Centre for Research and Advanced Study at IPN

Department of Mathematics

Master' Degree Program Admission Examination

December 2009

Instructions: Solve all exercises in sections 1 and 2 and the ones you can in section 3. Justify you answers.

1. Linear Algebra

1.1 Consider the usual internal product $\mathbb{R}^n$, $\langle u, v \rangle$. Prove that for every matrix n x n, $A \in M_n(\mathbb{R})$ and vectors $u, v \in \mathbb{R}^n$ meets:

$$\langle Au, v \rangle = \langle u, A^t v \rangle,$$

where At is that traversal matrix of A:

1. A matrix $A \in M_n(\mathbb{R})$ it is said that orthogonal if $AA^t = I_n$. Prove that the following conditions are equivalent:

a) A is orthogonal.
b) $\langle Au, Av \rangle = \langle u, v \rangle, \quad \forall u, v \in \mathbb{R}^n.$
c) $\|Au\| = \|u\|, \quad \forall u \in \mathbb{R}^n.$
d) Columns in A form an orthogonal basis of $\mathbb{R}^n$.

1.2 Give an example of the orthogonal matrix 2 x 2 that is not the identity matrix.

1.3 Let the A be a matrix with different appropriate values $\lambda_1, \lambda_2, \lambda_3$ and corresponding appropriate values v_1, v_2, v_3. Prove that $\{v_1, v_2, v_3\}$ is a linearly independent set.

2. Calculus

2.1 Let $f, g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions of order n. Prove that the n derivative of the product is given by the formula.

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

where $f^{(k)}$ represents the k^{th} derivative of f and $f^{(0)} = f$.

2.2 Prove that the values of the following expressions do not depend on x.

$$(a).\ \int_0^x \frac{1}{1+t^2}dt + \int_0^{\frac{1}{x}} \frac{1}{1+t^2}dt \qquad (b).\ \int_{-cosx}^{senx} \frac{1}{\sqrt{1-t^2}}dt.$$

2.3 Determine the values for α following series is convergent:

$$\sum_{n=0}^{\infty} e^{\alpha n}.$$

3. Additional Problems

1. Let $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^t = I_n\}$ be a set of orthogonal matrices n x n.

a) Prove that O(n) is a group with product of matrices.
b) Prove that with induced topology for $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, $O(n)$ is compact.
c) Prove that if $A \in O(n)$ then $\det(A) = \pm 1$.
d) Let $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$ be the set of all rotations in set of all rotations in $\mathbb{R}^n$ (also known as the special orthogonal group).Prove that SO(n) is connected.

2. Find the value of the integral

$$\int_{-\infty}^{\infty} e^{-x^2}dx.$$

3. Let E be a normed vector space on C. Prove that the norm $\|.\|$ comes from an scalar product if and only if satisfies the identity of the parallelogram

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \forall x, y \in E.$$