Linear syzygies and birational combinatorics

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Abstract

Let F be a finite set of monomials of the same degree $d \geq 2$ in a polynomial ring $R = k[x_1, \ldots, x_n]$ over an arbitrary field k. We give some necessary and/or sufficient conditions for the birationality of the ring extension $k[F] \subset R^{(d)}$, where $R^{(d)}$ is the dth Veronese subring of R. One of our results extends to arbitrary characteristic, in the case of rational monomial maps, a previous syzygy-theoretic birationality criterion in characteristic zero obtained in [1].

1 Introduction

By the expression "birational combinatorics" we mean the theory of characteristic-free rational maps $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{m-1}$ defined by monomials, along with natural criteria for such maps to be birational onto their image varieties. Both the theory and the criteria are intended to be simple and typically reflect the monomial data, as otherwise one falls back in the general theory of birational maps in projective spaces (cf., e.g., [12], [15]).

A first incursion in this kind of theory was made in [16]. There one focused mainly on monomial rational maps whose base ideal (ideal theoretic base locus) was normal. Though the results were fairly complete and some of the techniques used there are repeated here, one felt that normality was a special case obscuring the general picture.

In the present paper we envisage a general theory focusing on the underlying combinatorial elements rather than on special algebraic properties of the base ideal. In this sense, what we accomplish goes in the opposite direction of recent work on birational maps, where the emphasis fell on special behavior of the base locus. On the other hand, we did draw upon [12] and [15] (also upon the ongoing [1]) by invoking the role played by the so-called linear syzygies of the coordinates of the rational map. The methods in the first two of these references are specially suited for the explicit computation of the inverse map of a birational map onto the image. To compromise between the two approaches, we show a bridge between them by means of comparing the respective linear algebra gadgets - from modules over the ground polynomial ring $k[x_1, \ldots, x_n]$ to modules over $\mathbb{Z}$. The challenge remains as to how one computes the inverse map by a purely combinatorial method.

We now describe the content of the paper in more detail. It goes without saying that the language throughout is algebraic or combinatorial, although we do add frequent remarks as to the geometric meaning of the results.

Section 2 sets up the scenario for the basic pertinent integer combinatorics. We emphasize two criteria of birationality in this setup - the *arithmetical principle of birationality* and the *determinantal principle of birationality*. These criteria were used in [16] and seem to be part of the folklore

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in the scattered literature. Then, we introduce the various versions of matrices that will play a distinctive role in the theory and, in particular, replay in more generality the passage from the transposed Jacobian matrix to the log-matrix of a set of monomials, as devised in [14]. Since we wish to remain characteristic-free, we take the formal Jacobian matrix rather than the ordinary one, as is explained in the section. The so-obtained numerical matrices allow for a first birationality criterion (Proposition 2.3). We then proceed to a full arithmetical characterization of birationality (Theorem 2.6).

Section 3 deals with the role of the Fitting ideals of monomial structures. We expand on the topic only enough in order to compare ranks between matrices over $k[x_1, \ldots, x_n]$ and matrices over $\mathbb{Z}$. As a side bonus, we characterize totally unimodular log-matrices in terms of Fitting ideals of the formal Jacobian matrix. The main result of the section is Theorem 3.7, which extends one of the results of [1] to all characteristics for monomial rational maps.

Section 4 focuses on the case of monomials of degree 2. Here, we give complete results, covering all previously known results and establishing facts that do not extend to higher degrees. We introduce the notion of *cohesiveness* for rational maps of any degree inspired by the graph theoretic concept of connectedness. We show, preliminarily, that the lack of cohesiveness is an early obstruction for birationality and for the existence of "enough" linear syzygies. If, moreover, the degree is 2 we show that cohesiveness is a necessary and sufficient condition for having a linear syzygy matrix of maximal rank (Proposition 4.6). We proceed to one of the main theorems of the section (Theorem 4.7) saying that a rational map of degree 2 is birational onto its image if and only if it is cohesive and the corresponding log-matrix has maximal rank. This comes to us as a bit of a surprise as it says that any cohesive coordinate projection of the 2-Veronesean that preserves dimension is birational onto the image; moreover, this holds in any characteristic. We have not met any explicit mention of this fact in the previous literature. We give examples to show how easily this fails for non-monomial rational maps and for monomial ones in degrees ≥ 3. Finally we care to translate the results into the language of graphs with loops.

The last section has the purpose of describing sufficiently ample classes of monomial rational maps that are birational. It is further subdivided in two subsections, the first of which is entirely devoted to classes of Cremona maps. We characterize Cremona transformations of degree 2 as those cohesive ones whose log-determinant is nonzero. The corresponding graph theoretic characterization is suited to construct other Cremona transformations of higher degree via a certain duality principle. The second subsection is a pointer to a recently studied class of combinatorial objects called polymatroidal monomial sets. This class includes the toric algebras of Veronese type which, from the geometric angle, constitutes a vast class of dimension preserving projections of the ordinary Veronese embeddings.

2 Birationality of monomial subrings

Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. As usual we set $x^\alpha := x_1^{a_1} \cdots x_n^{a_n}$ if $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$. In the sequel we consider a finite set of distinct monomials $F = \{x^{v_1}, \ldots, x^{v_q}\} \subset R$ of the same degree $d \geq 2$ and having no non-trivial common factor. We also assume throughout that F is not *conic*, i.e., that every x_i divides at least one member of F. By trivially contracting to less variables, any set of monomials can be brought to this form.

Two integer matrices naturally associated to F are:

$$A = (v_1, \ldots, v_q) \quad \text{and} \quad A' = \begin{pmatrix} v_1 & \cdots & v_q \\ 1 & \cdots & 1 \end{pmatrix},$$

where the v_i's are regarded as column vectors. We will often refer to A as the *log-matrix* of F.

If C is an integer matrix with r rows, we denote by $\mathbb{Z}C$ (resp. $\mathbb{Q}C$) the subgroup of $\mathbb{Z}^r$ (resp. subspace of $\mathbb{Q}^r$) generated by the columns of C. $\Delta_r(C)$ will denote the greatest common divisor of all the nonzero $r \times r$ minors of C.

An extension $D' \subset D$ of integral domains is said to be birational if it is an equality at the level of the respective fields of fractions. In the sequel let $\mathbf{x}_d$ denote the set of *all* monomials of degree d in R. Then $k[\mathbf{x}_d]$ is the *dth* Veronese subring $R^{(d)}$ of R. Our main aim is the birationality of the ring extension $K[F] \subset k[\mathbf{x}_d]$.

For convenience of reference, we quote the following easy results stated in [16]:

Lemma 2.1 (Arithmetical Principle of Birationality (APB)) *Let F and G be finite sets of monomials of R such that $F \subset G$, and let A, B be their respective log-matrices. Then $k[F] \subset k[G]$ is a birational extension if and only if $\mathbb{Z}A = \mathbb{Z}B$.*

Proof. In this situation, the ring extension is birational if and only every monomial of G can be written as a fraction whose terms are suitable power products of the monomials of F. Clearing denominators of such a fraction and taking log of both members establishes the required equivalence. □

Lemma 2.2 (Determinantal Principle of Birationality (DPB)) *Let F be a finite set of monomials of the same degree $d \geq 1$. Then $k[F] \subset k[\mathbf{x}_d]$ is a birational extension if and only if $\Delta_n(A) = d$.*

Proof. See [16, Proposition 1.2]. □

By $e_i, 1 \leq i \leq n$, we denote the canonical basis vectors of the vector space $\mathbb{R}^n$ (sometimes of the free module $\mathbb{Z}^n$, respectively, the $\mathbb{Q}$-vector space $\mathbb{Q}^n$). Let, as before, $F = \{x^{v_1}, \ldots, x^{v_q}\} \subset R$ be a set of monomials of the same degree $d \geq 2$.

