Admissions Examination for Master's / Doctorate
15 December 2017

Name: ____

Instructions: For each question encircle the correct answers. There may be several correct solutions for a single question (choose all, but points will be taken off for incorrectly chosen options). You may make calculations on the paper you are given, but you do not need to hand that in. The examination has 30 questions. We suggest reading all of the statements first. **Calculators or cell phones not allowed.**

Duration of the examination: 2 hours

1. Which of the following sets is a vector space?

   ◯ $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - 3x_3 = 2\}$;

   ◯ the set of matrices 2×2 such that $\det(A) = 0$;

   ◯ the set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)dx = 0$;

   ◯ the set of real numbers with the sum given by $x \oplus y = xy$ and the scalar multiplication by $a \otimes x = x^a$.

2. Let V be the vector space of the real numbers over the field of the rational numbers. Which of the following statements are true?

   ◯ $\dim(V)$ is countable;

   ◯ $\dim(V)$ is not countable;

   ◯ $\dim(V) = 1$;

   ◯ V has no basis.

3. Let $S := \{(1, 1, 1, x), (1, 1, x, 1), (1, x, 1, 1,), (x, 1, 1, 1,)\}$. For how many distinct values of x, is S not a basis for $\mathbb{R}^4$?

   ◯ 0;

   ◯ 1;

   ◯ 2;

   ◯ any value of x.

4. Consider k linear equations in n variables, which written in matrix form result in the equation $AX = Y$. Which of the following statements are true?

   - ◯ if $n = k$ then there is always at most one solution;
   - ◯ if $n > k$ then $AX = Y$ can always be solved;
   - ◯ of $n < k$ then for some Y there is no solution of $AX = Y$.
   - ◯ if $n < k$ then the only solution of $AX = 0$ is $X = 0$.

5. Let A be a square real matrix. The statement "A is invertible if and and only if zero is not a eigenvalue." is

   - ◯ false;
   - ◯ true.

6. The statement "Every monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ es the characteristic polynomial of some matrix $n \times n$" is:

   - ◯ false;
   - ◯ true.

7. Let V be the vector space of real valued continuous functions in the interval $[-\pi, \pi]$ with inner product defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\mathrm{d}t$.
   Let $S = \{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots\}$. Then

   - ◯ S is orthogonal;
   - ◯ S is orthonormal;
   - ◯ S is a basis for V.

8. Consider $V = \mathbb{Z}_3^n$ as a vector space over $\mathbb{Z}_3$. How many subspaces of dimensión 1 does V have?

   - ◯ $(3^n - 1)$;
   - ◯ $3n$;
   - ◯ $(3^n - 1)/2$;
   - ◯ 1.

9. Find the characteristic polynomial of $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 1 \\ -4 & 0 & -2 \end{pmatrix}$.

- ◯ $-t^3 + t^2 + 10t - 4$;
- ◯ $t^3 + t^2 + 10t - 4$;
- ◯ $t^3 - t^2 - 2t - 4$;
- ◯ $t^3 - t^2 + 2t + 4$.

10. A graph G is a pair (V, E) where $V = \{1, \ldots, n\}$ es a set of vertices, and E is a set of pairs of vertices called *edges*. If $\{i, j\} \in E$ we say that i and j are adjacent. Let A be the $n \times n$ matrix where $A_{ij} = 1$ if i is adjacent to j and $A_{ij} = 0$ otherwise. Suppose tht every vertex of G is adjacent to exactly d other vertices. Then:

- ◯ $(1, \ldots, 1)$ is an eigenvector of A.
- ◯ A is always invertible;
- ◯ A is upper triangular ;

11. Suppose that the matrix A is similar to the matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then:

- ◯ $A^2 = A$;
- ◯ $\det(A) = 0$;
- ◯ $\operatorname{trace}(A) = 1$;
- ◯ $\lambda = 0$ is an eigenvalue of A;
- ◯ $\lambda = 1$ is an eigenvalue of A.

12. Let A be an $n \times n$ matrix and λ an eigenvalue of A with eigenvector v. Which of the following statements is true?

- ◯ $-v$ is an eigenvector of $-A$ with eigenvalue $-\lambda$.
- ◯ If B is an $n \times n$ matrix and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB.
- ◯ Let c be a scalar. Then $(\lambda + c)^2$ is an eigenvalue of $A^2 + 2cA + c^2 I$.
- ◯ If μ is an eigenvalue of an $n \times n$ matrix B , then $\lambda + \mu$ is an eigenvalue of $A + B$.
- ◯ $-\lambda$ is a root of the characteristic polynomial of A.

13. Calculate

$$\lim_{x\to\infty} \sqrt{\frac{x^3+7x}{4x^3+5}}.$$

- ○ ∞;
- ○ $-\infty$;
- ○ 0;
- ○ $\frac{1}{2}$.

14. Calculate

$$\lim_{x\to 0^+} x^x.$$

- ○ ∞;
- ○ 1;
- ○ 0;
- ○ e.

15. Let α be a real number. Consider the series

$$\sum_{n=1}^{\infty} n^{\alpha n}.$$

For which values of α does this series converge?

- ○ $\alpha \leq 0$;
- ○ $\alpha < 0$;
- ○ $\alpha \leq -1$;
- ○ $\alpha < -1$.

16. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that its Taylor series converges to $f(x)$ for every real number x. If $f(0) = 2$, $f'(0) = 2$ and $f^{(n)}(0) = 3$ for every $n \geq 2$, then $f(x)$ is equal to

- ○ $3e^x + 2x - 1$;
- ○ $e^{3x} + 2x + 1$;
- ○ $3e^x - x - 1$.
- ○ $e^{3x} - x + 1$.

17. Let $\{a_n\}$ be the sequence defined recursively as follows: $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$. Then the sequence $\{a_n\}$

- ◯ diverges;
- ◯ converges to $\frac{2}{\sqrt{2}}$;
- ◯ converges to 2;
- ◯ converges to e.

18. Let $f(x) = 1/(1+x)$, for $x \neq -1$. What is the n-ésima derivative of $f(x)$?

- ◯ $(-1)^n n!/(1+x)^{n+1}$.
- ◯ $n!(1+x)^{n+1}$;
- ◯ $-n!/(1+x)^{n+1}$;
- ◯ $n!/(1+x)^{n+1}$;

19. What is the maximum of the function $f(x,y) = x^2 y$, dado que $x^2 + y^2 = 1$?

- ◯ $\frac{4\sqrt{3}}{27}$;
- ◯ $\frac{2\sqrt{3}}{9}$;
- ◯ $\frac{\sqrt{3}}{9}$;
- ◯ $\frac{2}{3}$.

20. Calculate the improper integral $\int_0^\infty e^{-x^2} dx$.

- ◯ $\sqrt{\pi}$;
- ◯ $\frac{\pi}{2}$;
- ◯ $\frac{\sqrt{\pi}}{2}$;
- ◯ 1.

21. Calculate the integral defined by $\int_0^{\pi/4} x^2 \cos x dx$.

- ◯ $(\pi^2 + 8\pi - 32)$;
- ◯ $\frac{\sqrt{2}}{32}(\pi^2 + 8\pi - 32)$;
- ◯ $\sqrt{2}(\pi^2 + 8\pi - 32)$;
- ◯ $\frac{1}{32}(\pi^2 - 8\pi + 32)$.

22. Let C be curve in $\mathbb{R}^3$ defined by the parametric equations

$$x(t) = \cos(e^t);$$
$$y(t) = \sin(e^t);$$
$$z(t) = e^t;$$

with $t \in [0, 2]$ What is the length of the curve?

- ◯ $\int_0^2 \sqrt{1 + e^{2t}} dt$;
- ◯ $e^4 - 1$;
- ◯ $\frac{e^4+3}{2}$.
- ◯ $\sqrt{2}(e^2 - 1)$;

23. What is the volume of the closed region in $\mathbb{R}^3$ bounded by $z = 9 - x^2 - y^2$ and $z = 0$?

- ◯ $\frac{27\pi}{2}$
- ◯ 18π;
- ◯ $\frac{81\pi}{2}$
- ◯ 81π.

24. $\frac{\partial \sin(xy)}{\partial x^2 \partial y} =$

- ◯ $-xy^2 \cos(xy) - 2y \sin(xy)$;
- ◯ $-x^2 y \sin(xy) - 2y \sin(xy)$;
- ◯ $-x^2 y \sin(xy) - 2x \cos(xy)$;
- ◯ $x^2 y \sin(xy)$.

25. Let G be a finite group such that for every pair of subgrups H, K of G we have $H \subset K$ or $K \subset H$. Which of the following statements are always true?

- ◯ G is cyclic prime of order.
- ◯ G could be nonabelian.
- ◯ G is a cyclic group of order a power of a prime.
- ◯ G has only two subgroups.

26. Let S_n be the group of permutations of $\{1, \dots, n\}$. Let H be the group generated by the permutations $(1, 2)$ and $(1, 2, 3, \dots, n)$. Which of the following statements is valid?

- ○ H is abelian;
- ○ H the dihedral group D_n;
- ○ H is the alternating group A_n;
- ○ H is all of S_n.

27. How many of its elements generate $\mathbb{Z}_9$?

- ○ 1;
- ○ 6;
- ○ 9;
- ○ 8.

28. Which of the following functions define a metric in $\mathbb{R}$?

- ○ $d(x, y) = xy$;
- ○ $d(x, y) = 0$ si $x = y$ and $d(x, y) = 1$ si $x \neq y$.
- ○ $d(x, y) = \max\{|x|, |y|\}$
- ○ $d(x, y) = (x - y)^2$.

29. Which of the following are compact subsets of $\mathbb{R}$?

- ○ $[0, 1] \cup [5, 6]$;
- ○ $\{x \in \mathbb{R} : x \geq 0\}$;
- ○ $\{x \in \mathbb{R} : 0 \leq x \leq 1$ x is irracional $\}$;
- ○ $\{\frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$.

30. Which of the following functions are uniformly continuous?

- ○ $f(x) = \ln x$ in the interval $(0, 1)$;
- ○ $f(x) = x \sin x$ in the interval $[0, \infty]$;
- ○ $f(x) = \sqrt{x}$ in the interval $[0, \infty]$;
- ○ $f(x) = \frac{1}{x^2+1}$ in the interval (∞, ∞).