrix}$.