Consider the following basic matrices:

(a) the matrix $\mathcal{LS}(F)$ of the so-called *linear syzygies* of F, whose columns are the set of vectors of the form $x_i e'_j - x_k e'_\ell$ such that $x_i x^{v_j} = x_k x^{v_\ell}$;

(b) the *numerical linear syzygy matrix* S obtained from $\mathcal{LS}(F)$ by making the substitution $x_i = 1$ for all i;

(c) the matrix M whose columns are the set of difference vectors $e_i - e_k$ such that $e_i - e_k = v_j - v_\ell$, for some pair of indices $j, \ell \in \{1, \ldots, q\}$ – in other words, $M = AS$;

(d) the *formal* Jacobian matrix

$$\Theta(F) = \left(\frac{\partial x^{v_j}}{\partial x_i}\right)_{\substack{1 \leq j \leq q \\ 1 \leq i \leq n}}$$

A word in order to explain the last matrix. The notion of derivative of a polynomial $f \in k[x_1, \ldots, x_n]$ usually requires the specification of a base field. However, if f is an ordinary monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ its *formal* partial derivative with respect to x_i is defined to be

$$a_i x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{a_i - 1} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}$$

regarded as a term in the polynomial ring $\mathbb{Z}[\mathbf{x}]$ (in particular it is always nonzero provided $a_i \geq 1$). The *formal Jacobian matrix* of $x^{v_1}, \ldots, x^{v_q}$ is accordingly defined. Of course, by applying the unique homomorphism from $\mathbb{Z}$ to k we find the ordinary partial derivatives and the ordinary Jacobian matrix over this ring.

Notice that the matrices in the first row of the diagram:

$$\begin{array}{ccc} \Theta(F)^t & \mathcal{LS}(F) & \mathcal{M} := \Theta(F)^t\mathcal{LS}(F) \\ \downarrow & \downarrow & \downarrow \\ A & S & M := AS \end{array}$$

specialize to the matrices in the second row by making $x_i = 1$ for all i. The matrices $\mathcal{LS}(F)$ and S have order $q \times r$, while the matrices $\mathcal{M}$ and M have order $n \times r$.

Here is a couple of uses of these matrices. The following notion will be used in the proof below: a matrix C is called *totally unimodular* if each $i \times i$ minor of C is 0 or ± 1 for all $i \geq 1$.

Proposition 2.3 *Let F be a finite set of monomials of the same degree $d \geq 2$.*

(i) *If* $\operatorname{rank}(M) = n - 1$, *then* $k[F] \subset k[\mathbf{x}_d]$ *is a birational extension.*

(ii) *If* $\operatorname{rank}(S) = q - 1$ *and* $\operatorname{rank}(A) = n$, *then* $\operatorname{rank}(M) = n - 1$. *In particular* $k[F] \subset k[\mathbf{x}_d]$ *is birational.*

Proof. (i) Let $a = (a_i) \in \mathbb{N}^n$ such that $|a| = \sum_i a_i = d$. By APB (Lemma 2.1) it suffices to prove that $a \in \mathbb{Z}A$. Let $w_1, \ldots, w_r$ be the column vectors of the matrix M. Each w_m is of the form $e_i - e_k = v_j - v_\ell$ for a unique pair $i \neq k$, $1 \leq i < k \leq n$ and suitable $j \neq \ell$, $1 \leq j < \ell \leq q$. Hence $\operatorname{rank}(A) = n$ because $v_1 \notin \mathbb{Q}M$. Therefore we can write

$$\lambda a = \lambda_1 w_1 + \cdots + \lambda_r w_r + \mu v_1 \quad (\lambda, \mu, \lambda_i \in \mathbb{Z}).$$

Taking inner product with $\mathbf{1} = (1, \ldots, 1)$ yields

$$\begin{aligned} \lambda d = \lambda|a| = \lambda_1|w_1| + \cdots + \lambda_r|w_r| + \mu|v_1| = \mu d \;\Rightarrow\; \lambda = \mu \\ \Rightarrow\; \lambda(a - v_1) = \lambda_1 w_1 + \cdots + \lambda_r w_r. \end{aligned} \tag{1}$$

Consider the digraph $\mathcal{D}$ with vertex set $X = \{x_1, \ldots, x_n\}$ such that the directed edges (x_i, x_k) correspond bijectively to the column vectors $e_i - e_k$ of M. The incidence matrix of $\mathcal{D}$ is M, thus M is totally unimodular [13, p. 274] and $\mathbb{Z}^n/\mathbb{Z}M$ is torsion-free. Hence from Eq. (1) we get $a - v_1 \in \mathbb{Z}M$ and $a \in \mathbb{Z}M + v_1 \subset \mathbb{Z}A$, as required.

(ii) Consider the $\mathbb{Q}$-linear maps

$$\mathbb{Q}^r \xrightarrow{S} \mathbb{Q}^q \xrightarrow{A} \mathbb{Q}^n.$$

Letting A_1 denote the restriction of A to $\operatorname{im}(S)$, we have a linear map

$$\operatorname{im}(S) \xrightarrow{A_1} \operatorname{im}(AS) = \operatorname{im}(M) \longrightarrow 0.$$

By hypothesis, $\dim(\operatorname{im}(A)) = n$ and $\dim(\operatorname{im}(S)) = q - 1$. Hence

$$\begin{aligned} q - 1 &= \dim(\operatorname{im}(S)) = \dim(\ker(A_1)) + \dim(\operatorname{im}(M)), \\ q - n &= \dim(\ker(A)) \geq \dim(\ker(A_1)). \end{aligned}$$

Therefore $\dim(\operatorname{im}(M)) \geq n - 1$. On the other hand, since $\operatorname{im}(M)$ is generated by vectors of the form $e_i - e_j$, certainly $e_1 \notin \operatorname{im}(M)$, hence $\dim(\operatorname{im}(M)) = n - 1$. $\square$

Remark 2.4 Let $\mathcal{D}$ be the digraph in the proof of Proposition 2.3(i). Then according to [7, Theorem 8.3.1] we have

$$\operatorname{rank}(M) = n - c,$$

where c is the number of connected components of $\mathcal{D}$. In particular M has rank $n-1$ if and only if $\mathcal{D}$ is connected.

To proceed with a full arithmetical characterization of birationality, we will need the following results on modules over $\mathbb{Z}$.

Lemma 2.5 (i) *Let $e_i, 1 \leq j \leq n$ be the canonical basis vectors of the free $\mathbb{Z}$-module $\mathbb{Z}^n$ and let $E \subset \mathbb{Z}^n$ be the submodule generated by the difference vectors $e_i - e_k, 1 \leq i < k \leq n$. Then E is freely generated by $\{e_1 - e_k \,|\, 2 \leq k \leq n\}$ and the quotient $\mathbb{Z}^n/E$ is torsionfree of rank one.*
(ii) *Let $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}^n$ be arbitrarily given. Then the injective $\mathbb{Z}$-homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n+1}$, $\alpha \mapsto (\alpha, 0)$, induces an injective homomorphism of $\mathbb{Z}$-modules*

$$\mathbb{Z}^n/\mathbb{Z}\left(\alpha_2 - \alpha_1, \ldots, \alpha_m - \alpha_1\right) \hookrightarrow \mathbb{Z}^{n+1}/\mathbb{Z}\left((\alpha_1, 1), \ldots, (\alpha_m, 1)\right),$$

which is an isomorphism at the level of the respective torsion submodules.

Proof. (i) This is simply the fact that E is the kernel of the $\mathbb{Z}$-homomorphism $\mathbb{Z}^n \to \mathbb{Z}$, $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$.

(ii) Clearly, there is an induced map as argued – because $\alpha_j - \alpha_1$ maps to $(\alpha_j - \alpha_1, 0) = (\alpha_j, 1) - (\alpha_1, 1)$ – and the induced map is injective – because the two equations $a_1 + \cdots + a_m = 0$ and $\alpha = a_1\alpha_1 + \cdots + a_m\alpha_m$ easily imply that $\alpha \in \mathbb{Z}\left(\alpha_2 - \alpha_1, \ldots, \alpha_m - \alpha_1\right)$.

