On terminal delta–wye reducibility of planar graphs

Isidoro Gitler[1] **Feliú Sagols**[2]

Departamento de Matemáticas

Centro de Investigación y de Estudios Avanzados del IPN

Apartado Postal 14–740

07000 México City, D.F.

igitler@math.cinvestav.mx fsagols@math.cinvestav.mx

Abstract

A graph is terminal $\Delta - Y$*-reducible* if, it can be reduced to a distinguished set of terminal vertices by a sequence of series-parallel reductions and $\Delta - Y$-transformations. Terminal vertices (o terminals for short) cannot be deleted by reductions and transformations. Reducibility of terminal graphs is very difficult and in general not possible for graphs with more than three terminals (even planar graphs). Terminal reducibility plays an important role in decomposition theorems in graph theory and in important applications, as for example, network reliability. We prove terminal reducibility of planar graphs with at most three terminals. The most important consequence of our proof is that this implicitly gives an efficient algorithm, of order $O(n^4)$, for reducibility of planar graphs with at most three terminals that also can be used for restricted reducibility problems with more terminals. It is well known that these operations can be translated to operations on the medial graph. Our proof makes use of this translation in a novel way, furthermore terminal vertices now seen as terminal faces and by duality of the reductions and transformations, the set of terminals can be taken as a set of vertices and a set of faces of the original graph.

[1]Corresponding author. Partially supported by CONACyT grants U40201, 45256 and SNI.

[2]Partially supported by CONACyT grants U40201, 45256 and SNI.

Key words and phrases. Delta-Wye transformations, medial graph, planar graphs and topological graph theory.

1 Background

A graph is planar if it can be drawn in the plane without edge crossings, and it is a plane graph if it is so drawn in the plane. The drawing separates the rest of the plane into regions called *faces*. We consider graphs which may include *loops* (edges whose two end vertices are identical), *parallel edges* (two edges with the same end vertices) and parallel loops.

$\Delta \leftrightarrow \mathcal{Y}$ Operations: A class of graphs $\mathcal{Q}$, is said to be $(\Delta \leftrightarrow \mathcal{Y})$ *reducible* to a canonical simple graph structure P if, any $\mathcal{G} \in \mathcal{Q}$ can be reduced to P, by repeated application of the following four reductions and two transformations :

R0 *Loop reduction.* Delete a loop.

R1 *Degree-one reduction.* Delete a degree one vertex and its incident edge.

R2 *Series reduction.* Delete a degree two vertex y and its two incident edges xy and yz, and add a new edge xz.

R3 *Parallel reduction.* Delete one of a pair of parallel edges.

Each of these reductions decreases the number of vertices or edges in a graph. Two other transformations of graphs are important. A *wye* ($\mathcal{Y}$) is a vertex of degree three. A *delta* (Δ) is a cycle $\{x, y, z\}$ of length three. The transformations are:

$\mathcal{Y}\Delta$ *Wye-delta transformation.* Delete a wye w and its three incident edges wx, wy, wz, and add in a delta $\{x, y, z\}$ with edges xy, yz, and zx.

$\Delta\mathcal{Y}$ *Delta-wye transformation.* Delete the edges of a delta $\{x, y, z\}$, add in a new vertex w and new edges wx, wy, and wz.

Terminals are distinguished vertices that cannot be deleted by reductions and transformations. When a specified subset $\mathcal{A}$ of the nodes is distinguished as terminal nodes, we require that for any $a \in \mathcal{A}$, a cannot take the place of the degree one or degree two vertex in reductions $R1$ and $R2$, or the degree three vertex in the $\mathcal{Y}\Delta$ transformation, described above.

Working with terminals introduces the question of deciding what should be considered as an adequate list of irreducible configurations. In [5], Feo and Provan introduce an operation which we call an *FP-assignment*: it reassigns a degree one terminal vertex a incident to a non-terminal vertex b. This is done by eliminating the vertex a and then changing the status of b considering it a terminal vertex.

FP – assignment (see Figure 1). Eliminate a terminal vertex a of degree one that is the end of a pendant edge and then change the status of the other end vertex of the edge considering it as a terminal vertex.

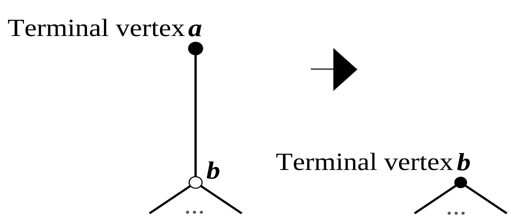

Figure 1: FP-assignment

This transformation implicitly re-embeds terminal pendant edges, but it remains consistent by keeping track of these reassignments. In other words, when we apply this transformation we "forget" how some pendant edges were originally embedded in order to be able to perform further reductions. In applications, one must keep track of the embedding of pendant edges in order to reverse correctly the reduction process.

Allowing FP-assignments, the set of irreducible configurations is decreased substantially without changing the nature of the reduction problem (this is particularly useful in applications and reduction algorithms), with an unnatural long list of irreducible configurations as Figure 2 shows.

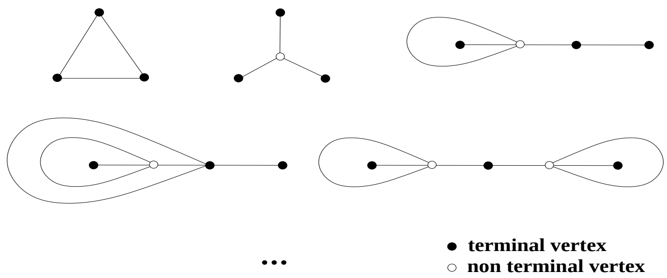

Figure 2: Artificial irreducible 3-terminal plane graphs

Definition 1.1 *A connected plane graph is* terminal $\Delta \leftrightarrow \mathcal{Y}$ reducible *if, it can be reduced to eliminate all non-terminal vertices by using* **R0***,* **R1***,* **R2***,* **R3***,* $\mathcal{Y}\boldsymbol{\Delta}$*,* $\boldsymbol{\Delta}\mathcal{Y}$*, and* **FP-assignments***.*

Contracting an edge $e = uv$, consists of deleting e and identifying its two endpoints $u = v$ to make a single vertex. A *minor* of $\mathcal{G}$ is a graph formed by a sequence of edge deletions, edge contractions and deletion of isolated vertices.

Combining the previous two concepts, we can have a minor $\mathcal{H}$ of a graph $\mathcal{G}$ with terminals. Specifically, a *terminal minor* is formed using the same three minor operations as above, except that we forbid contracting an edge joining two terminals and deleting an isolated terminal. It follows that $\mathcal{H}$ has the same number of terminals as $\mathcal{G}$.

In 1966 G.V. Epifanov proved the following important theorem :

Theorem 1.1 *(Epifanov [4]). Each connected plane graph with two terminals is $\Delta \leftrightarrow \mathcal{Y}$ reducible to a single edge between the terminals.*

This theorem has the following corollary:

Corollary 1.2 *(Epifanov [4]), Grünbaum [7]). Each connected plane graph is* $\Delta \leftrightarrow \mathcal{Y}$ *reducible to a vertex.*

Since Epifanov's and Grünbaum's work simpler proofs have been found (see [5] and [11]). For generalizations see for instance [10]. It is important to note that not every connected graph is $\Delta \leftrightarrow \mathcal{Y}$ reducible, one such graph is the complete bipartite graph $K_{4,4}$.

Theorem 1.3 *(Truemper [11], Gitler [6], and Archdeacon,etal [2]).*

Suppose that $\mathcal{H}$ *is a terminal minor of* $\mathcal{G}$*. If* $\mathcal{G}$ *is terminal* $\Delta \leftrightarrow \mathcal{Y}$ *reducible, then* $\mathcal{H}$ *is terminal* $\Delta \leftrightarrow \mathcal{Y}$ *reducible.*

It follows that if a connected graph $\mathcal{G}$ is $\Delta \leftrightarrow \mathcal{Y}$ reducible, then each connected minor $\mathcal{H}$ of $\mathcal{G}$ is $\Delta \leftrightarrow \mathcal{Y}$ reducible as well.

Theorem 1.4 *(Gitler [6]). A two connected plane graph with three terminals is* $\Delta \leftrightarrow \mathcal{Y}$ *reducible to a copy of* K_3 *with the original three terminals as vertices.*

We first describe an outline of the proof given by Gitler [6]. Let $\mathcal{G}$ be a two connected 3-terminal plane graph. The first step is to show that $\mathcal{G}$ can always be embedded on some grid $\mathcal{F}$ which is also a 3-terminal plane graph, and has $\mathcal{G}$ as a minor. Then, using the corner and modified corner algorithms appearing in [6], the second step is to show that $\mathcal{F}$ is $\Delta \leftrightarrow \mathcal{Y}$ reducible to a specific 3-terminal graph, called a perfect mirror M. The third step consists in showing that M is $\Delta \leftrightarrow \mathcal{Y}$ reducible to K_3. Finally, the minor $\mathcal{G}$ of $\mathcal{F}$ is $\Delta \leftrightarrow \mathcal{Y}$ reducible to K_3 by Theorem 1.3.

Connected plane graphs with more than three terminals are in general not reducible ([2, 6]), see the example in Figure 3.

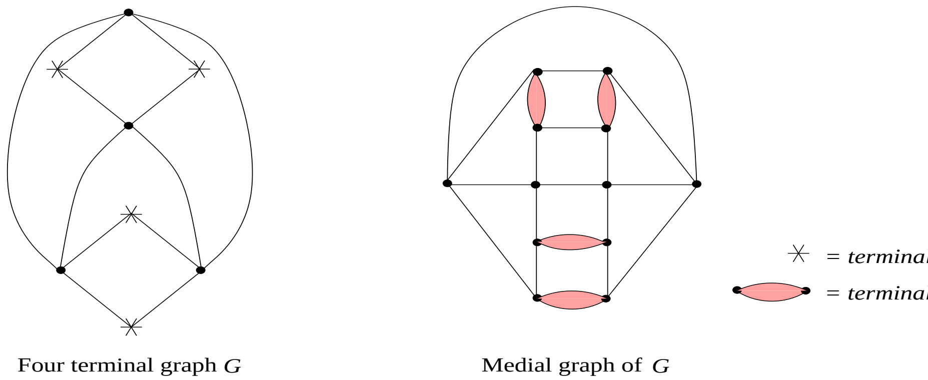

Figure 3: Non reducible 4 terminal plane graph

2 Preliminaries

Given a connected plane graph $\mathcal{G}$, its *medial graph* $\mathcal{M}(\mathcal{G})$ is defined as follows. The vertices of $\mathcal{M}(\mathcal{G})$ are the edges of $\mathcal{G}$. Each face $f = e_1, \ldots, e_r$ of length r in $\mathcal{G}$ determines r edges $\{e_i e_{i+1} : 1 \leq i \leq r-1\} \bigcup \{e_r e_1\}$ of $\mathcal{M}(\mathcal{G})$. In this definition, a loop e that bounds a face is viewed as a face of length one, and so determines one edge of $\mathcal{M}(\mathcal{G})$, which is a loop on e (similarly for pendant edges).

