**Centro de Investigación y de Estudios Avanzados del IPN**
Departament of Mathematics

**Admissions Examination for the Master's Program**

10 January 2014

Name: ____________________________________
Area: ______________________________________
Advisor: ____________________________________

**Instructions:** Solve all problems of sections 1 and 2 and as many as possible from section 3. All solutions must be justified appropriately. The examination will last for three hours.

1. **Linear algebra**

   **Notación:** Given a matrix $A$, $A^T$ means its transpose.

   1.1 We say that an $n \times n$ symmetric matrix $A$ is positive definite if $\langle Ax, x \rangle > 0, \forall x \in \mathbb{R}^n - \{0\}$. Prove that $A$ is symmetric positive definite if and only if $A = P^T P$ for some invertible matrix $P$.

   1.2 Let $A$ and $B$ be two $n \times n$ matrices such that $a_{ij},\ b_{ij} \in K$, where $K$ is a field. $A$ is said to be equivalent to $B$, written $A \sim B$, if there exists a matrix $C$ invertible over $K$ such that $A = CBC^{-1}$. Prove that:

   **(a)** "$\sim$" is an equivalence relation.

   **(b)** $P_A(\lambda) = P_B(\lambda)$, where $P_Q(\lambda)$ denotes the characteristic polynomial of the matrix $Q$.

   1.3 Let $V = \mathbb{R}^4$ and let $\langle \cdot, \cdot \rangle$ be its usual internal product. Let $v_1 = (1, 0, 1, 0)^T$, $v_1 = (1, 1, 2, 1)^T$, and $v_1 = (0, 1, 1, 2)^T$ be three vectores in $V$. Use the Gram-Schmidt process to find an orthonormal basis in $V$ starting from $v_1$, $v_2$ y $v_3$.

2. **Calculus**

2.1 For each $m \in \mathbb{N}$, define the function

$$f_m(x) = x^3 + 3x + m.$$

Prove that $f_m$ cannot have two distinct roots in the interval $[0, 1]$.

2.2 Let $f : \mathbb{R} \to \mathbb{R}_+$ be defined as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Define $Q(h) := f(h)/h$, if $h \neq 0$.

**(a)** Prove that $Q(h) \to 0$ when $h \to 0$.

**(b)** Prove that $f$ has a derivative at $0$ and calculate $f'(0)$.

2.3 Let $f : \mathbb{N} - \{0\} \to \mathbb{N} - \{0\}$ be an injective function. Prove that

$$\sum_{n \geq 1} \left( \frac{1}{(n+1)!} \prod_{k=1}^{n} f(k) \right)$$

diverges.

3. **Optional problems**

3.1 Tell whether the differential equation

$$\dot{x}(t) = 3x(t)^{2/3}, \quad t \geq 0,\ x(0) = 0,$$

has a solution and whether it is unique.

3.2 Let $X$ be a given set. For $x, y \in X$, define

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Prove that $(X, d)$ is a metric space.

3.3 Prove that if $H$ is an abelian normal subgroup in $G$ and $G/H$ is abelian, then $G$ is not necessarily abelian.

3.4 Let $\gamma$ be the perimeter of the square formed by the points $0$, $1$, $1 + i$, $i$ and let $z = x + iy$. Calculate

$$\int_\gamma x dz.$$