Next, clearly any homomorphism maps torsion to torsion, so it remains to check surjectivity at the torsion level. Let then (α, b) be a torsion element of $\mathbb{Z}^{n+1}/\mathbb{Z}((\alpha_1, 1), \ldots, (\alpha_m, 1))$. This implies a relation

$$s(\alpha, b) = \lambda_1(\alpha_1, 1) + \cdots + \lambda_m(\alpha_m, 1) \qquad (\lambda_i \in \mathbb{Z}),$$

where $0 \neq s \in \mathbb{N}$, $\alpha \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$. Then

$$\begin{aligned} s\alpha &= \lambda_1\alpha_1 + \cdots + \lambda_m\alpha_m, \\ sb &= \lambda_1 + \cdots + \lambda_m, \\ s(\alpha - b\alpha_1) &= \lambda_2(\alpha_2 - \alpha_1) + \cdots + \lambda_m(\alpha_m - \alpha_1). \end{aligned}$$

Hence it follows that the class $\overline{\alpha - b\alpha_1} \in \mathbb{Z}^n/\mathbb{Z}(\alpha_2 - \alpha_1, \ldots, \alpha_m - \alpha_1)$ is a torsion element and maps to $\overline{(\alpha, b)}$, as required. □

Theorem 2.6 *Let F be a finite set of monomials of the same degree $d \geq 2$. The following conditions are equivalent*

(a) *$k[F] \subset k[\mathbf{x}_d]$ is birational.*

(b) *$\mathbb{Z}^n/\mathbb{Z}(\{v_1 - v_j \,|\, 2 \leq j \leq q\})$ is free of rank 1.*

(c) *The log-matrix A of F has maximal rank and $\mathbb{Z}(\{v_1 - v_j \,|\, 2 \leq j \leq q\}) = \mathbb{Z}(\{e_1 - e_k \,|\, 2 \leq i \leq n\})$.*

Proof. First we observe that, quite generally, there is an exact sequence of finite abelian groups

$$0 \to T(\mathbb{Z}^{n+1}/\mathbb{Z}A') \xrightarrow{\varphi} T(\mathbb{Z}^n/\mathbb{Z}A) \xrightarrow{\psi} \mathbb{Z}_d \to 0 \tag{2}$$

(here $\varphi(\overline{(\alpha, b)}) = \overline{\alpha}$ and $\psi(\overline{\alpha}) = \overline{\langle \alpha, \mathbf{1} \rangle}$, for $\alpha \in \mathbb{Z}^n, b \in \mathbb{Z}$) – see [16, Proof of Theorem 1.1].

If, moreover, A has full rank then $\mathbb{Z}^n/\mathbb{Z}A$ is torsion, hence $\mathbb{Z}^n/\mathbb{Z}A \simeq \mathbb{Z}_d$ if and only if $\mathbb{Z}^{n+1}/\mathbb{Z}A'$ is torsionfree, and in this case the $0th$ Fitting ideal $\Delta_n(A)$ of $\mathbb{Z}^n/\mathbb{Z}A$ is the same as that of $\mathbb{Z}_d$, i.e., $\Delta_n(A) = (d)$.

On the other hand, we have an exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z}A/\mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\}) \to \mathbb{Z}^n/\mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\}) \to \mathbb{Z}^n/\mathbb{Z}A \to 0 \tag{3}$$

Again, if A has full rank then the leftmost module has rank 1 and, since the rightmost module is torsion, the mid module has rank 1. Now apply Lemma 2.5(ii) with $m = q$ and $\alpha_j = v_j$ to get

$$T(\mathbb{Z}^n/\mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\})) \simeq T(\mathbb{Z}^{n+1}/\mathbb{Z}A').$$

Therefore, the equivalence (a) $\iff$ (b) follows from DBP of Lemma 2.2.

It remains to show that (b) $\iff$ (c). First, (c) $\Rightarrow$ (b) is clear by Lemma 2.5(i). For the reverse implication, since the mid term of the sequence (3) is assumed to be torsionfree of rank one and $\mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\}) \neq \mathbb{Z}A$, then A must have full rank and, moreover, $\mathbb{Z}A/\mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\})$ is torsionfree of rank one. In particular, there is a splitting $\mathbb{Z}A \simeq \mathbb{Z}(\{v_1 - v_j|\, 2 \leq j \leq q\}) \oplus \mathbb{Z}$ which, after extending to $\mathbb{Q}$, implies

$$\mathbb{Q}(\{v_i - v_j|1 \leq i < j \leq q\}) = \mathbb{Q}(\{e_i - e_j|1 \leq i < j \leq n\}). \tag{4}$$

Hence we get the desired equality because of the torsion freeness hypothesis. Notice that Eq. (4) also follows directly. Indeed if $\{v_1, \ldots, v_n\}$ is a basis for the column space of A, then

$$\{v_1 - v_n, v_2 - v_n, \ldots, v_{n-1} - v_n, v_n\}$$

is also a basis because $|v_i| = d$ for all i. Hence each $e_i - e_j$ can be written as

$$e_i - e_j = a_1(v_1 - v_n) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_n v_n \quad (a_i \in \mathbb{Q}).$$

Taking inner products with the vector $\mathbf{1} = (1, \ldots, 1)$ yields $a_n = 0$. Therefore we have shown the containment "$\supset$" in Eq. (4). A symmetric argument proves the equality. □

3 When are the Fitting ideals monomial ideals?

In [14, Lemma 1.1] was shown that the minors of the Jacobian matrix of a set of monomials are always monomials (possibly zero). The following result extends and clarifies the above assertion.

Proposition 3.1 *Let R be a graded ring with grading given by an additive abelian monoid $\mathcal{Z}$. Let N be a finitely generated $\mathcal{Z}$-graded module over R. Then the Fitting ideals of N are homogeneous ideals of R.*

Proof. By assumption, there is an exact sequence of $\mathcal{Z}$-graded modules over R

$$\sum_{\mathfrak{z}_j \in \mathcal{Z}} R(\mathfrak{z}_j) \xrightarrow{\phi} \sum_{\mathfrak{w}_i \in \mathcal{Z}} R(\mathfrak{w}_i) \longrightarrow N \to 0.$$

A Fitting ideal of N is an ideal $I_t(\phi)$ generated by the t-minors of ϕ, for a suitable t. This ideal is the image of the well-known induced $\mathcal{Z}$-graded homomorphism

$$\bigwedge^t \sum_{\mathfrak{z}_j \in \mathcal{Z}} R(\mathfrak{z}_j) \otimes_R \bigwedge^t \sum_{\mathfrak{w}_i \in \mathcal{Z}} R(\mathfrak{w}_i) \longrightarrow R.$$

Therefore, $I_t(\phi)$ is a homogeneous ideal of R. □

Corollary 3.2 *Let $R = k[x_1, \ldots, x_n]$ be given the standard multigrading (i.e., the $\mathbb{Z}^n$-grading with x_i of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$). If N is a finitely generated multigraded R-module, then the Fitting ideals of N are monomial ideals. In particular, any minor of the Jacobian matrix, respectively, of the syzygy matrix of arbitrary order, of a finite set of monomials is a monomial.*

Proof. Apply Proposition 3.1 while noticing that a homogeneous polynomial in the standard multigrading is necessarily a monomial. □

We can also apply the previous result in the case of the standard multigraded ring $\mathbb{Z}[x_1, \ldots, x_n]$, with $\mathbb{Z}$ in degree $\mathbf{0} = (0, \ldots, 0)$. The result is that, in particular, the formal Jacobian matrix of a finite set of monomials has monomial Fitting ideals. We wish to emphasize this in the following form:

Corollary 3.3 *The formal Jacobian matrix and the log-matrix of a finite set F of monomials have the same number of zero or nonzero minors. In particular, these matrices have the same rank. Also, there are at most finitely many field characteristics over which the Jacobian matrix of F over these characteristics has rank strictly smaller than the rank of the corresponding log-matrix.*

There is also a consequence tied up with the notion of a unimodular matrix.

Corollary 3.4 *The following are equivalent for a finite set F of monomials.*

(i) *The log-matrix of F is totally unimodular*

(ii) *Every nonzero minor of the formal Jacobian matrix of F has unit leading coefficient*

(ii) *The formal Jacobian matrix of F has characteristic-free Fitting ideals (i.e., the Fitting ideals of F over any field are generated by the same set of nonzero monomials).*

As for the syzygies of F, we observe that, in particular, any minor of the first Taylor syzygy matrix of F (see [4] for an explanation of the Taylor complex) is a monomial with coefficient ± 1. We next include an alternative elementary proof of this fact alone, as the method of the proof might be useful in some other context.