The graph $\mathcal{M}(\mathcal{G})$ is four regular and plane. Let $\mathcal{G}^*$ denote the dual of the graph $\mathcal{G}$, then $\mathcal{M}(\mathcal{G}) \equiv \mathcal{M}(\mathcal{G}^*)$; $\mathcal{G}$ and $\mathcal{G}^*$ are the *face graphs* of $\mathcal{M}(\mathcal{G})$. Any connected four regular plane graph is the medial graph of some pair of dual plane graphs. When speaking about the medial graph $\mathcal{M}(\mathcal{G})$ we always take as reference the face graph $\mathcal{G}$ that does not contain a vertex corresponding to the infinite face in $\mathcal{M}(\mathcal{G})$, and we refer to it as the *black graph* of $\mathcal{M}(\mathcal{G})$. The faces in $\mathcal{M}(\mathcal{G})$ corresponding to the vertices in $\mathcal{G}$ are *black faces* and the others *white faces*.

Let $\mathcal{M}$ be a 4-regular graph embedded on the plane. The *straight decomposition* $\mathcal{K}(\mathcal{M})$ of $\mathcal{M}$ is the decomposition of the edges of $\mathcal{M}$ into closed curves $C_1, \ldots, C_k$, (called closed geodesic arcs) in such a way that, each edge is traversed exactly once for these curves and in each vertex v of $\mathcal{M}$ if e_1, e_2, e_3 and e_4 are the edges incident to v in cyclic order, then $e_1 v e_3$

are traversed consecutively (in one way or the other); in this case e_1 has as *direct extension* e_3 (e_3 has as direct extension e_1). Similarly e_2ve_4 are traversed consecutively (in one way or the other). The straight decomposition is unique up to choice of the beginning vertex of curves, up to reversing the curves and up to permuting the indexes of $C_1, ..., C_k$.

We always view a given plane connected graph $\mathcal{G}$ with k-terminal vertices, through its medial graph as a black graph, together with its straight decomposition $\mathcal{K}(\mathcal{M}(\mathcal{G}))$ into closed geodesic arcs and we call the faces in $\mathcal{M}(\mathcal{G})$ corresponding to the k terminal vertices of $\mathcal{G}$, the *terminal faces* of $\mathcal{M}(\mathcal{G})$.

2.1 Delta-Wye operations on the medial graph

The $\Delta \leftrightarrow \mathcal{Y}$ operations introduced in section 2, have a direct translation on to $\mathcal{M}(\mathcal{G})$ as four medial reductions and two medial transformations (see [3]). If a $\Delta \leftrightarrow \mathcal{Y}$ operation $\mathcal{O}$ (transformation or reduction) is applied on $\mathcal{G}$ giving $\mathcal{O}(\mathcal{G})$ then the medial graph of the resulting graph $\mathcal{M}(\mathcal{O}(\mathcal{G}))$ is $\mathcal{M}(\mathcal{G})$ after the application of the corresponding medial operation $\mathcal{O}_{\mathcal{M}}$. In other words $\mathcal{M}(\mathcal{O}(\mathcal{G})) \equiv \mathcal{O}_{\mathcal{M}}(\mathcal{M}(\mathcal{G}))$. In general we denote a medial operation with its $\Delta \leftrightarrow \mathcal{Y}$ name followed by $\mathcal{M}$ as subscript.

The $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ operations are defined as (see Figure 4):

R0$_{\mathcal{M}}$ *Medial loop reduction.* Topologically collapse a white loop (a loop enclosing a white face) in $\mathcal{M}(\mathcal{G})$ to a single vertex of degree two and then omit this vertex.

R1$_{\mathcal{M}}$ *Medial degree-one reduction.* Topologically collapse a black loop (a loop enclosing a black face) in $\mathcal{M}(\mathcal{G})$ to a single vertex of degree two and then omit this vertex.

R2$_{\mathcal{M}}$ *Medial series reduction.* Topologically collapse a digon enclosing a black face in $\mathcal{M}(\mathcal{G})$, to a single vertex of degree four, thus identifying the end vertices of the digon.

R3$_{\mathcal{M}}$ *Medial parallel reduction.* Topologically collapse a digon enclosing a white face in $\mathcal{M}(\mathcal{G})$, to a single vertex of degree four, thus identifying the end vertices of the digon.

Each of these reductions decreases the number of vertices, edges or faces in $\mathcal{M}(\mathcal{G})$. The other two transformations on $\mathcal{M}(\mathcal{G})$ are:

$(\boldsymbol{\mathcal{Y}\Delta})_{\mathcal{M}}$ *Medial* $\boldsymbol{\mathcal{Y}\Delta}$ *transformation.* Topologically collapse a black triangle face to a single vertex of degree six, thus identifying the vertices of the triangle, then expand this vertex to a white triangle whose edges are incident to the black regions.

$(\boldsymbol{\Delta\mathcal{Y}})_{\mathcal{M}}$ *Medial* $\boldsymbol{\Delta\mathcal{Y}}$ *transformation.* Topologically collapse a white triangle face to a single vertex of degree six, thus identifying the vertices of the triangle, then expand this vertex to a black triangle whose edges are incident to the white regions.

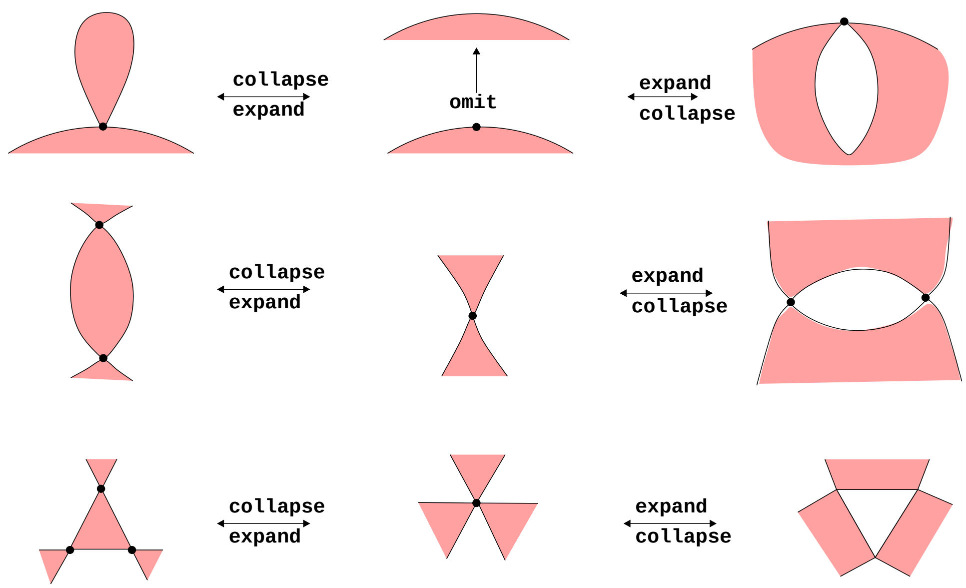

Figure 4: Medial $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ transformations

As before, when a specified subset F of black faces is distinguished as terminal faces, we require that for any $f \in F$, f cannot take the place of the black loop or the black face in the medial operations $R1_{\mathcal{M}}$ and $R2_{\mathcal{M}}$, or the black triangle in the medial operation $(\mathcal{Y}\Delta)_{\mathcal{M}}$, described above.

We now state the formulation of the **FP-assignment** on the medial graph:

FP$_{\mathcal{M}}$ *Medial FP assignment* (see Figure 5). Eliminate a black terminal loop a provided that its cone b (the cone of a loop (or $1 - lens$) is defined in Section 3) corresponds to a non-terminal face, henceforth b is considered a terminal face.

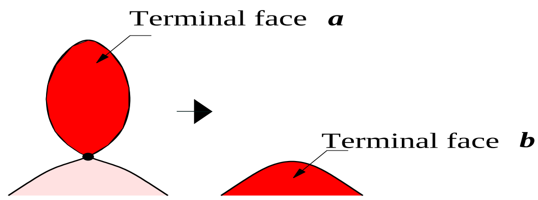

Figure 5: FP-assignment

If a medial graph with terminal faces can be reduced by using the operations defined above, to eliminate all non-terminal black faces then it is *terminal* $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ *reducible*.

3 k-lenses

Throughout the remaining part of this paper G will denote a four regular, connected, plane graph (a medial graph), together with its straight decomposition $\mathcal{K}(G)$. We follow [7].

A path $v_0v_1 \ldots v_n$ in G is a *geodesic arc*, if and only if, $v_{i-1}v_i$ has v_iv_{i+1} as direct extension for $1 \le i < n$; for a *closed geodesic arc*, $v_0 = v_n$, and $v_{n-1}v_n$ has v_0v_1 as direct extension. We call *simple* paths or *simple* geodesic arcs, those paths or geodesic arcs which do not have self intersections.

Given an integer $k > 0$, a subgraph $\mathcal{L}$ of G is called a *k-lens* provided:

L1: $\mathcal{L}$ consists of a simple closed path (see Figure 6.a) $R = v_{0,0}v_{0,1}\ \ldots\ v_{0,i_0}v_{1,0}v_{1,1}\ \ldots\ v_{1,i_1}$ $\ldots\ v_{k-1,0}v_{k-1,1}\ \ldots v_{k-1,i_{k-1}}v_{0,0}$ called the *boundary* of $\mathcal{L}$, and all the vertices and edges of G contained in one of the connected components of the complement of R in the 2-sphere. We call *interior* of $\mathcal{L}$ to one such connected component, the vertices and edges contained in it are the *inner* vertices and edges of R.

L2: For $0 \leq j < k$ $v_{j,0}v_{j,1}\ldots v_{j,i_j}v_{(j+1) \bmod k,0}$ are simple geodesic arcs, called *geodesic boundary arcs*. No inner edge of R is incident to the vertices $v_{0,0}, v_{1,0}, \ldots, v_{k-1,0}$, which are called the *poles* of the k-lens.

