Centre for Research and Advanced Study at IPN

Department of Mathematics

Master' Degree Program Admission Examination

July 17, 2000

1. Linear Algebra

1.1 Consider the matrix given by:

$$A = \begin{pmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{pmatrix}$$

a) Prove that matrix A is not similar to a diagonal matrix on real numbers.
b) Prove that matrix A is not similar to a diagonal matrix on complex numbers.

Remember that a matrix A of size n x n is similar to a diagonal matrix on the real numbers (or the complex numbers) if they exist D matrix diagonal and P reversible matrix both n x with real entries (complex entries, respectively) such that $A = PDP^{-1}$

1.2 Let $\{v1....v_n\}$ an orthonormal set of $\mathbb{R}^n$ that is, $<v_i, v_i> = 1$ for each i, and $<vi, vj> = 0$ if $i \neq j$. Prove that {v1, ..., vn} is a basis of $\mathbb{R}^n$.

1.3 Let $T : V \longrightarrow W$ be a linear transformation between two real vector spaces of finite dimension. Prove that

$$\dim V = \dim T(V) + \dim T^{-1}(0).$$

2. Calculus

2.1 Calculate the derivative of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \int_{\sqrt{1+x^2}}^{x^3} t(t+1)dt$$

2.2 Let $n \geq 1$ an integer number and $f : \mathbb{R} \longrightarrow \mathbb{R}$ a polynomial function of grade n, that is, there are real constants $a_n, \ldots, a_0,$ with $a_n \neq 0,$ such that f is given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$. Prove that f is uniformly continuous on $\mathbb{R}$ if and only if n = 1.

2.3 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function with a continuous derivative. Prove that the restriction of f is any bounded interval is Lipschitz. In other words, prove that for each bounded interval I there is a constant C (that depends of I) such that $|f(x) - f(y)| \leq C|x - y|$ for each $x, y \in I$.

3 optional problems

3.1 Let G be a finite group and H a subgroup of G of index 2. Prove that H is normal in G.

3.2 Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ a holomorphic function which has constants $A, B > 0$ and an integer $n \geq 0$ such that $|f(z)| \leq A|z|_n + B$ for each $z \in \mathbb{C}$. Prove that f is a polynomial of lesser grade or equals to n.

3.3 Let X be a connected topological space and locally arc-connected. Prove that X is arc-connected.

3.4 Let I be an interval of a real straight line. Prove that $L_2(I) \subset L_1(I)$ if and only if I is d finite length. Remember that for each real number $p \geq 1$ it is defined $L_p(I) = \{f : I \longrightarrow \mathbb{R} | f$ is measurable and $\int_I |f|^p < \infty\}$.