Lemma 3.5 *Let $\mathcal{T}(F)$ denote the Taylor syzygy matrix of F. Then any nonzero minor of $\mathcal{T}(F)$ is a monomial with coefficient ± 1.*

Proof. We proceed by induction on the size s of the minor. The case $s = 1$ being obvious, we assume that $s \geq 2$. We may clearly assume that the given minor is formed by the submatrix Z with the first s rows and columns of $\mathcal{T}(F)$. Let Z' denote the $q \times s$ submatrix of $\mathcal{T}(F)$ with the first s columns. By definition of the Taylor syzygy matrix of F, any column of the latter has exactly two nonzero entries. It follows that the complementary rows in Z' to the rows of Z cannot all be zero as otherwise Z would be a matrix of syzygies of the initial s monomials $\{x^{v_1}, \ldots, x^{v_s}\}$ of F, which is impossible since $\det(S) \neq 0$ while the entire syzygy matrix of these monomials has rank $s - 1$.

Thus, there must be a nonzero entry in some complementary row to Z in Z', say, the jth column, with $1 \leq j \leq s$. By the Taylor construction, there is exactly one further nonzero entry on the jth column. This entry must belong to Z as otherwise $\det(Z) = 0$. Also, this entry is again monomial with coefficient $\pm$. Expanding $\det(Z)$ by the jth column yields the product of this monomial by the minor of a suitable $(s-1) \times (s-1)$ submatrix of S. By induction, this minor has the required form, hence so does $\det(Z)$. □

Corollary 3.6 *Let Z be any submatrix of $\mathcal{T}(F)$ and let $\mathbb{T}(Z)$ denote the specialized matrix over $\mathbb{Z}$ obtained by sending $x_i \mapsto 1$. Then* $\operatorname{rank}(Z) = \operatorname{rank} \mathbb{T}(Z)$.

The next result complements one of the results of [1], where a criterion is given for a rational map to be birational in characteristic zero. The present proposition extends the latter result in all characteristics for monomial rational maps.

Theorem 3.7 *Let F be a finite set of monomials of the same degree $d \geq 2$. If* $\operatorname{rank}(A) = n$ *and* $\operatorname{rank}(\mathcal{LS}(F)) = q - 1$, *then $k[F] \subset k[\mathbf{x}_d]$ is birational.*

Proof. By Corollary 3.6 (or by Corollary 3.2) the matrix S obtained from $\mathcal{LS}(F)$ by making $x_i = 1$ for all i has also rank $q - 1$. By Proposition 2.3(ii), the extension $k[F] \subset k[\mathbf{x}_d]$ is birational. □

Corollary 3.8 *If the log-matrix A has maximal rank and the ideal $I = (F) \subset R$ has a linear presentation, then $k[F] \subset k[\mathbf{x}_d]$ is birational.*

Proof. It follows at once from Theorem 3.7 because in this case the rank of $\mathcal{LS}(F)$ is $q - 1$. □

4 Monomials of degree two

The birational theory of monomials of degree two can be completely established using elementary graph theory as we show in the sequel.

We start with a general auxiliary result which holds, more generally, for any rational map between projective spaces.

Lemma 4.1 *Let $F = \{f_1, \ldots, f_q\} \subset R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be forms of fixed degree $d \geq 2$. Suppose one has a partition $\mathbf{x} = \mathbf{y} \cup \mathbf{z}$ of the variables such that $F = G \cup H$, where the forms in the set G (respectively, H) involve only the $\mathbf{y}$-variables (respectively, $\mathbf{z}$-variables). If neither G nor H is empty then:*

(i) *The extension $k[F] \subset k[\mathbf{x}_d]$ is* NOT *birational*

(ii) *The linear syzygy matrix of F does* NOT *have maximal rank.*

Proof. (i) Suppose to the contrary, i.e., that $k(F) = k(\mathbf{x}_d)$. Since clearly $k(F) = k(G, H) \subset k(\mathbf{y}_d, \mathbf{z}_d)$, it follows that $k(\mathbf{x}_d) = k(\mathbf{y}_d, \mathbf{z}_d)$. Say $\mathbf{y} = \{y_1, \ldots, y_r\}$ and $\mathbf{z} = \{z_1, \ldots, z_s\}$. Then one has $k(\mathbf{y}_d) = k(y_2/y_1, \ldots, y_r/y_1, y_1^d)$ and, similarly, $k(\mathbf{z}_d) = k(z_2/z_1, \ldots, z_s/z_1, z_1^d)$. But this is a contradiction as, e.g., $y_1^{d-1}z_1 \notin k(y_2/y_1, z_2/z_1, \ldots, y_r/y_1, z_s/z_1, y_1^d, z_1^d)$ (for instance, by APB (Lemma 2.1)).

(ii) The linear syzygy matrix of F is a block-diagonal $(r + s) \times m$ matrix

$$\mathcal{LS}(F) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are the linear syzygy matrices of G and H, respectively. Since $\operatorname{rank}(A) \leq r - 1$ and $\operatorname{rank}(B) \leq s - 1$, then $\operatorname{rank}(\mathcal{LS}(F)) \leq r + s - 2 \leq q - 2$. □

Definition 4.2 A set $F = \{f_1, \ldots, f_q\}$ of forms of fixed degree ≥ 2 will said to be *cohesive* if the forms have no non-trivial common factor and F cannot be disconnected as in the hypothesis of the previous lemma.

Remark 4.3 The reason to assume that the forms have no non-trivial common factor is technical: multiplying a set of forms of the same degree by a given form yields the same rational map. To make the rational map correspond uniquely to a set of forms, one usually assumes that their gcd is one, i.e., that the ideal generated by these forms in the polynomial ring has codimension at least two (for further details on this and similar matters see [15]).

Yet another concept that fits the scene is a convenient extension of the notion of an ideal of linear type.

Definition 4.4 Let $F = \{f_1, \ldots, f_q\} \subset R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be forms of fixed degree $d \geq 2$, with $q \geq n$. Consider a presentation of the Rees algebra $\mathcal{R}_R(I) \simeq k[\mathbf{x}, \mathbf{y}]/\mathcal{J}$ where $\mathbf{y} = \{y_1, \ldots, y_q\}$ and $\mathcal{J}$ is a bihomogeneous ideal. We will say that $I = (F)$ is *of residual linear type* if $\mathcal{J}$ is generated in bidegrees $(*, 1)$ and $(0, *)$, where $*$ denotes an arbitrary integer ≥ 1.

Ideals of residual linear type are called ideals of *fiber type* in [11], it is shown there that polymatroidal ideals (see Section 5) are of fiber type. Thus, $I = (F)$ is of residual linear type if its relations are generated by the relations that define the symmetric algebra $\mathcal{S}_R(I)$ and the polynomial relations of I with coefficients in the base field k. A conjecture – perhaps only a question – regarding these ideals can be phrased as follows.

Conjecture 4.5 *Let F be a finite set of $q \geq n$ monomials of the same degree $d \geq 2$ such that the ideal $(F) \subset k[\mathbf{x}]$ is of residual linear type. Then the following conditions are equivalent:*

(i) *Both the log-matrix and the linear syzygy matrix of F have maximal rank.*

(ii) *The extension $k[F] \subset k[\mathbf{x}_d]$ is birational.*

A comment on the reasonableness of the conjecture. The implication (i) $\Rightarrow$ (ii) is just Theorem 3.7.

The reverse implication (ii) $\Rightarrow$ (i) follows from the principle of linear obstruction [15, Proposition 3.5] (see also [1]) in the case of an ideal of linear type (necessarily, $q = n$). In order to suitably extend to ideals of residual linear type, one could in principle use the main criterion of [15] and the terminology thereof. Let ϕ_1 denote the linear syzygy matrix of F. Thus, the *weak Jacobian matrix* ψ of F [15, Definition 2.2] can be thought of as the $\mathbf{y}$-Jacobian matrix of the quadrics in $k[\mathbf{y}]$ obtained by replacing every product $x_i y_k$ in $\mathbf{y} \cdot \phi_1$ by $y_i y_k$, if $1 \leq i < k \leq n$, and by $(1/2) y_k^2$ if $1 \leq i = k \leq n$ (thus, we need $\mathrm{char}(k) \neq 2$). In the case $q = n$, an easy strong duality works here to yield that ψ^t and the Jacobian dual of ψ^t define the same cokernel, hence have the same rank. But the Jacobian dual of ψ^t is ϕ_1, hence $\mathrm{rank}(\psi) = \mathrm{rank}(\phi_1) = n - 1$. For ideals of residual linear type, one needs an analogue that says $\mathrm{rank}_S(\psi) = n - 1 \Rightarrow \mathrm{rank}_{k[\mathbf{x}]}\phi_1 = q - 1$, where $S = k[\mathbf{y}]/P \simeq k[F]$, with P a (prime) toric ideal. Since F is monomial, a sufficiently elaborated application of Corollary 3.2 shows that the minors of ψ are monomials. Since P is toric, then $\mathrm{rank}_S(\psi) = \mathrm{rank}_{k[\mathbf{y}]}(\psi)$. Therefore, we are reduced to show that $\mathrm{rank}_{k[\mathbf{y}]}\psi = n - 1 \Rightarrow \mathrm{rank}_{k[\mathbf{x}]}\phi_1 = q - 1$. It is this the missing argument, for which one may have to bring in the other underlying facts of birationality – e.g., the log-matrix of F has maximal rank and, moreover, $\mathrm{coker}_S(\psi^t)$ is torsion free as S-module (the latter issues from the criterion in [15]).