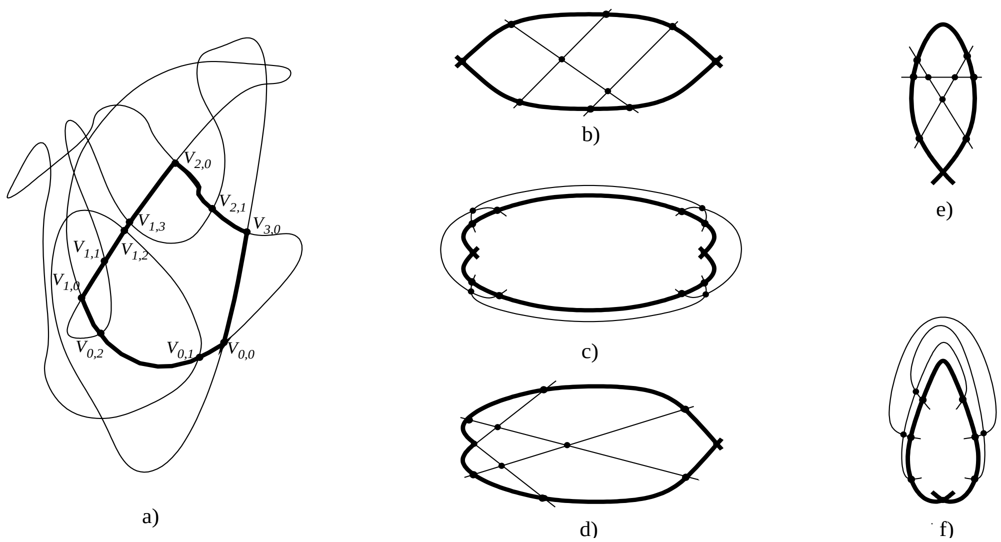

Figure 6: Lenses examples

Figure 6 depicts examples of k-lenses. a) a 4-lens, b) and c) two 2-lenses, d) a configuration that is not a 2-lens and e) and f) two 1-lenses.

In a generic way, a *lens* is a k-lens for some value of k. Let $\mathcal{L}$ be a lens, it is *singular* if, it does not contain any inner vertex or edge; otherwise it is *non singular*.

A *chord* of $\mathcal{L}$ is a simple geodesic arc $P = p_0p_1\ldots p_k$ such that p_0 and p_k belong to the boundary of $\mathcal{L}$ but the other vertices and edges in P are interior to $\mathcal{L}$.

The *rays* of a pole w in $\mathcal{L}$ are the edges in G which are incident to w but do not belong to the boundary of $\mathcal{L}$. By definition, the rays of a pole in $\mathcal{L}$ are not in the interior of $\mathcal{L}$.

Given a pole w of a lens $\mathcal{L}$ there is a unique face F interior to $\mathcal{L}$ incident to w. Of the remaining faces incident to w exactly one, say H, shares color with F (since the boundary of a lens is a Jordan curve, F and H are distinct faces). We call H the *cone* associated to the pole w of $\mathcal{L}$.

When we talk about the *rays* and *cone* of a 1-lens we do not make any explicit reference to the pole.

A k-lens ($k \geq 2$) $\mathcal{L}$ in G is *indecomposable*, if and only if, it does not contain properly a 1-lens or a 2-lens. A 1-lens is *indecomposable*, if and only if, it does not contain properly in its interior a 1-lens or a 2-lens (see Figure 7). A k-lens which is not indecomposable is *decomposable*. All lenses in Figure 6 are indecomposable, the 1-lens in Figure 12.b is not.

The following result is a consequence of this definition.

Lemma 3.1 *A k-lens is indecomposable, if and only if, the following conditions are true:*

I1: Every inner edge in $\mathcal{L}$ belongs to a chord of $\mathcal{L}$.

I2: Two different chords in $\mathcal{L}$ meet in at most one vertex.

I3: If $k > 1$ then each chord in $\mathcal{L}$ intersects the boundary of $\mathcal{L}$ in exactly two vertices: each of them in a different geodesic boundary arc.

We say that an indecomposable k-lens ($k \geq 2$) can be *emptied*, if after a finite sequence of $(\mathcal{Y}\Delta)_{\mathcal{M}}$ or $(\Delta\mathcal{Y})_{\mathcal{M}}$ transformations it becomes singular. Note that after applying any of these transformations to an indecomposable $k-lens$ the resulting $k-lens$ is indecomposable.

The following result is used several times in our proofs.

Lemma 3.2 *For every graph G, any indecomposable k-lens $\mathcal{L}$ (with $k = 2$ or 3) in G can be emptied through a finite sequence of $(\mathcal{Y}\Delta)_{\mathcal{M}}$ or $(\Delta\mathcal{Y})_{\mathcal{M}}$ operations.*

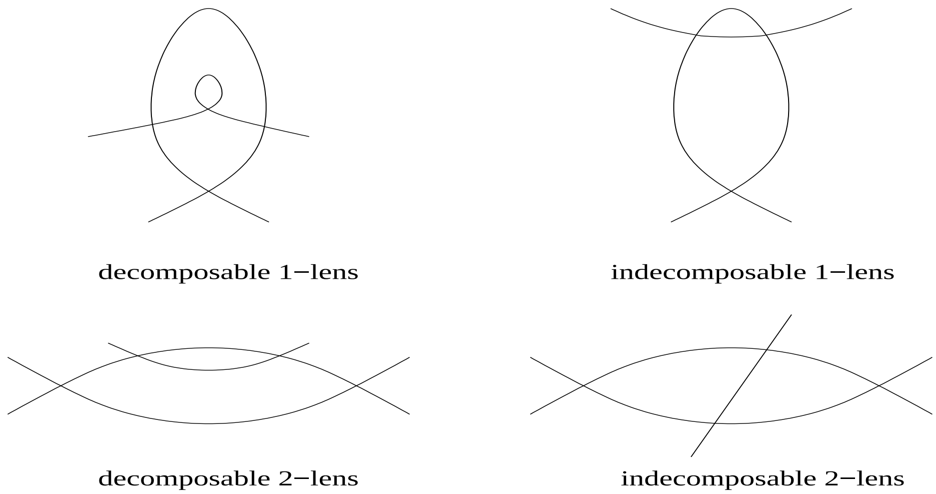

Figure 7: Decomposable and indecomposable k-lens ($k = 1, 2$)

Proof: The proof for $k = 2$ was given by Grünbaum [7].

Now assume that $\mathcal{L}$ is a 3-lens. We first prove two elementary results.

Proposition 3.3 *Let $\mathcal{L}$ be an indecomposable 3-lens with geodesic boundary arcs A, B, and C. Suppose $\mathcal{L}$ does not contain chords meeting both A and B, then there exists a triangular face R in $\mathcal{L}$ sharing an edge with C (see Figure 8).*

Proof: We give an algorithmic proof. Let $\mathcal{C}$ be the set of chords in $\mathcal{L}$ including the geodesic boundary arcs A and B. We orientate each geodesic path in $\mathcal{C}$ from its common vertex with A or B to its common vertex with C (see Figure 9).

Let C_0 be a geodesic path in $\mathcal{C}$ chosen arbitrarily. Inductively suppose we have built C_i ($i \geq 0$), we will construct C_{i+1} as follows. We represent by w_i the intersection vertex between C_i and C and by v_i the vertex along C_i (following its orientation) appearing just before w_i, vertex v_i always exists (probably on A or/and B). We designate C_{i+1} the geodesic arc in $\mathcal{C}$ which is distinct to and meets C_i at v_i. We stop this inductive construction when v_i is equal to v_{i+1}.

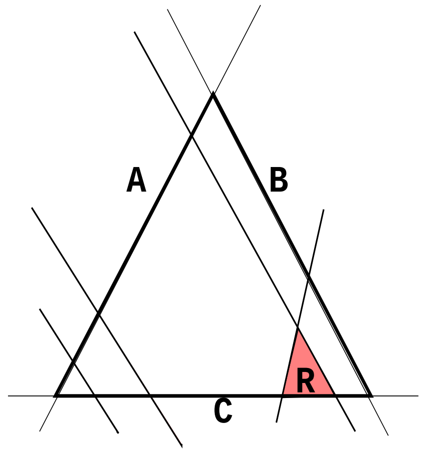

Figure 8: Illustration of Lemma 3.3

Geodesic paths $C_0, C_1, C_2, \ldots$ are pair-wise distinct, otherwise a 1- or 2-lens should be contained in $\mathcal{L}$ contradicting that $\mathcal{L}$ is indecomposable. Since $\mathcal{C}$ is finite the sequence $C_0, C_1, C_2, \ldots$ is finite too, so there exists a maximum index n such that C_n belongs to the sequence. It follows from the construction that v_{n-1}, w_{n-1} and w_n are the vertices of a triangular face in $\mathcal{L}$ which share the edge $w_{n-1}w_n$ with C and the proposition is true. ■

An example of this construction appears in Figure 9. In this case $C_0 = A$, $n = 2$ and the final triangular face is $v_1 w_1 w_2$.

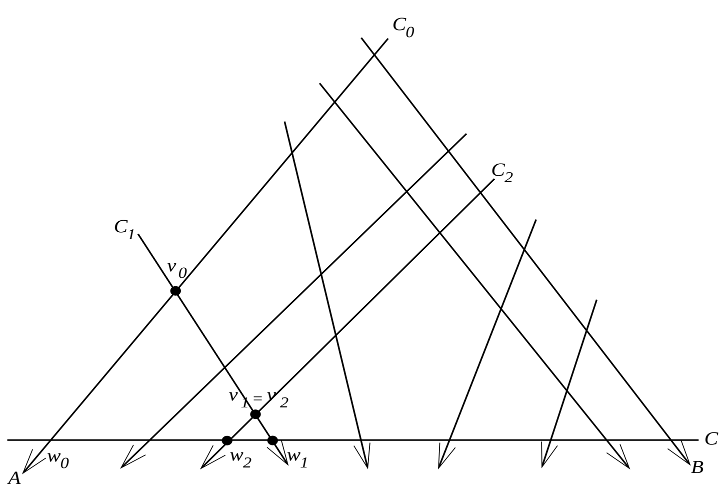

Figure 9: Example of the construction in Proposition 3.3.

