Centro de Investigación y de Estudios Avanzados del IPN
Departament of Mathematics

Admissions Examination for the Master's Program

1 July 2013

Name:____
Area: _________________________
Advisor: ____

Instructions: Solve all problems of sections 1 and 2 and as many as possible from section 3. All solutions must be justified appropriately. The examination will last for three hours.

1. **Linear algebra**

   1.1 An $n \times n$ square matrix A is called **nilpotent** if $A^r = 0$ for some integer $r \geq 1$. Let A, B be nilpotent matrices of the same dimension and suppose that $AB = BA$. Prove that AB and $A + B$ are nilpotent matrices

   1.2 Let $L\colon V \to W$ be una linear transformation, where V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Prove that the kernel of L is not $\{0\}$.

   1.3 Let T be a linear transformation on a vector space V of dimension n. Prove that if T has n **distinct** eigenvalues, then T is diagonalizable.

2. **Calculus**

   2.1 Let $f\colon I \to \mathbb{R}$. We say that f is **convex** if for all $a, b \in I$ and for any $0 < t < 1$,

   $$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

   Prove that if $I \subseteq \mathbb{R}$ is open and f is convex on I, then f is continuous.

   2.2 Prove that the following sequences $\{x_n\}_n$ and $\{y_n\}_n$ converge and find their respective limits, where

(a) $x_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}, \quad n \geq 1$

(b) $y_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n}), \quad n \geq 1.$

2.3 Find the extreme values of the function

$$f(x, y, z) = x^2 + y^2 - z,$$

subject to the restriction

$$2x - 3y + z - 6 = 0.$$

3. **Optional problems**

3.1 Let X and Y be topological spaces and suppose further that X is compact. If $f : X \to Y$ is a continuous function, prove that $f(X)$ is a compact set.

3.2 Let $\{f_n\}$ be a sequence of integrable functions $f_n\colon [a, b] \to \mathbb{R}$ on $[a, b]$ such that $\{f_n\}$ converges uniformly to a function $f\colon [a, b] \to \mathbb{R}$. Prove that f is integrable on $[a, b]$.

3.3 Prove that every group of order ≤ 5 is abelian.

3.4 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \left\{ \begin{array}{cc} cx(x-y) & \text{if} \quad 0 < x < 2, \quad -x < y < x \\ 0 & \text{otherwise,} \end{array} \right\}$$

where $c > 0$ is constant. Calculate the value of c for which f is a probability density.