Henceforth we assume that $\deg(x^{v_i}) = 2$ for all i. It is convenient to interpret a set of monomials of degree two in terms of graphs, possibly with loops. Thus, consider the graph $\widetilde{\mathcal{G}}$ on the vertex set $X = \{x_1, \ldots, x_n\}$ whose set of edges and loops correspond bijectively to the pairs $\{x_i, x_j\}$ such that $x_i x_j \in F$ (possibly $i = j$). Denote by $\mathcal{G}$ the underlying simple graph obtained by omitting all loops. Notice that, in our situation, the log-matrix A of F is the incidence matrix of $\widetilde{\mathcal{G}}$ and the monomial subring $k[F]$ is the edge subring $k[\widetilde{\mathcal{G}}]$ of the graph $\widetilde{\mathcal{G}}$.

One basic result for cohesive sets of monomials in degree $d = 2$ reads as follows.

Proposition 4.6 *If $F = \{x^{v_1}, \ldots, x^{v_q}\}$ is a set of forms of degree 2 with no non-trivial common factor. Then* $\operatorname{rank} \mathcal{LS}(F) = q - 1$ *if and only if F is cohesive.*

Proof. One implication follows immediately from Lemma 4.1. For the reverse implication, assume that F is cohesive. Then the corresponding graph $\widetilde{\mathcal{G}}$ as above is connected, hence the underlying simple graph $\mathcal{G}$ has a spanning tree $\mathcal{T}$. Being a tree, $\mathcal{T}$ has $n-1$ edges. The required result is easily verified in this case by induction on the number n of vertices: consider the subtree $\mathcal{T} \setminus x_n$ obtained by removing a vertex of degree one and the corresponding edge, say, $x_i x_n$. By the inductive assumption, $\operatorname{rank} \mathcal{LS}(\mathcal{T} \setminus x_n) = n-3$, so let L denote an $(n-2) \times (n-3)$ submatrix thereof of rank $n-3$. If $x_i x_j$ is any edge of $\mathcal{T} \setminus x_n$ then, by restoring the removed vertex and edge, yields a linear syzygy of $\mathcal{T}$ involving edges $x_i x_j$ and $x_i x_n$ and a submatrix of $\mathcal{LS}(\mathcal{T})$ formed by bordering L with the corresponding column syzygy and a last rows of zeros. It is clear that this $(n-1) \times (n-2)$ has rank $n-2$.

This takes care of the spanning tree $\mathcal{T}$. Next, one successively restores edges and loops on to $\mathcal{H}$ in order to recover the whole $\widetilde{\mathcal{G}}$, this time with no new vertices. By a similar token, adding one such edge or loop at a time to the connected subgraph $\mathcal{H}$, will increase by one the rank of the new submatrix of $\mathcal{LS}(F)$ formed by bordering as before the previous one with the column corresponding to the added edge or loop. $\square$

Before we set ourselves to state the main result of this section, the following observation seems pertinent. Quite generally, as used in the proof of Lemma 4.1 and easily shown, the field of fractions of the d-Veronese algebra is generated by the fractions $x_2/x_1, x_3/x_1, \ldots, x_n/x_1$ and the pure power x_1^d. Thus, a simple necessary condition in order that $k[F] \subset k[\mathbf{x}_d]$ be birational is that x_1^d be expressed as a fraction whose terms are products of the monomials in F. Now, in particular, if all these monomials are squarefree then a reasonable *tour de force* may be needed in order to accomplish it. Thus, e.g., for $d = 2$ it is not difficult to guess that the corresponding simple graph must have a cycle of odd length. At the other end of the spectrum it is possible, by such elementary considerations, to guess sufficient conditions under which one has enough fractions x_i/x_1 out of the monomials in F.

We chose to follow a more conceptual thread.

The next result generalizes [16, Corollary 3.2] and gives a complete answer for monomial birationality in degree two.

Theorem 4.7 *Let $F \subset R$ be a finite set of monomials of degree two having no non-trivial common factor and let $\mathcal{G} \subset \widetilde{\mathcal{G}}$ denote the corresponding graphs as above. Let A denote the incidence matrix of $\widetilde{\mathcal{G}}$. The following conditions are equivalent:*

(i) *F is cohesive and A has maximal rank.*

(ii) *The extension $k[F] \subset k[\mathbf{x}_2]$ is birational.*

(iii) *$\mathcal{G}$ is connected and, moreover, either it is non bipartite or else it is bipartite and $\widetilde{\mathcal{G}} \setminus \mathcal{G} \neq \emptyset$.*

Proof. We first show that (i) and (ii) are equivalent. Clearly, (ii) implies that $\operatorname{rank}(A) = \dim k[F] = n$ and cohesiveness follows from Proposition 4.6. The converse is a consequence of Theorem 3.7 and Proposition 4.6.

We next show that (ii) and (iii) are equivalent.

First, (iii) $\Rightarrow$ (ii).

Since $\mathcal{G}$ is connected, there is a spanning tree T of $\mathcal{G}$ containing all the vertices of $\mathcal{G}$, see [9].

If $\mathcal{G}$ is a bipartite graph and x_n is a loop of $\widetilde{\mathcal{G}}$. We may then regard T as a tree with a loop at x_n. Notice that T has exactly $n-1$ simple edges plus a loop. The incidence matrix B of T has order n, is non singular, and we may assume that the last column of B is the transpose of $(0,0,\ldots,0,2)$. Consider the matrix B' obtained from B by removing the last column. The matrix B' is totally unimodular because it is the incidence matrix of a simple bipartite graph [13, p. 273]. Therefore $\det(B) = \pm 2$ and $\mathrm{rank}(A) = n$. From Lemma 2.2 we obtain that $k[T] \subset k[\mathbf{x}_2]$ is birational, hence $k[G] \subset k[\mathbf{x}_2]$ is birational as well.

Now, let $\mathcal{G}$ be a non bipartite graph. Then $\mathrm{rank}(A) = n$. Since $\mathcal{G}$ has a spanning tree and G has at least one odd cycle ([9, pp. 37-39 and p. 42]), then $\mathcal{G}$ admits a connected simple subgraph $\mathcal{H}$ with n vertices and n edges with a unique cycle of odd length. By [16, Corollary 3.2] the extension $k[\mathcal{H}] \subset k[\mathbf{x}_2]$ is birational, hence so is $k[\mathcal{G}] \subset k[\mathbf{x}_2]$.

Finally, we show the implication (ii) $\Rightarrow$ (iii).

By Proposition 4.6, F must be cohesive, i.e., $\mathcal{G}$ is connected. We have already seen that $\mathrm{rank}\, A = \dim k[F] = n$. Suppose that $\mathcal{G}$ is bipartite. Then the log-matrix of $\mathcal{G}$ has rank $n-1$, hence $\widetilde{\mathcal{G}}$ has at least one loop. $\square$

Example 4.8 A geometer would summarize the result of Theorem 4.7 by saying that any cohesive coordinate projection of the 2-Veronesean that preserves dimension is birational onto the image. This is clearly false if the projection is a non coordinate cohesive projection, e.g., if F is a set of 2-forms forming a cohesive regular sequence (the simplest example with $n=2$ would be $F = x_1x_2, x_1^2 - x_2^2$). At the other end, for $d > 2$, a cohesive coordinate projection of the d-Veronesean preserving dimension can fail to be birational for the simple reason that it may be composed with a non-cohesive set. The simplest example of this phenomenon is $F = \{x_1^4, x_1^2x_2^2, x_2^4\}$. Here, $k[F] \subset k[(x_1,x_2)_4]$ is not birational, but its "reparametrization" $F' = \{y_1^2, y_1y_2, y_2^2\}$ is the 2-Veronesean. For $n > 2$, one of the simplest examples is $F = \{x_1^3, x_1^2x_2, x_2x_3^2\}$, which is cohesive of maximal rank, non-reparametrizable and non-birational: the ideal $(F) \subset k[x_1,x_2,x_3]$ is of linear type, but the linear syzygy matrix is of rank 1, hence falls below the needed value 2 (of course, this apparatus in such a simple example is worthless since one immediately sees that x_3^3 does not belong to the field of fractions of $k[x_1^3, x_1^2x_2, x_2x_3^2]$).