Corollary 3.4 *Any indecomposable 3-lens $\mathcal{L}$ with boundary arcs A, B, and C which does not contain chords meeting simultaneously A and B can be emptied through medial $(\mathcal{Y}\Delta)_{\mathcal{M}}$ or $(\Delta\mathcal{Y})_{\mathcal{M}}$ operations.*

Proof: The triangular face R determined by Lemma 3.3 can be eliminated from $\mathcal{L}$ applying a $(\mathcal{Y}\Delta)_{\mathcal{M}}$ or a $(\Delta\mathcal{Y})_{\mathcal{M}}$ operation. The resulting 3-lens satisfies the conditions in the hypothesis of Proposition 3.3 and, again, there exists a triangular face R' which can be eliminated. We may continue in this manner until the original 3-lens is emptied. ∎

Now, to prove Lemma 3.2 let A, B and C be the boundary arcs of $\mathcal{L}$. Among the chords in $\mathcal{L}$ meeting simultaneously A an B we choose one (in general there are several possibilities), named D, such that in the 3-lens with boundary arcs A, B and D no chord meets simultaneously A and B. We empty this 3-lens using the method in Corollary 3.4 and then we eliminate D from the interior of the lens by a $(\mathcal{Y}\Delta)_{\mathcal{M}}$ or $(\Delta\mathcal{Y})_{\mathcal{M}}$ transformation. We continue in this way until the original 3-lens becomes singular. This completes the proof. ∎

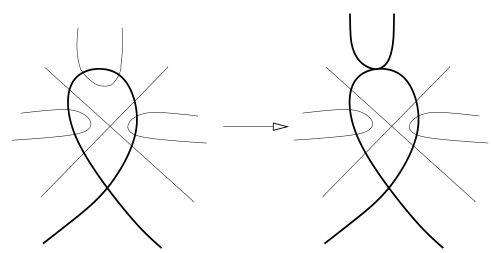

Figure 10: After a medial series or parallel reduction a 1-lens becomes a 2-lens

The last Lemma is not valid for $k = 1$: in Figure 10 we show a 1-lens to which only a medial series or parallel reductions can be applied, notice that after applying any of these reductions, the 1-lens is transformed into a 2-lens and the structure of the original 1-lens is lost. So, “emptied” does not apply in this context.

The Lemma does not hold for $k > 3$. Figure 11 depicts a 4-lens which cannot be emptied

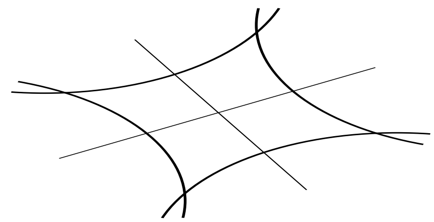

Figure 11: A 4-lens which cannot be emptied by $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ transformations

by the application of any $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ transformations.

4 One-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducibility

We study how to obtain a reduction in a medial graph with terminal faces by $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ operations.

Definition 4.1 *A medial graph G is* one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible *if, it admits a sequence of $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ operations in which all of them are $(\Delta\mathcal{Y})_{\mathcal{M}}$ or $(\mathcal{Y}\Delta)_{\mathcal{M}}$ except the last one, which is a reduction of type $R0_{\mathcal{M}}, R1_{\mathcal{M}}, R2_{\mathcal{M}}, R3_{\mathcal{M}}$, or $FP_{\mathcal{M}}$.*

A *t-scheme* in a graph G is a 1-lens $\mathcal{T}$ containing a chord P. Let $S = w_0 \dots w_j w_0$ and $P = p_0 \dots p_i$ be the boundary and the chord of $\mathcal{T}$, respectively. Assume that $p_0 = w_s$ and $p_i = w_r$ $(0 < r < s \leq j)$. The *2-lens of* $\mathcal{T}$, denoted $\mathcal{L}^2(\mathcal{T})$, is the 2-lens with boundary $p_0 \dots p_{i-1} w_r w_{r+1} \dots w_{s-1} p_0$. The *3-lens of* $\mathcal{T}$, denoted $\mathcal{L}^3(\mathcal{T})$ is the 3-lens with boundary $w_0 \dots w_{r-1} p_i \dots p_1 w_s \dots w_0$. (See Figure 12).

Lemma 4.1 *Let $\mathcal{L}$ be an arbitrary k-lens in a graph G with $k = 1$ or $k = 2$. Then it must contain a singular 1-lens or an indecomposable 2-lens.*

Proof: We start with $k = 2$. If $\mathcal{L}$ is indecomposable the result is true, otherwise it contains a 1-lens or a 2-lens. If $\mathcal{L}$ contains a 1-lens, say $\mathcal{M}$, we have two possibilities:

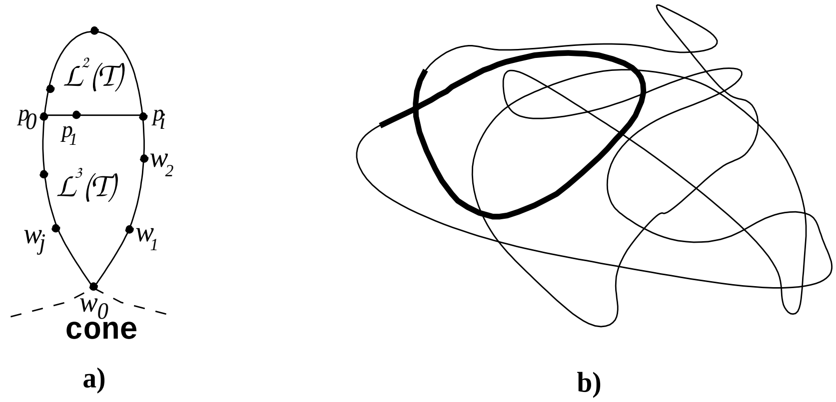

Figure 12: (a) A t-scheme $\mathcal{T}$. (b) A t-scheme in a four regular plane graph

L.1 $\mathcal{M}$ is singular and the result is true.

L.2 $\mathcal{M}$ is not singular. Now, if $\mathcal{M}$ is not a t-scheme then it contains a geodesic arc which intersects itself in its interior, so $\mathcal{M}$ contains a 1-lens $\mathcal{T}$ which we take in place of M, and we repeat this replacement until an empty loop or a t-scheme is found. If $\mathcal{M}$ is a t-scheme then consider $\mathcal{L}^2(\mathcal{M})$ instead of $\mathcal{L}$ (we mean, $\mathcal{L}$ has been replaced now by $\mathcal{L}^2(\mathcal{M})$). If the new $\mathcal{L}$ is indecomposable the proof is over, otherwise we start again the analysis in a recursive manner. Since G is finite we finish after a finite number of steps.

If $\mathcal{L}$ does not contain a 1-lens then, since $\mathcal{L}$ is decomposable, must contain a 2-lens $\mathcal{N}$ which takes the place of $\mathcal{L}$ and we continue as before, in a recursive way. This completes the analysis for $k = 2$.

For $k = 1$ the analysis corresponds to Cases L.1 and L.2 given before. ■

This result has the following corollary.

Corollary 4.2 *Let $\mathcal{L}$ be an arbitrary terminal free k-lens in a graph G, with $k = 1$ or $k = 2$. Then G is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: By Lemma 4.1 $\mathcal{L}$ contains a singular 1-lens or an indecomposable 2-lens. In the former case G reduces its number of edges and faces after the application of a $R0_{\mathcal{M}}$ or a $R1_{\mathcal{M}}$ operation. In the latter case the indecomposable 2-lens can be emptied by Lemma 3.2, and G reduces the sum of the number of vertices, edges and faces, after the application of a $R2_{\mathcal{M}}$ or a $R3_{\mathcal{M}}$ operation. ∎

For 3-lenses we have a similar corollary:

Corollary 4.3 *Let $\mathcal{L}$ be an arbitrary terminal free 3-lens in a graph G. If $\mathcal{L}$ is not indecomposable then G is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: Since $\mathcal{L}$ is not indecomposable it contains a 1-lens or a 2-lens. The result follows from Corollary 4.2. ∎

Lemma 4.4 *If G has a non singular 1-lens $\mathcal{T}$ containing at most one terminal face then G is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: Suppose that $\mathcal{T}$ is not a t-scheme, hence $\mathcal{T}$ has no chords, and thus it contains at least a 1-lens $\mathcal{T}'$. If $\mathcal{T}'$ is singular and terminal free then the result follows because we can apply a $R0_{\mathcal{M}}$ or a $R1_{\mathcal{M}}$ operation. If $\mathcal{T}'$ is singular and contains a terminal face, then the cone of $\mathcal{T}'$ is not a terminal face and so we can apply the $FP_{\mathcal{M}}$ reduction and the result is true. If $\mathcal{T}'$ is not singular and terminal free then the result follows from Corollary 4.2. If $\mathcal{T}'$ is not singular and contains at most one terminal face then $\mathcal{T}$ is replaced with $\mathcal{T}'$ and we start the analysis in a recursive manner.

Now, if $\mathcal{T}$ is a t-scheme then one of $\mathcal{L}^2(\mathcal{T})$ or $\mathcal{L}^3(\mathcal{T})$ is terminal free. In the first case the result follows from Lemma 3.2. In the second case, if $\mathcal{L}^3(\mathcal{T})$ is not indecomposable then the result is true by Corollary 4.3. If $\mathcal{L}^3(\mathcal{T})$ is indecomposable then we empty it using the method given in the proof of Corollary 3.4 (in order to simplify the explanation, we still denote by $\mathcal{T}$ the resulting 1-lens). After this $\mathcal{L}^3(\mathcal{T})$ becomes a triangular non terminal face which can be eliminated from $\mathcal{T}$ by the application of a $(\Delta\mathcal{Y})_{\mathcal{M}}$ or $(\mathcal{Y}\Delta)_{\mathcal{M}}$ operation. After this operation the cone of $\mathcal{T}$ is not a terminal face and we have two possible situations:

1. The 1-lens $\mathcal{T}$ is a loop containing a terminal face. Then its cone does not contain a terminal face and we can apply an $FP_{\mathcal{M}}$ reduction.

2. The 1-lens $\mathcal{T}$ is not singular and contains at most one terminal face. In this case we start again with the analysis on $\mathcal{T}$ in a recursive way. ■

Corollary 4.5 *If G has a 1-lens $\mathcal{T}$ containing at most one terminal face, and the cone of $\mathcal{T}$ is terminal free then G is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: If $\mathcal{T}$ is not singular then the result follows from Lemma 4.4. Otherwise $\mathcal{T}$ satisfies one of the following:

1. $\mathcal{T}$ is terminal free, so the result is true since we can apply one of the reductions R0 or R1.

