Centre for Research and Advanced Study at IPN

Department of Mathematics

Master' Degree Program Admission Examination

January 15, 2009

1. Linear Algebra

1.1 Let S be the vector subspace of M n x n (the vector space of the real matrices of n x n) generated by all matrices of the form AB – BA in $M_{n\times n}(\mathbb{R})$. Prove that:

$$\dim(S) = n^2 - 1.$$

1.2 Let $P, Q \in M_{n\times n}(\mathbb{R})$ be such $P^2 = P$, $Q^2 = Q$ and $I_{n\times n} - P - Q$ a reversible matrix. Prove that P and Q have the same range.

1.3 Let $A \in M_{n\times n}(\mathbb{R})$ be such that $A^3 = I_{n\times n}$ and $A \neq I_{n\times n}$.

a) Which of the following eigenvalues of A?
b) Give an example of a matrix that satisfies these conditions.

2. Calculus

2.1 Let $f : [1, \infty) \to \mathbb{R}$ be a real function, such f(1) = 1 and

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that

$$\lim_{x\to\infty} f(x)$$

exists and that is minor than $1 + \frac{\pi}{4}$.

Note: $f'(x)$ is the derivative of F in x.

2.2 Let $f, g : [0,1] \to [0, \infty)$ continuous functions that satisfy :

$$\sup_{0\leq x\leq 1} f(x) = \sup_{0\leq x\leq 1} g(x).$$

Prove that $t \in [0,1]$ exist such as $f(t)^2 + 3f(t) = g(t)^2 + 3g(t).$

2.3 Prove that as an counter example for each of the following sentences:

a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be real functions such

$$\lim_{t \to a} g(t) = b \text{ y } \lim_{t \to b} f(t) = c.$$

Then $\lim_{t \to a} f(g(t)) = c$

b) if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and U an open set of R, then $f(U)$ is an open set of $\mathbb{R}$.

3. Optional Problems

3.1 Let n > 1 be an integer. Prove that the sum:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

it is not an integer

3.2 Let $\mathbb{Z}[x]$ be the ring of polynomials of a variable on the integers and

$$I = \langle 5, x^2 + 2 \rangle$$

the ideal of $\mathbb{Z}[x]$ generated by 5 and $x^2 + 2$. Prove that the ideal I is a maximal ideal.

3.3 Calculate the following integral:

$$I = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{\operatorname{sen} 4z},$$

where the direction of integration is counter-clockwise.

3.4 Calculate the following integral:

$$\int_0^{2\pi} e^{(e^{i\theta} - i\theta)} d\theta.$$