Corollary 4.9 *Let $\mathcal{G}$ be a connected simple bipartite graph. Assume that $x_{n-1}x_n$ is an edge of an even cycle of $\mathcal{G}$. Then $k[\widetilde{\mathcal{G}\setminus x_n}] \subset k[(\mathbf{x}\setminus x_n)_2]$ is a birational extension, where $\widetilde{\mathcal{G}\setminus x_n}$ is the graph on the vertices $X \setminus x_n = \{x_1,\ldots,x_{n-1}\}$ obtained by contracting the edge $x_{n-1}x_n$ to a loop around the vertex x_{n-1}.*

Proof. By the contracting-looping transformation, the resulting graph $\widetilde{\mathcal{G}\setminus x_n}$ acquires an odd cycle. Therefore, the simple subgraph induced by $\widetilde{\mathcal{G}\setminus x_n}$ is non-bipartite and the assertion follows from Theorem 4.7. $\square$

Remark 4.10 (a) The fact that a connected graph on n vertices having exactly n edges and a unique cycle of odd length induces a birational (Cremona) map had been guessed in [12, Conjecture 2.8] and proved in [16, Corollary 3.3] in a characteristic-free way. In characteristic zero, the more general context envisaged in [1] includes this result.

(b) If $q = n$, Corollary 4.9 has a pretty geometric interpretation. The given ring extension $k[\mathcal{G}] \subset k[\mathbf{x}_2]$ ($\mathcal{G}$ bipartite) translates into a rational map

$$\mathcal{F}\colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$$

whose image is $\mathrm{Proj}(k[\mathcal{G}])$, after normalizing the grading of $k[\mathcal{G}]$. The induced ring extension $k[\widetilde{\mathcal{G}\setminus x_n}] \subset k[(\mathbf{x}\setminus x_n)_2]$ corresponds to the restriction of $\mathcal{F}$ to the hyperplane L defined by $\widetilde{x_{n-1}-x_n}=0$ and its image can be identified with the image of $\mathcal{F}$ (actually, the algebras $k[\mathcal{G}]$ and $k[\widetilde{\mathcal{G}\setminus x_n}]$ are isomorphic as graded k-algebras by the contracting isomorphism $k[\mathbf{x}]/L \simeq k[\mathbf{x}\setminus x_n]$ sending $x_i \mapsto x_i$ for $1 \leq i \leq n-1$ and $x_n \mapsto x_{n-1}$). Thus, $\mathcal{F}$ restricts to a birational map of $L \simeq \mathbb{P}^{n-2}$ onto $\mathrm{im}(\mathcal{F})$.

5 Hall of examples

5.1 Monomial Cremona transformations

Among monomial birational maps, the Cremona ones form a well-known distinguished class. A Cremona map is a birational map of $\mathbb{P}^{n-1}$ onto itself. A recent surprising result ([8]) showed that the monomial Cremona transformations of $\mathbb{P}^{n-1}$ is generated by the ones of degree 2 and by the projective linear group, thus partially extending the classical result of M. Noether to higher dimension. The question as to which are the "standard ones" in dimension ≥ 3, if any at all, remains open as far as we know.

5.1.1 Monomial Cremona transformations of degree 2

We add a tiny contribution towards further understanding the structure of such maps. The next result extends a bit [16, Corollary 3.3] and likewise clarifies the algebraic/combinatorial background of the involved Cremona maps.

Proposition 5.1 *Let $F \subset k[x_1, \ldots, x_n]$ be a cohesive finite set of monomials of degree two having no non-trivial common factor and let $\mathcal{G} \subset \widetilde{\mathcal{G}}$ denote the corresponding graphs as above. Let A denote the $n \times n$ incidence matrix of $\widetilde{\mathcal{G}}$. The following conditions are equivalent:*

(i) $\det A \neq 0$

(ii) *F defines a Cremona transformation of $\mathbb{P}^{n-1}$*

(iii) *Either*

  (a) *$\widetilde{\mathcal{G}} = \mathcal{G}$ (i.e., no loops), $\mathcal{G}$ has a unique cycle and this cycle has odd length;*
  or else

  (b) *$\widetilde{\mathcal{G}}$ is a tree with exactly one loop.*

(iv) *The ideal $(F) \subset k[x_1, \ldots, x_n]$ is of linear type.*

Proof. The equivalence of (i) through (iii) follows immediately from Theorem 4.7, by noticing that if the underlying simple graph $\mathcal{G}$ is bipartite and $\widetilde{\mathcal{G}}$ has exactly n edges and loops, then the latter has to be a tree with exactly one loop. To see that the first three conditions are also equivalent to (iv), notice that (iv) implies (i) since the generators of an ideal of linear type are analytically independent, hence algebraically independent as they are forms of the same degree. Now, when $\widetilde{\mathcal{G}} = \mathcal{G}$, the implication (iii)(a) $\Rightarrow$ (iv) is part of [16, Corollary 3.3] but has really been noticed way before in [17, Corollary 3.2] (see also [18, Corollary 8.2.4]). Thus, it remains to see that (iii)(b) $\Rightarrow$ (iv) in the case where $\widetilde{\mathcal{G}}$ effectively has loops. This follows from Lemma 5.2 below using induction and noticing that an edge with a loop is clearly of linear type. $\square$

Lemma 5.2 *Let $F = \{f_1, \ldots, f_q\} \subset R$ be a set of monomials of degree two and let $f_{q+1} = x_i x_{n+1}$ be a monomial in $R' = R[x_{n+1}]$, where x_{n+1} is a new variable and $1 \leq i \leq n$. If $I = (F)$ is of linear type, then $I' = (I, f_{q+1})$ is of linear type.*

Proof. Let $R'[I't]$ be the Rees algebra of I' over the extended polynomial ring R'. Let J' denote the presentation ideal of $R'[I't]$, i.e., the kernel of the graded epimorphism:

$$\varphi\colon\ B' = R'[t_1, \ldots, t_{q+1}] \longrightarrow R'[I't] \longrightarrow 0 \qquad (t_i \longmapsto f_i t).$$

We may assume that J' extends the presentation ideal J of the Rees algebra $R[It]$ over R via the natural inclusion $R[It] \subset R'[I't]$. We know that $J' = \oplus_{s \geq 1} J'_s$ is a graded ideal in the standard $\mathbb{Z}$-grading of $R'[I't]$ with $(R'[I't])_0 = R'$. To show that I' is of linear type we have to show that $J'_s \subset B'J'_1$ for all $s \geq 1$. We proceed by induction on s, the result being vacuous for $s = 1$. Thus, assume $s \geq 2$. Since J' is a toric ideal, it is generated by binomials. Therefore, by the inductive hypothesis, it suffices to show that any binomial in J' belongs to $B'J'_{s-1}$. Let

$$h = x^\alpha t_{i_1}^{a_1} \cdots t_{i_k}^{a_k} - x^\beta t_{j_1}^{b_1} \cdots t_{j_r}^{b_r}$$

be a binomial in J'_s, where $i_1, \ldots, i_k, j_1, \ldots, j_r$ are distinct integers between 1 and $n+1$, $a_i > 0, b_i > 0$ for all i, j and $a_1 + \cdots + a_k = b_1 + \cdots + b_r = s$. We may assume that $f_{i_k} = f_{q+1} = x_i x_{n+1}$, otherwise $h \in BJ_1 \subset B'J'_1$ because I is of linear type. From the equality

$$x^\alpha f_{i_1}^{a_1} \cdots f_{i_k}^{a_k} = x^\beta f_{j_1}^{b_1} \cdots f_{j_r}^{b_r}$$