2. $\mathcal{T}$ contains exactly one terminal face. The result is true again because we can apply the $FP_{\mathcal{M}}$ reduction. ■

5 Main Theorem

We prove that any graph with three terminal black faces is $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible to one of the graphs $\mathcal{M}(P_3)$ or $\mathcal{M}(P_3')$ given in Figure 13. We call these graphs the *irreducible* $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ *graphs* or simply *irreducible graphs.*

In order to reduce a graph with three terminal black faces, we first show (Theorem 5.2) that any graph not isomorphic to the irreducible graphs must contain at least one of four

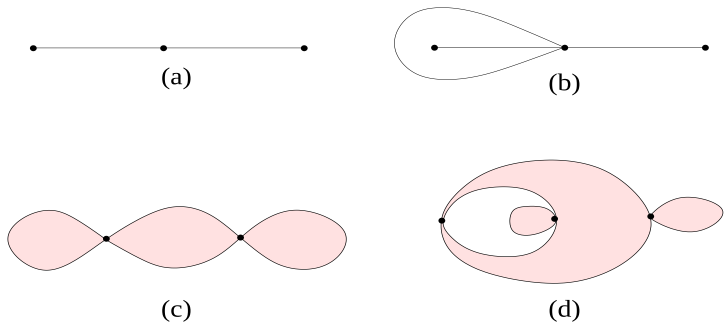

Figure 13: Irreducible graphs: a) P_3, b) P_3', c) $\mathcal{M}(P_3)$ and d) $\mathcal{M}(P_3')$

configurations. Next we show that each of these four configurations is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible (Theorem 5.11). For this we use the results of sections 3 and 4 to show that there always exist a terminal free indecomposable $k - lens$ ($k = 1, 2$) or a non singular $1 - lens$ containing at most one terminal face, or a singular $1 - lens$ containg a terminal face with terminal free cone. Since each of these is one step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible, a reduction is obtained. Hence the sum of the number of vertices, edges and faces is decreased. Then the main result (Theorem 5.12) follows by successive application of a finite sequence of one-step medial reductions.

Henceforth G_3 will denote a connected four regular plane graph with three terminal black faces, together with its straight decomposition. Let C be a closed geodesic arc in G_3. A vertex x in C is a *self-intersection vertex* of C, if all the edges which are incident to x belong to C. The *self-intersection number* of C is the number of self-intersection vertices in C.

Lemma 5.1 *A closed geodesic arc C in a graph G_3 contains the boundary of a 1-lens, if and only if, its self-intersection number is greater or equal to one.*

Proof: Let us assume that C contains the boundary of a 1-lens $\mathcal{L}$. All the edges which are incident to the pole of $\mathcal{L}$ belong to C. This pole is a self-intersection vertex of C and the result follows.

Suppose now that the self-intersection number of C is greater than zero. Let us assume that $C = v_0 v_1 \ldots v_n$ with $v_0 = v_n$. The smallest number k such that the sequence $v_0, \ldots, v_k$ contains a repeated vertex is lower than n, otherwise the self-intersection number of C would be equal to zero. If $v_i = v_k$ for some $0 \le i < k$ then $v_i v_{i+1} \ldots v_k$ is the boundary of a 1-lens contained in C. ■

Theorem 5.2 *Let G_3 be a non-empty graph, then it (see Figure 14):*

C1 contains a closed geodesic arc C with self-intersection number greater or equal than two, such that there exists a non singular 1-lens whose boundary is contained in C, or

C2 contains a closed geodesic arc C with self-intersection number one, such that there exists a non singular 1-lens whose boundary is contained in C, or

C3 has at least one $1-lens$ and all its $1-lenses$ are singular, or

C4 contains only closed geodesic arcs with self-intersection number zero.

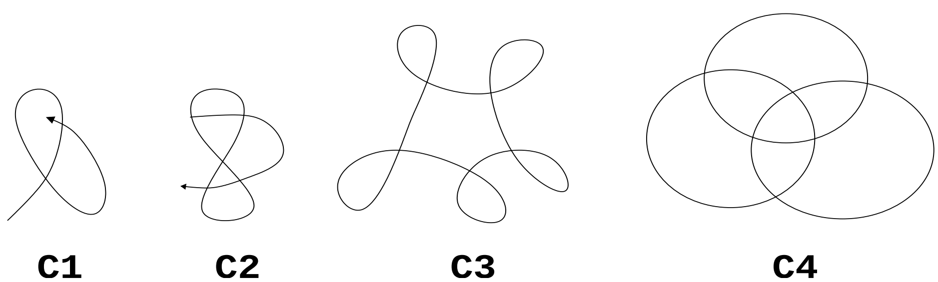

Figure 14: Configurations in Theorem 5.2

Proof: If G_3 contains a closed geodesic arc C such that there exists a non singular 1-lens whose boundary is contained in C, then from Lemma 5.1 the self-intersection number n of

C is greater than zero. If $n \geq 2$ then C1 is satisfied. If $n = 1$ then C2 is true.

Otherwise, no geodesic arc in G_3 contains the boundary of a non sigular 1-lens.

Thus, if one closed geodesic arc D in G_3 has self-intersection number greater than zero, we have from Lemma 5.1 that D contains the boundary of a 1-lens. This 1-lens is singular as well as any other 1-lens contained in G_3, so C3 is true.

In the remaining case no geodesic arc in G_3 has self-intersection number greater than zero and C4 is satisfied. ∎

We will prove now that any graph G_3 which is not isomorphic to one of the irreducible graphs $\mathcal{M}(P_3)$ and $\mathcal{M}(P_3')$ is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible (see Theorem 5.11 below). We will prove this result for each case in Theorem 5.2. The key idea is to determine, (for $k \leq 3$):

1. Four k-lenses (with $k = 0$ or 1) having pairwise disjoint interiors, or
2. two k-lenses (with $k = 0$ or 1) $\mathcal{L}$ and $\mathcal{N}$ and one 1-lens $\mathcal{T}$ having pairwise disjoint interiors and the cone of $\mathcal{T}$ not contained in the interior of the three lenses, or
3. two 1-lenses with disjoint interiors and cones disjoint of both interiors.

In each case we try to locate four disjoint regions, if we are successful then one of them should be terminal free (because G_3 contains at most three terminals). The terminal free region is used to prove that G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. If we are not able to find a terminal free region we prove that G_3 is isomorphic to one of the irreducible graphs.

In order to complete the proof we need some structural results about the existence of lenses with disjoint interiors. The first one is a direct consequence of Jordan's Theorem curve.

Lemma 5.3 *Let C_1 and C_2 be two Jordan curves on the plane such that $C_1 \subset ext(C_2) \bigcup C_2$ and $C_2 \subset ext(C_1) \bigcup C_1$. Then $int(C_1) \bigcap int(C_2) = \emptyset$.*

Here $\mathrm{int}(C_1)$ and $\mathrm{ext}(C_1)$ denote the interior and exterior regions of C_1, respectively. It is used that $\mathrm{int}(C_1)$ be the bounded connected region of C_1 and $\mathrm{ext}(C_1)$ the other one. But for a 1, 2 or 3-lens $\mathcal{L}$, as we established in Section 3, $\mathrm{ext}(\partial(\mathcal{L}))$ is the connected region containing the rays of the lens and $\mathrm{int}(\partial(\mathcal{L}))$ is the other connected region.

Lemma 5.4 *Let G_3 be a graph containing a non sigular 1-lens $\mathcal{T}$. Then one of the following cases is satisfied (see Figure 15).*

Case 5.4.1. There exist a 2-lens $\mathcal{L}$ in G_3 whose interior is disjoint to the interior of $\mathcal{T}$.

Case 5.4.2. There exist a 1-lens $\mathcal{N}$ in G_3 whose interior and cone are disjoint to the interior of $\mathcal{T}$.

Proof: Let v_1v_2 be an edge in G_3 contained in the exterior of $\mathcal{T}$ in such a way that v_1 belongs to the boundary of $\mathcal{T}$ but v_1 is not the pole of $\mathcal{T}$. Vertex v_1 exists because $\mathcal{T}$ is non singular.

Let $v_0v_1v_2 \dots v_k$ be the geodesic arc containing v_1v_2 such that k is the minimum $k > 1$ satisfying one of the conditions: i) v_k is in the boundary of $\mathcal{T}$, or ii) $v_k = v_j$ for some index $0 < j < k$. Vertex v_k does exist because the edge v_0v_1 is in the interior of $\mathcal{T}$ (since v_1 is not the pole of $\mathcal{T}$), $v_1v_2 \in \mathrm{ext}(\mathcal{T})$ and the geodesic arc $v_1v_2\dots$ is closed, then in some vertex different from v_1, say $v_{k'}$, this geodesic arc should cross again the boundary of $\mathcal{T}$ and thus k should be lower or equal than k'.

We have two possibilities.

Condition i) is satisfied. Since v_1 and v_k belong to the boundary of $\mathcal{T}$ there exist a geodesic arc $t_1, \dots, t_l$ contained in the boundary of $\mathcal{T}$ with $v_1 = t_1$ and $v_k = t_l$. In this way $v_1 \dots v_k t_{l-1} \dots t_1$ is the boundary of a 2-lens $\mathcal{L}$. From the construction we have that

$\partial(\mathcal{L}) \subset \mathrm{ext}(\mathcal{T}) \bigcup \partial(\mathcal{T})$ and $\partial(\mathcal{T}) \subset \mathrm{ext}(\mathcal{L}) \bigcup \partial(\mathcal{L})$; so, from Lemma 5.3 the interior of $\mathcal{T}$ is disjoint to the interior of $\mathcal{L}$, and Case 5.4.1 is satisfied.

Condition ii) is satisfied. Then $v_j \dots v_k$ is the boundary of a 1-lens $\mathcal{N}$. Since all the edges $v_j v_{j+1}, \dots, v_{k-1} v_k$ are in the exterior of $\mathcal{T}$ we have that $\partial(\mathcal{N}) \subset ext(\mathcal{T})$. In the same manner we can prove that $\partial(\mathcal{T}) \subset ext(\mathcal{N})$. We conclude from Lemma 5.3 that the interior of $\mathcal{T}$ is disjoint to the interior of $\mathcal{N}$, and Case 5.4.2 is true. ∎

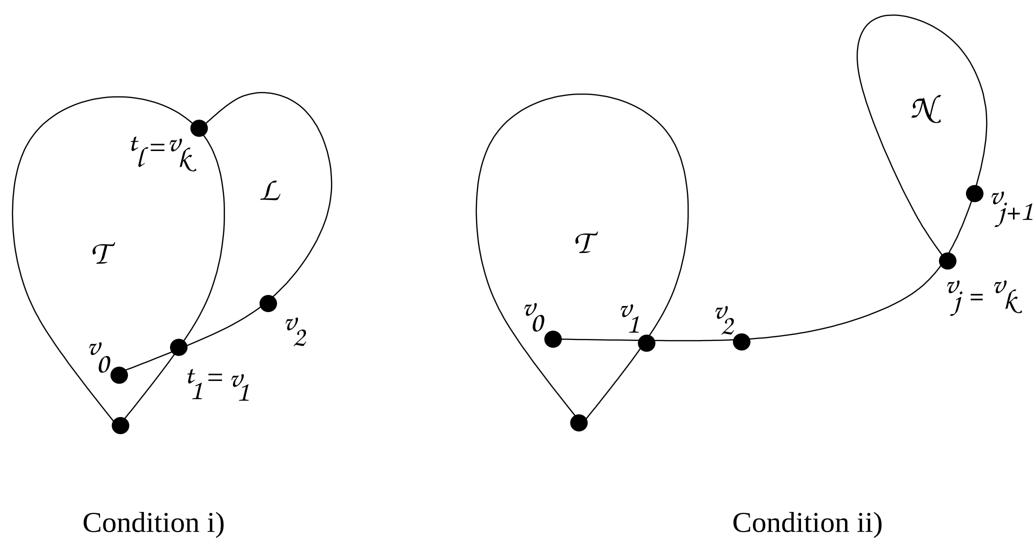

Figure 15: Illustration of Lemma's 5.4 proof.

Assertions like $\partial(\mathcal{L}) \subset \mathrm{ext}(\mathcal{T}) \bigcup \partial(\mathcal{T})$ in previous lemma deserve some further analysis. For instance, from the construction for Condition i), part of $\partial(\mathcal{L})$ is the geodesic arc $t_1, \dots, t_l$ which is contained in $\partial(\mathcal{T})$, the remaining part of $\partial(\mathcal{L})$ is the geodesic arc $v_1 v_2 \dots v_k$ which is formed by edges contained in the exterior of $\mathcal{T}$ because $v_1 v_2$ is in the exterior of this lens and the geodesic arc $v_1 \dots v_k$ never crosses the boundary of $\mathcal{T}$, so $\partial(\mathcal{L}) \subset \mathrm{ext}(\mathcal{T}) \bigcup \partial(\mathcal{T})$ is hold. These type of analysis however will not be done henceforth, because they are straightforward and it helps to make shorter proofs. At some point we will go beyond and simply say "the interiors of $\mathcal{T}$ and $\mathcal{L}$ are disjoint".

Lemma 5.5 *Let G_3 be a graph containing a closed geodesic arc C with self-intersection number greater than one such that there exists a non-sigular 1-lens $\mathcal{T}$ whose boundary is contained in C. Then one of the following is satisfied.*

Case 5.5.1. There exist a non-singular 1-lens $\mathcal{O}$ whose interior is disjoint to the interior of $\mathcal{T}$.

Case 5.5.2. There exist a 1-lens $\mathcal{N}$ whose interior and cone are disjoint to the interior of $\mathcal{T}$.

Proof: Let v_1 be the pole of $\mathcal{T}$ and let v_1v_2 be one of the rays of $\mathcal{T}$; by definition v_1v_2 are in the exterior of $\mathcal{T}$. Let $v_1v_2 \dots v_k$ be the geodesic arc containing v_1v_2 such that k is the minimum $k > 1$ satisfying one of the conditions: i) v_k is in the boundary of $\mathcal{T}$, or ii) $v_k = v_j$ for some index $0 < j < k$. Vertex v_k does exist because the closed geodesic arc C has at least two self-intersection vertices, one is v_1 and the other one is found when $v_1v_2 \dots$ intersects itself or reaches the boundary of $\mathcal{T}$.

Under condition ii) we can prove that Case 5.5.2 is hold exactly as we did with Case 5.4.2 at Lemma 5.4. So we only need to prove Case 5.5.1.

Let us assume that condition i) is hold and let $v_1u_2 \dots u_{l-1}v_1$ the boundary of $\mathcal{T}$, since $\mathcal{T}$ is not singular $l > 3$. We know that v_k is in the boundary of $\mathcal{T}$ and it is different to v_1, so there exists an index $1 < j < l$ such that $v_k = u_j$. The geodesic arc $u_ju_{j+1} \dots u_{l-1}v_1v_2 \dots v_k$ is the boundary of a 1-lens $\mathcal{O}$. From this construction we have that $\partial(\mathcal{O}) \subset \mathrm{ext}(\mathcal{T}) \bigcup \partial\mathcal{T}$ and $\partial(\mathcal{T}) \subset \mathrm{ext}(\mathcal{O}) \bigcup \partial\mathcal{O}$ and conclude from Lemma 5.3 that the interior of $\mathcal{T}$ is disjoint to the interior of $\mathcal{O}$. Finally $\mathcal{O}$ is not singular because the pole of $\mathcal{T}$ (v_1) is on the boundary of $\mathcal{O}$ and it is distinct to the pole of $\mathcal{O}$ (v_k). We conclude that Case 5.5.1 is true. ∎

For Case 5.5.1 it is possible to prove something stronger: besides the 1-lens $\mathcal{O}$ we can find a 2-lens $\mathcal{L}$ (with boundary $v_1u_2 \dots u_jv_{k-1} \dots v_1$) such that the interiors of $\mathcal{T}$, $\mathcal{O}$, and $\mathcal{L}$ are pairwise disjoint. However the existence of $\mathcal{L}$ is irrelevant when we work with at most three

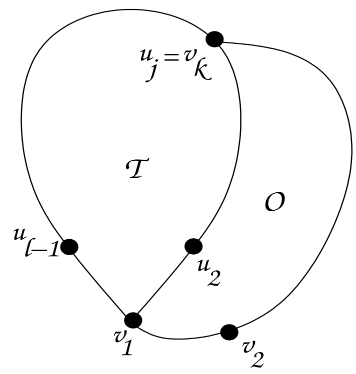

Figure 16: Illustration of Lemma's 5.5 proof.

terminals.

Proposition 5.6 *Let G_3 be a graph containing a closed geodesic arc C with self-intersection number greater or equal than two, such that there exists a non singular 1-lens $\mathcal{T}$ whose boundary is contained in C. Then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. In other words, if G_3 satisfies condition C1 of Theorem 5.2, then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: If $\mathcal{T}$ contains at most one terminal face then the proposition follows from Lemma 4.4. Otherwise $\mathcal{T}$ contains two or more terminal faces.

From Lemma 5.5 we know that one of the cases 5.5.1 or 5.5.2 is hold.

If Case 5.5.1 is true then G_3 contains a singular 1-lens $\mathcal{O}$ with interior disjoint to $\mathcal{T}$. Terminal faces in $\mathcal{T}$ are not in $\mathcal{O}$ (on the contrary the interiors of $\mathcal{T}$ and $\mathcal{O}$ would intersect at common terminal faces). In this way $\mathcal{O}$ contains at most one terminal face and the result follows applying Lemma 4.4 to $\mathcal{O}$.

If Case 5.5.2 is satisfied then G_3 contains a 1-lens $\mathcal{N}$ whose interior and cone are disjoint to $\mathcal{T}$. If the cone of $\mathcal{N}$ is terminal free then $\mathcal{N}$ contains at most one terminal face and the result follows from Corollary 4.5. If the cone of $\mathcal{N}$ contains a terminal face then $\mathcal{N}$ is terminal free and the result follows from Corollary 4.2. ∎

Proposition 5.7 *Let G_3 be a graph containing a closed geodesic arc C with self-intersection number one, such that there exists a non singular 1-lens $\mathcal{T}_1$ whose boundary is contained in C. Then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. In other words, if G_3 satisfies condition C2 of Theorem 5.2, then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: Since C has self-intersection number one, it contains two 1-lenses with disjoint interiors. One of these is $\mathcal{T}_1$, name the other $\mathcal{T}_2$.

If $\mathcal{T}_1$ contains at most one terminal face then the result follows from Lemma 4.4. Otherwise $\mathcal{T}_1$ contains two or more terminal faces and if $\mathcal{T}_2$ is non singular, for sure, it contains at most one terminal face and again the result follows from Lemma 4.4.

We can assume now that $\mathcal{T}_1$ contains two or more terminal faces and that $\mathcal{T}_2$ is singular. If $\mathcal{T}_2$ is terminal free then the result follows from Corollary 4.2.

The remaining case is when $\mathcal{T}_1$ contains exactly two terminal faces, $\mathcal{T}_2$ is singular and contains a terminal face. From Lemma 5.4 (applied to $\mathcal{T}_1$) we know that there exist a 1- or 2-lens $\mathcal{T}_3$ whose interior is disjoint to $\mathcal{T}_1$. On the other hand $\partial(\mathcal{T}_3) \subset \mathrm{ext}(\mathcal{T}_2)$ and $\partial(\mathcal{T}_2) \subset \mathrm{ext}(\mathcal{T}_3)$; from Lemma 5.3 we know that the interior of $\mathcal{T}_3$ is disjoint to the interior of $\mathcal{T}_2$. It means that $\mathcal{T}_3$ is terminal free. The result follows from Corollary 4.2. ■

Let G_3 be a graph. If v_1v_2 is one edge in G_3 we say that v_1v_2 is *between* v_1w_1 and v_1w_2 *with respect to* v_1, if and only if: i) the edges v_1w_1 and v_1w_2 are in G_3, ii) none of the edges v_1w_1 and v_1w_2 is the direct extension of v_1v_2. For instance, in Figure 17.a the edge e is between sq_1 and sp_1 with respect to s. This concept is used in the proof of the next proposition.

Proposition 5.8 *Let G_3 be a graph non-isomorphic to the irreducible graphs $\mathcal{M}(P_2)$, $\mathcal{M}(P_3)$, or $\mathcal{M}(P_3')$, containing at least one 1-lens, and in which all 1-lenses are singular. Then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. In other words, if G_3 satisfies condition C3 of Theorem 5.2, then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: If G_3 contains at least three 1-lenses then one of them is terminal free or one of their cones is terminal free. If one of the 1-lenses is terminal free the result follows from Corollary 4.2. Otherwise it follows from Corollary 4.5.