follows that x_{n+1} divides x^β, since no f_j on the right side of this equality involves the variable x_{n+1}. Thus there is a relation

$$x^\alpha f_{i_1}^{a_1} \cdots f_{i_{k-1}}^{a_{k-1}} f_{q+1}^{a_{q+1}-1} = x^\delta f_{j_1}^{c_1} \cdots f_{j_r}^{c_r} \tag{5}$$

where one of the c_i's may be zero and $c_1 + \cdots + c_r = s - 1 \geq 1$. We may assume that $c_1 > 0$ because not all c_i's are zero. Consider the equality

$$h = t_{i_{q+1}} F_1 + t_{j_1} F_2, \tag{6}$$

where $F_1 = x^\alpha t_{i_1}^{a_1} \cdots t_{i_{k-1}}^{a_{k-1}} t_{i_{q+1}}^{a_{q+1}-1} - x^\delta t_{j_1}^{c_1} \cdots t_{j_r}^{c_r}$ and $F_2 = x^\delta t_{i_{q+1}} t_{j_1}^{c_1-1} t_{j_2}^{c_2} \cdots t_{j_r}^{c_r} - x^\beta t_{j_1}^{b_1-1} t_{j_2}^{b_2} \cdots t_{j_r}^{b_r}$. Since $F_1 \in J'$ because of (5), then $t_{j_1} F_2 \in J'$, hence $F_2 \in J'$ as J' is prime. Therefore, (6) expresses h as an element of $B'J'_{s-1}$, as required. □

Example 5.3 $\{x_1x_2, x_1x_3, x_2x_3\}$ and $\{x_1x_2, x_1x_3, x_3^2\}$ are examples of each of the subcases (a) and (b) in Proposition 5.1. They respectively define the standard Cremona plane maps with 3 distinct base points and with 2 base points and one infinitely near point. The third type of standard Cremona map is a double structure on one single point, hence is not monomial.

5.1.2 Squarefree monomial Cremona transformations

Let $F \subset k[x_1, \ldots, x_n]$ be a set of n squarefree monomials of degree d and let A denote the log-matrix of these monomials.

For convenience, a set F of monomials with no common factor defining a Cremona transformation will be said to be a *Cremona set*. Since F has no common factor, the corresponding Cremona map determines F uniquely. Likewise, we will call the *inverse Cremona set* the set of monomials that define the inverse map. We say that two squarefree monomial Cremona sets are *permutable* – to mean "equivalent" in the lack of better terminology – if they coincide up to a permutation of the source and of the target variables. This is supposedly the equivalent of saying that the two squarefree monomial Cremona maps are geometrically one and the same.

Obviously, for a given pair n, d, where n is the number of variables and d is the degree of the monomials, there are a finite number of mutually non-permutable Cremona sets with these values. Classifying means finding this complete list.

Classifying squarefree Cremona sets looks within grasp since necessarily $d \leq n-1$. Up to permutability, the only Cremona transformation of degree $d = n-1$ in n variables whose terms are squarefree monomials is the analogue of the classical Steiner plane inversion, given by $F = \{x_1 \cdots x_{n-1},\, x_1 \cdots x_{n-2}x_n,\, \ldots,\, x_2 \cdots x_n\}$. For degrees $d \leq n-2$, the classification becomes more involved. Our purpose in this part is to convey the impact of combinatorics on birationality by examining some scattered examples for low values of n and degrees $d \leq n-2$. The case where $d = 2$ was completely covered by case (a) of Proposition 5.1.

Recall the following notion of combinatorial nature. If F is a set of monomials of the same degree with log-matrix $A = (a_{ij})$, its *dual complement* is the set $\widehat{F}$ of monomials whose log-matrix is $\widehat{A} = (1 - a_{ij})$. The following basic principle guides us into further simplification.

Proposition 5.4 (Duality Principle) *Let F be a set of monomials in n variables, of the same degree d, with no common factor. Then F is a Cremona set if and only if $\widehat{F}$ is a Cremona set.*

Proof. There is a known equality that works for all n and d (see for instance [5]): $(n-d)\det(A) = (-1)^{n-1} d \det(\widehat{A})$. A simple proof of this equality consists in adding the rows of A to get $\det(A) = d\det(A')$, where:

$$A' = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n} \\ 1 & \cdots & 1 \end{bmatrix}.$$

Similarly adding the rows of $\widehat{A}$ we get $\det(\widehat{A}) = (n-d)\det(\widehat{A}')$, where:

$$\widehat{A}' = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n} \\ 1 & \cdots & 1 \end{bmatrix}.$$

Then $\widehat{A}'$ is obtained from A' by subtracting the row $\mathbf{1} = (1, \ldots, 1)$ from each of the first $n-1$ rows of A' and making a change of sign at each step. Thus the determinants of A' and $\widehat{A}'$ differ by (at most) a sign given by $(-1)^{n-1}$. Thus

$$\det(A) = d\det(A') = (-1)^{n-1} d \det(\widehat{A}') = (-1)^{n-1} \frac{d}{n-d} \det(\widehat{A}),$$

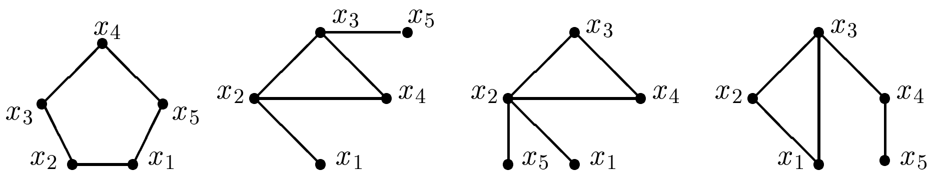
which yields the required formula.

Now, if F is Cremona then $|\det(A)| = |d|$ by DPB. Taking absolute values, this formula yields $|\det(\widehat{A})| = |n-d|$. Thus $\widehat{F}$ is a Cremona set by DPB. The reverse implication is obtained by a symmetrical argument. $\square$

Proposition 5.5 *Up to permutation of the variables, the complete list of distinct squarefree Cremona sets of degree 3 in 5 variables is as follows:*

- $F = \{x_3x_4x_5,\ x_1x_4x_5,\ x_1x_2x_5,\ x_1x_2x_3,\ x_2x_3x_4\}$ ($\mathcal{DB}$)
- $F = \{x_3x_4x_5,\ x_1x_4x_5,\ x_1x_2x_5,\ x_1x_3x_5,\ x_1x_2x_4\}$ (p-involutive)
- $F = \{x_3x_4x_5,\ x_1x_4x_5,\ x_1x_2x_5,\ x_1x_3x_5,\ x_1x_3x_4\}$ (p-involutive)
- $F = \{x_3x_4x_5,\ x_1x_4x_5,\ x_1x_2x_5,\ x_2x_4x_5,\ x_1x_2x_3\}$ (apocryphal)

Proof. According to Proposition 5.4, the required complete list is the complete list of the dual-complements. The latter is the list of all squarefree degree 2 Cremona sets obtained from Proposition 5.1(a). Their corresponding graphs are shown below:



To conclude, we explain the appended terminology. A set F of squarefree monomials is called *d-doubly-stochastic* (short: $\mathcal{DB}$) if its log-matrix $A = (a_{ij})$ is doubly-stochastic, i.e., the entries of each column sum up to d (i.e., the monomials have fixed degree d) and so do the entries of each row (i.e., no variable is privileged or, the "incidence" degrees of the variables is also d). A Cremona set is called *p-involutive* if it coincides with its inverse set up to permutability. Finally, a Cremona set is called *apocryphal* if its inverse set has at least one non-squarefree monomial. □

It may be easier to classify $\mathcal{DB}$ Cremona sets. For instance, the following simple result considerably reduce the possibilities.

Proposition 5.6 *If $A = (a_{ij})$ is doubly stochastic and $|\det(A)| = d$, then* $\gcd\{n, d\} = 1$.

Proof. Adding the first $n-1$ rows of A to its last row and factoring out d we get:

$$\det(A) = d \det \begin{bmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{n-11} & \cdots & a_{n-1n-1} & a_{n-1n} \\ 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Next we add the first $n-1$ columns of the matrix occurring in the right hand side of this equality to its last column to get

$$\det(A) = d \det \begin{bmatrix} a_{11} & \cdots & a_{1n-1} & d \\ \vdots & & \vdots & \vdots \\ a_{n-11} & \cdots & a_{n-1n-1} & d \\ 1 & \cdots & 1 & n \end{bmatrix}.$$

Hence since $\det(A) = \pm d$ we obtain that n and d are relatively prime. □

Another useful tool is the following.

Lemma 5.7 (Inductive principle for $\mathcal{DB}$) *Let $F = u_1, \ldots, u_n \subset k[x_1, \ldots, x_n]$ be a $\mathcal{DB}$ set of squarefree monomials of degree d. Then, given a permutation $\{i_1, \ldots, i_n\}$ of the indices such that the set $u_1/x_{i_1}, \ldots, u_n/x_{i_n}$ has no repeated monomials, then this set is a $\mathcal{DB}$ set of squarefree monomials of degree $d-1$.*

Proof. The proof follows immediately from a close inspection of the corresponding log-matrices. □

Of course, the result of the lemma can be read backwards, i.e., from degree $d-1$ up to degree d by multiplying by variables out of $\{x_{i_1}, \ldots, x_{i_n}\}$.

Corollary 5.8 *For $n = 6$ the only $\mathcal{DB}$ squarefree Cremona set of degree d is the involutive Steiner inversion, given by*

$$x_1x_2x_3x_4x_5,\ x_1x_2x_3x_4x_6\ ,\ x_1x_2x_3x_5x_6\ ,\ x_1x_2x_4x_5x_6\ ,\ x_1x_3x_4x_5x_6\ ,\ x_2x_3x_4x_5x_6$$

Proof. It follows readily from Proposition 5.6. In particular the inductive principle above does not preserve the rank of the log-matrix. □

To classify the squarefree Cremona sets with $n = 6$ we only need to look at degree $d = 3$, since $d = 2$ follows from Proposition 5.1 and $d = 4$ goes by duality. We give some instances of Cremona sets with $n = 6$ and $d = 3$, with special care for their linear syzygy behavior. These examples will hopefully give some measure of the theoretical hardship in classifying squarefree Cremona sets for $n \geq 6$.

Example 5.9 The set $F = \{x_1x_2x_6,\ x_2x_3x_6,\ x_1x_3x_6,\ x_1x_3x_4,\ x_1x_4x_5,\ x_3x_4x_6\}$ is a Cremona set: both the log and the linear syzygy matrices have maximal rank. A calculation using the method of [15] shows that F is p-involutive. Its dual complement $\widehat{F}$ is also a p-involutive Cremona set which is not permutable equivalent to F.

Example 5.10 The set $F = \{x_1x_2x_3,\ x_2x_3x_4,\ x_3x_4x_5,\ x_1x_3x_6,\ x_2x_5x_6,\ x_4x_5x_6\}$ is a Cremona set: both the log and the linear syzygy matrices have maximal rank. A calculation as in the previous example shows that F is apocryphal with degree 4 inverse set $\{y_1^2y_6^2,\ y_1y_2y_6^2, \ldots\}$ (the dots stand for squarefree monomials), a rather weird turnout.

Example 5.11 The set $F = \{x_1x_2x_4,\ x_2x_3x_5,\ x_3x_4x_6,\ x_1x_4x_5,\ x_1x_4x_6,\ x_2x_5x_6\}$ has log matrix of maximal rank, but not so the linear syzygy matrix whose rank is 4 (though the corresponding syzygy submodule is 5-generated). Nevertheless, a calculation as before shows that F acquires an extra $\mathbf{x}$-linear relation (of higher $\mathbf{y}$-degree) which suffices to derive birationality. Moreover, as it turns out, F is apocryphal with degree 5 inverse set $\{y_2^2y_6^3,\ y_2^2y_3y_6^2,\ y_1y_3^2y_4y_5,\ y_2y_3y_5y_6^2, \ldots\}$, an even weirder turnout.

5.2 Monomial birational maps from other combinatorial constructs

The following class of sets of monomials was considered in [10].

Definition 5.12 A set $F = \{\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_q}\}$ of monomials of degree d minimally generating the ideal $(F) \subset k[\mathbf{x}]$ is called *polymatroidal* if the following condition is satisfied: given any two $\mathbf{x}^u, \mathbf{x}^v \in F$, if $u_i > v_i$ for some index i, then there is an index j with $u_j < v_j$ such that $\frac{x_j}{x_i}\mathbf{x}^u \in F$.

If F is polymatroidal or even matroidal, the dimension of $k[F]$ may be less than n. For instance if $k[F]$ is the edge subring of a complete bipartite graph on n vertices, then $\dim(k[F]) = n-1$.

The definition of polymatroidal set is somewhat tailored for having enough linear syzygies. This is expressed in a slightly different way in [2], where it has been shown that, provided F is ordered in the reverse lex order, it has *linear quotients*, i.e., the ideals $(\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_{i-1}}) : \mathbf{x}^{v_i}$ are generated by a set of variables, for every i. Clearly, this result implies that the ideal (F) is in fact linearly presented. Therefore, one has:

Proposition 5.13 *Let $F \subset k[\mathbf{x}]$ be a set of monomials of degree d minimally generating the ideal (F) and whose log-matrix is of maximal rank. If F is polymatroidal, then $k[F] \subset k[\mathbf{x}_d]$ is birational.*

Proof. This follows from Corollary 3.8 because (F) is linearly presented as discussed above. $\square$

Example 5.14 Fix an integer d and a sequence of integers $1 \leq s_1 \leq \cdots \leq s_n \leq d$. Let

$$F = \{x^{a_1} \cdots x^{a_n} \mid a_1 + \cdots + a_n = d;\ 0 \leq a_i \leq s_i \,\forall\, i\}.$$

Then F is a polymatroidal set of maximal rank (see [3]). The subalgebra of $k[\mathbf{x}]$ generated by F is said to be of *Veronese type*. It includes, as special cases, the Veronese algebra of $k[\mathbf{x}]$ of order d and the algebra of squarefree products of d variables. The birationality of $k[F] \subset k[\mathbf{x}_d]$ follows directly from Proposition 5.13 or from [16] using the fact that $R[Ft]$ is normal [6].

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References

[1] C. Ciliberto, F. Russo and A. Simis, Cremona maps, ideals of linear type and linear syzygies, in preparation.

[2] A. Conca and J. Herzog, Castelnuovo-Mumford regularity of products of ideals, Collect. Math. **54** (2003), 137-152.

[3] E. de Negri and T. Hibi, Gorenstein algebras of Veronese type, J. Algebra **193** (1997), 629–639.

[4] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, 1995.

[5] C. Escobar, *Normal monomial subrings, unimodular matrices and Ehrhart rings*, PhD thesis, Cinvestav–IPN, 2004.

[6] C. Escobar, R. H. Villarreal and Y. Yoshino, Torsion freeness and normality of blowup rings of monomial ideals, in *Commutative algebra with a focus on geometric and homological aspects*, Proceedings: Sevilla and Lisbon (A. Corso et al., Eds.), Lecture Notes in Pure and Appl. Math. **244**, Taylor & Francis, Philadelphia, 2005, pp. 69-84.

[7] C. Godsil and G. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics **207**, Springer, New York, 2001.

[8] G. Gonzalez-Sprinberg and I. Pan, On the monomial birational maps of the projective space, An. Acad. Brasil. Ciênc. **75** (2003), 129-134.

[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1972.

[10] J. Herzog and T. Hibi, Discrete polymatroids, J. Algebraic Combin. **16** (2002), 239–268.

[11] J. Herzog, T. Hibi and M. Vladoiu, Ideals of fiber type and polymatroids, Osaka J. Math. **42** (2005), 1–23.

[12] F. Russo and A. Simis, On birational maps and Jacobian matrices, Compositio Math. **126** (2001), 335–358.

[13] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1986.

[14] A. Simis, On the jacobian module associated to a graph, Proc. Amer. Math. Soc. **126** (1998), 989–997.

[15] A. Simis, Cremona transformations and related algebras, J. Algebra **280** (2004), 162–179.

[16] A. Simis and R. H. Villarreal, Constraints for the normality of monomial subrings and birationality, Proc. Amer. Math. Soc. **131** (2003), 2043–2048.

[17] R. H. Villarreal, Rees algebras of edge ideals, Comm. Algebra **23** (1995), 3513–3524.

[18] R. H. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001.

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