Graph G_3 cannot contain less than two 1-lenses. If G_3 contains exactly two 1-lenses $\mathcal{S}$ and $\mathcal{T}$ whose cones c_s and c_t are different faces then $\mathrm{int}(c_s) \cap \mathrm{int}(\mathcal{T}) = \mathrm{int}(c_t) \cap \mathrm{int}(\mathcal{S}) = \emptyset$ (otherwise G_3 would be isomorphic to $\mathcal{M}(P_2)$). It means that the interiors of $\mathcal{S}$, $\mathcal{T}$, c_s and c_t are pairwise disjoint and thus one of them is terminal free. If $\mathcal{S}$ or $\mathcal{T}$ is terminal free the result follows from Corollary 4.2, if c_s or c_t is terminal free, it follows from Corollary 4.5.

We only need to analyze the case in which G_3 contains exactly two 1-lenses $\mathcal{S}$ and $\mathcal{T}$ whose cones are in a common face, say f (see Figure 17). If one of $\mathcal{S}$, $\mathcal{T}$, or f are terminal free then the proposition follows from Corollaries 4.2 or 4.5 like in the previous paragraphs. So we assume that the three terminals are in $\mathcal{S}$, $\mathcal{T}$, and f.

Both 1-lenses $\mathcal{S}$ and $\mathcal{T}$ should belong to the same closed geodesic arc C (since the minimum number of 1-lenses in a closed geodesic arc with self-intersection number greater than zero is two, if $\mathcal{S}$ and $\mathcal{T}$ are in different closed geodesic arcs then the number of 1-lenses in G_3 is at least four but this is not possible), so the rays of $\mathcal{S}$ and $\mathcal{T}$ should be connected by two geodesic paths P and Q going from s to t (the poles of $\mathcal{S}$ and $\mathcal{T}$, respectively). All edges in C are in $P \bigcup Q \bigcup \mathcal{S} \bigcup \mathcal{T}$.

Since s and t belong to the boundary of their common face f there is a continuous curve in $\mathbb{R}^2$ going from s to t crossing f such that no edge in G_3 crosses that curve. We can think this curve as the embedding of some edge e going from s to t through f. Since P has no self-intersection vertices (because otherwise G_3 would contain more than two 1-lenses) the embedding of $P \bigcup \{e\}$ is a Jordan curve separating $\mathbb{R}^2$ into two connected regions R_1 and R_2. We have two cases: i) both 1-lenses $\mathcal{S}$ and $\mathcal{T}$ are in the same region, say R_1; or ii) the 1-lens $\mathcal{S}$ is contained into R_1 and $\mathcal{T}$ is contained into R_2 (see Figure 17.a and .b).

Let sq_1 and $q_{l-1}t$ be the edges in Q which are incident to s and t, respectively. In the same manner let sp_1 and $p_{m-1}t$ be the first and last edges in P. Edge e should be between sq_1 and sp_1 with respect to s and edge e should be between $q_{l-1}t$ and $p_{m-1}t$ with respect to t.

So, in case i) both edges sq_1 and $q_{l-1}t$ should be in R_1, it means that the number of times that Q crosses P (in vertices different to s and t) is even. When this number is zero and G_3 only contains one geodesic curve, G_3 is isomorphic to the irreducible graph $\mathcal{M}(P_3)$ (compare Figure 13.c and Figure 17.a). If the number of intersections among internal vertices of P and Q is greater than zero then two consecutive intersections between internal vertices of P and Q define a 2-lens $\mathcal{L}$ whose interior is disjoint to $\mathcal{S}$, $\mathcal{T}$ and f. The 2-lens $\mathcal{L}$ is terminal free and the result follows from Corollary 4.2.

In case ii) the edge sq_1 should be in R_1 but $q_{l-1}t$ should be in R_2 (see Figure 17.b), it means that the number of times that Q crosses P is odd. When this number is one and G_3 only contains one geodesic curve, G_3 is isomorphic to the irreducible graph $\mathcal{M}(P_3')$ (compare Figure 13.d and Figure 17.b). If the number of intersections among internal vertices of P and Q is greater than one then two consecutive intersections between internal vertices of P and Q define a 2-lens $\mathcal{L}'$ whose interior is disjoint to $\mathcal{S}$, $\mathcal{T}$ and f. The 2-lenses $\mathcal{L}'$ is terminal free and the result follows from Corollary 4.2.

Certainly G_3 could contain more closed geodesic curves besides C. Any additional closed geodesic curve should have self-intersection number zero because otherwise G_3 would contain more than two 1-lenses. If G_3 contains more closed geodesic curves, one of them, say D, should intersect C in a vertex v (because G_3 is connected). Without losing generality we may assume that $v \in P$. Then D is first going into R_1 crossing P, then at some different point v' in P D should leave R_1; between these two intersection vertices a 2-lens terminal free should be formed and the result follows from Corollary 4.2. ■

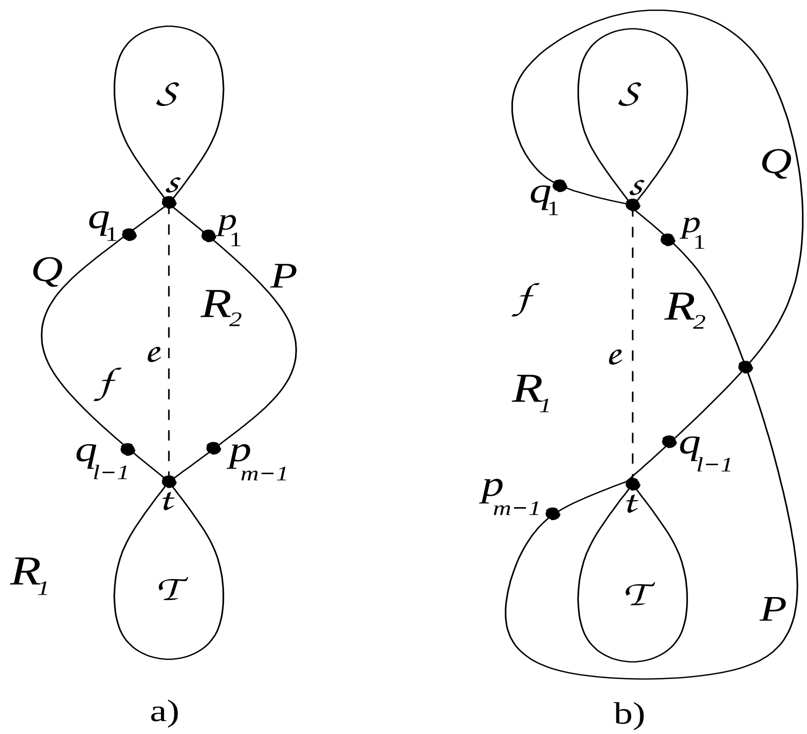

Figure 17: Illustration of Proposition 5.8

We need an additional structural result to complete the proof of Theorem 5.2.

Lemma 5.9 *Let G_3 be a graph containing two closed geodesic arcs C and D, both of them with self-intersection number zero and having at least one vertex in common then G_3 contains four 2-lenses whose interiors are pairwise disjoint.*

Proof: It follows from Jordan's theorem curve that C and D have an even number of vertices in common, and the minimum possibility is two. We distinguish two cases: i) C and D have exactly two intersection vertices and ii) C and D have more than two common vertices. Let us assume that $C = c_0c_1 \ldots c_{l-1}c_0$ and $D = d_0d_1 \ldots d_{m-1}d_0$ for some natural numbers l and m and $c_0 = d_0$.

- If case i) is true then there exist two indices l_1 and m_1 ($0 < l_1 < l$ and $0 < m_1 < m$) such that $c_{l_1} = d_{m_1}$. Denote $P_C = c_0 \ldots c_{l_1}, Q_C = c_{l_1} \ldots c_{l-1}c_0, P_D = d_0 \ldots d_{m_1}, Q_D = d_{m_1} \ldots d_{m-1}d_0$, then $P_CP_D^{-1}$, $P_CQ_D^{-1}$, $Q_CP_D^{-1}$ and $Q_CQ_D^{-1}$ are four 2-lenses whose interiors are pairwise disjoint (see Figure 18.a).
- If case ii) is satisfied then there exist four indices l_1, l_2, m_1 and m_2 ($0 < l_1 < l_2, < l$ and $0 < m_1 < m_2 < m$ such that $c_{l_1} = d_{m_1}$ and $c_{l_2} = d_{m_2}$. Denote $P_C = c_0 \ldots c_{l_1}$, $Q_C = c_{l_1} \ldots c_{l_2}$, $R_C = c_{l_2} \ldots c_{l-1}c_0$, $P_D = d_0 \ldots d_{m_1}$, $Q_D = d_{m_1} \ldots d_{m_2}$, then $P_CP_D^{-1}$, $Q_CQ_D^{-1}$, $R_CP_CQ_D$ and $Q_CR_CP_D$ are four 2-lenses whose interiors are pairwise disjoint (see Figure 18.b). ∎

Proposition 5.10 *Let G_3 be a graph in which all closed geodesic arcs have self-intersection number zero. Then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. In other words, if G_3 satisfies condition C4 of Theorem 5.2, then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: Any vertex in G_3 is the intersection of two closed geodesic arcs C and D, both of them with self intersection number zero. From Lemma 5.9 G_3 contains four 2-lenses whose

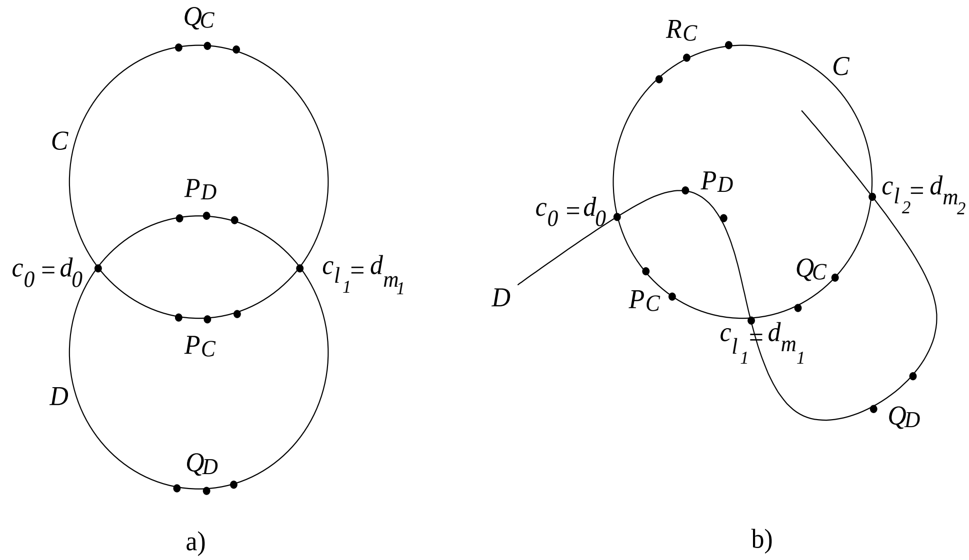

Figure 18: Illustration of Lemma 5.9.

interiors are pairwise disjoint. Since G_3 contains at most 3-terminals, one of these 2-lenses is terminal free and the result follows from Corollary 4.2. ∎

Theorem 5.11 *Let G_3 be a graph not isomorphic to one of the irreducible graphs $\mathcal{M}(P_3)$ and $\mathcal{M}(P_3')$, then G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible.*

Proof: We have proved respectively in Propositions 5.6, 5.7, 5.8 and 5.10 that for cases C1, C2, C3 and C4 of Theorem 5.2, G_3 is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible. Since these cases cover any possibility this ends the proof of Theorem 5.11. ∎

Theorem 5.12 *Any graph G_3 can be $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduced to one of the irreducible graphs.*

Proof: Since any graph G_3 which is not isomorphic to one of the irreducible graphs is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible and in each one-step reduction the sum of the number of vertices, edges and faces is decreased, after a finite number of one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reductions we reach one of the irreducible graphs.

It readily follows that any connected planar graph with at most three terminals is $(\Delta \leftrightarrow \mathcal{Y})$ reducible to a vertex, an edge, a path P_3, or the graph P_3'. Each vertex in the reduced graph is terminal. ∎

6 Implementation of the algorithm

It is worth noticing that the proof of Theorem 5.12 implicitly gives an algorithm for reducibility that can be implemented efficiently. We did one implementation in the C programming language just as it is described in the previous section and without any optimization. The program was used to test the proof completeness. In our experiments we reduced thousands of graphs randomly generated with about 200 vertices and faces. For each graph we made the reduction for any possible way to place the terminals and in all cases the algorithm found the reductions exactly as is described in the proof.

The program is available at

`http://aishia.math.cinvestav.mx/~deltawye/deltawye.html`

there you can find the instructions to submit planar graphs as well as to interpret the results. The source code is available requesting it directly to the authors of this work.

A simpler and more efficient implementation is the following one. We build a data structure that contains a representation of the indecomposable 1-lens and 2-lens. For each lens we take the number of interior faces as a measure of its complexity, we also save the number, if any, of the terminals it contains. Then we iteratively proceed as follows: we find a lens of minimum complexity that is one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducible and empty it. From theorems 5.2 and 5.11 we know such lens always exists. Since in each one-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduction the sum of the number of vertices, edges and faces is decreased, after a finite number of iterations we reach one of the irreducible graphs.

We finish this section giving some details about the constructive approach in the proof of Theorem 5.2 that we have implemented.

Algorithm 6.1 *[To $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduce a graph G_3]*

Input: *The graph G_3.*

Output: *The $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduction of G_3.*

Method:

1. $G \leftarrow G_3$

2. **While** *G is not $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduced* **do**

3. *Determine the closed geodesic arc pattern C1, C2, C3 o C4 in the statement of theorem 5.2 contained in G.*

4. *Identify the pairwise interior disjoint 1- and 2-lenses in P (use Lemma 5.4, 5.5 or 5.9 to arc patters C1, C2 and C4, respectively). For C3 follow Proposition 5.8.*

5. *Count internal and terminal faces inside the 1 or 2-lenses in P. Identify 1-lenses' cones. Finally determine the lens $\mathcal{L}$ in P that should be emptied.*

6. **If** *$\mathcal{L}$ is a t-scheme containing a terminal face* **then**

7. *Find recursively the singular terminal free 1-lens, or the indecomposable terminal free 2-lens or the 1-lens with exactly one terminal face and empty cone inside $\mathcal{L}$ to make a 1-step $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reduction (see Lemma 4.4 and Corollary 4.5).*

8. **If** *$\mathcal{L}$ is a 1-lens containing exactly one terminal face and the cone of $\mathcal{L}$ is terminal free* **then**

9. *Apply the medial* FP *assignment to $\mathcal{L}$.*

10. **else**

Empty and reduce $\mathcal{L}$ (see Corollary 4.2).

11. Replace G by the reduced graph.

12. Return G.

Proposition 6.1 *Algorithm 6.1 runs in $O(|V(G_3)|^4)$ worst time.*

Proof: In this proof we denote by $n_i = |V(G_3)| - i + 1$ the number of vertices in G in iteration i at step 2 in the algorithm. We denote by $T_j(n_i)$ the time complexity of step j to accomplish iteration i.

Steps 3 and 4 require traversing the edges in G along the geodesic curves. In the route every edge should be oriented in the traversing direction; then, for each geodesic we should count the number of self-intersections and look at the relative orientation of the geodesic in these vertices to determine the pattern P as well as the 1- and 2-lenses it contains. Since in the worst case every edge in G is reached and the number of edges in G is proportional to n_i, this process is completed in time $T_3(n_i) + T_4(n_i) = O(n_i)$

The counting of internal faces and terminal inside 1- or 2-lenses in P to accomplish step 5, could be done without altering the complexity $T_3(n_i)$ for faces meeting the geodesic arc in P. The other faces could be counted recursively by visiting faces which are adjacent to previously counted ones. Since we know the 1-lenses in P we can find their cones too. In this process we finally choose $\mathcal{L}$ as the 1- or 2-lens in P with zero terminal faces or the $t-$ *scheme* in P containing one terminal face. All this process could be completed in time $T_5(n_i) = O(n_i)$.

Step 7 requires to locate first a chord in $\mathcal{L}$ or a loop in $\mathcal{L}$ and then determine where the terminal is (in the 3-lens or the 2-lens of $\mathcal{L}$, or in a 1-lens contained in $\mathcal{L}$), this process could be completed in $O(n_i)$ time. Then we may need to empty a 3- or 2- lens, that would take additional time $O(n_i^2)$. After that we can bring out of $\mathcal{L}$ one edge and continue recursively. Since in this process we may require make $\mathcal{L}$ singular, the number of steps is $T_7(n_i) = O(n_i^3)$.

In step 10 we need to identify if there is a 1- or 2-lens contained in $\mathcal{L}$. This could be done by navigating all the geodesic arcs contained in $\mathcal{L}$, we give to each geodesic arc a number and assign to each vertex the numbers of the geodesic arcs meeting at that vertex. A 1-lens is contained in $\mathcal{L}$, if and only if, the pair of numbers assigned to a vertex are equal. A 2-lens is contained in $\mathcal{L}$, if and only if, two vertices inside $\mathcal{L}$ have the same pair of numbers assigned. All this process could be completed in time proportional to the number of edges inside $\mathcal{L}$ which is $O(n_i)$. Once we identify a 1- or 2-lens inside $\mathcal{L}$ we replace $\mathcal{L}$ by the new lens. We continue in this way until $\mathcal{L}$ becomes indecomposable, the whole process takes $O(n_i^2)$ time in the worst case. Finally we empty $\mathcal{L}$ by locating and $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ transforming triangles incident to the boundary. We finish the process in $T_{10}(n_i) = O(n_i^2)$ time.

The total time we need to run the algorithm is

$$\begin{aligned} T(|V(G_3)|) &= \sum_{n_i=1}^{|V(G_3)|} (T_3(n_i) + T_4(n_i) + T_5(n_i) + T_7(n_i) + T_{10}(n_i)) \\ &= O(|V(G_3)|^4) \end{aligned}$$

■

In this proof we made a direct implementation of the constructive approach in Theorem 5.2, the algorithm could be substantially improved by the application of specialized dynamic data structures and algorithms. In fact we conjecture that it could be improved to $O(|V(G_3)|^3)$ worst time.

As a concluding remark observe that given the duality of $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ operations: $\{\mathbf{R0}_{\mathcal{M}}$ and $\mathbf{R1}_{\mathcal{M}}\}$, $\{\mathbf{R2}_{\mathcal{M}}$ and $\mathbf{R3}_{\mathcal{M}}\}$, $\{(\mathcal{Y}\boldsymbol{\Delta})_{\mathcal{M}}$ and $(\boldsymbol{\Delta}\mathcal{Y})_{\mathcal{M}}\}$ it is natural to define the dual operation $(\mathbf{FP}_{\mathcal{M}})^*$ associated to $\mathbf{FP}_{\mathcal{M}}$ which can be applied to a subset of white faces distinguished as terminals. Then one may consider the problem of terminal $(\Delta \leftrightarrow \mathcal{Y})_{\mathcal{M}}$ reducing a medial graph with a set of distinguished black and white faces as terminals by using all of the eight operations on the medial graph. This is equivalent to the problem of $\Delta \leftrightarrow \mathcal{Y}$ reducing a plane graph, in which we have subsets of vertices and faces as possible terminals.

References

[1] S. B. Akers Jr., The use of wye-delta transformations in network simplification, Operations Research, 8, (1960), 311-323.

[2] D. Archdeacon, C.J. Colbourn, I. Gitler and J.S. Provan, Four terminal reducibility and projective planar $\Delta \leftrightarrow Y$ reducible graphs, J. Graph Theory 33, (2000), 83-93.

[3] Y. Colin de Verdiere, I. Gitler and D. Vertigan, Reseaux electriques planaires II, Comment. Math. Helvetici 71 (1996), 144-167.

[4] G. V. Epifanov, Reduction of a plane graph to an edge by a star-triangle transformation, Soviet Math. Doklady, 166, (1966), 13-17.

[5] T. A. Feo and J. S. Provan, Delta-wye transformations and the efficient reduction of two-terminal planar graphs, Oper. Res. 41 (1993), 572-582.

[6] I. Gitler, Delta-wye-delta transformations algorithms and applications, Ph.D. Thesis, Univ. Waterloo, 1991.

[7] B. Grünbaum, Convex Polytopes, Interscience Publishers, London, 1967.

[8] A.B. Lehman, J. Soc. Indust. Appl. Math. 11 (1963).

[9] A. B. Lehman, Wye-delta transformations in probabilistic networks, J. SIAM, 11, (1962), 773-805.

[10] A. Schrijver, On the uniqueness of Kernels, J. Combin. Theory, Series B 55, (1992), 146-160.

[11] K. Truemper, On the delta-wye reduction for planar graphs, J. Graph Theory, 13, (1989), 141-148.