Hypercyclic, Supercyclic Non-Archimedean Linear Operators

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1. INTRODUCTION

Linear dynamics is a young and rapidly evolving branch of functional analysis, which was probably born in 1982 with the Toronto Ph.D. thesis of C. Kitai [10]. It has become rather popular, thanks to the efforts of many mathematicians (see [5, 6]). In particular, hypercyclicity and supercyclicity of weighted bilateral shifts were characterized by Salas [16, 17]. In [18] Shkarin proved the existence of a bounded linear operator T satisfying the Kitai Criterion on each separable infinite-dimensional Banach space. For more detailed information about cyclic, hypercyclic linear operators we refer to [1].

We stress that all investigations on dynamics of linear operators were considered over the field of the real or complex numbers. On the other hand, non-Archimedean functional analysis is well-established discipline, which was developed in Monna's series of works in 1943. Last decades there have been published a lot of books devoted to the non-Archimedean functional analysis (see for example [14, 19]). In [12] a Non-Archimedean spectral theorem has been recently developed for normal operator linear operators on non-Archimedean Banach spaces.

In the present talk, we are going to discuss dynamics of linear operators defined on topological vector space over non-Archimedean valued fields. We will show that there does not exist any hypercyclic operator on a finite dimensional space. Moreover, we give sufficient and necessary conditions of hypercyclicity (supercyclicity) of linear operators on separable F-spaces. We will show that a linear operator T on topological vector space X is hypercyclic (supercyclic) if it satisfies Hypercyclic (resp. Supercyclic) Criterion. Note that the shift operators have many applications in many branches of modern mathematics (in real setting).

Kocubei in [13] has considered various functional models of the unilateral shift operator. For the sake of completeness, let us provide one an illustrative example.

Example 1.1. Let $\mathbb{Z}_p$ be the unit ball in $\mathbb{Q}_p$. By $C(\mathbb{Z}_p, \mathbb{C}_p)$ we denote the space of all continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ endowed with "sup"-norm. Consider a linear operator $T : C(\mathbb{Z}_p, \mathbb{C}_p) \to C(\mathbb{Z}_p, \mathbb{C}_p)$ defined by

$$(Tf)(x) = f(x+1) - f(x), \quad (x \in \mathbb{Z}_p), \quad f \in C(\mathbb{Z}_p, \mathbb{C}_p).$$

We note (see [11]) that the operator T can be interpreted as the annihilation operators in a p-adic representation of the canonical commutation relations of quantum mechanics.

It is well known [15] that the Mahler polynomials

$$P_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}, \quad n \in \mathbb{N}; \quad P_0(x) = 1,$$

form an orthonormal basis in $C(\mathbb{Z}_p, \mathbb{C}_p)$. The operator T acts on the Mahler polynomials as follows:

$$TP_n = P_{n-1}, \quad n \in \mathbb{N}; \quad TP_0 = 0.$$

It is known that the spaces $C(\mathbb{Z}_p, \mathbb{C}_p)$ and $c_0(\mathbb{N})$ are isomorphic via the isomorphism

$$\sum_{n=0}^{\infty} x_n P_n \to (x_0, x_1, \ldots, x_n, \ldots)$$

therefore, the operator T is transformed to the shift operator.

We stress that the non-Archimedean shift operators have certain applications in p-adic dynamical systems [8, 9]. These investigations motivate us to consider weighted shifts (which are more general). So, we study weighted backward shifts on $c_0(\mathbb{Z})$ and $c_0(\mathbb{N})$ spaces, respectively, and characterize hypercyclicity and supercyclicity of such kinds of operators. Our investigations will open further investigations of non-Archimedean analogous of Volterra integration operators.

2. DEFINITIONS AND PRELIMINARY RESULTS

All fields appearing in this paper are commutative. A valuation on a field $\mathbb{K}$ is a map $|\cdot| : \mathbb{K} \to [0, +\infty)$ such that:

(i) $|\lambda| = 0$ if and only if $\lambda = 0$,

(ii) $|\lambda\mu| = |\lambda| \cdot |\mu|$ (multiplicativity),

(iii) $|\lambda + \mu| \leq |\lambda| + |\mu|$ (triangle inequality), for all $\lambda, \mu \in \mathbb{K}$.

The pair $(\mathbb{K}, |\cdot|)$ is called a *valued field*. We often write $\mathbb{K}$ instead of $(\mathbb{K}, |\cdot|)$.

Definition 2.1. *Let $\mathbb{K} = (\mathbb{K}, |\cdot|)$ be a valued field. If $|\cdot|$ satisfies the strong triangle inequality: (iii') $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$, for all $\lambda, \mu \in \mathbb{K}$, then $|\cdot|$ is called non-Archimedean, and $\mathbb{K}$ is called a non-Archimedean valued field*

Remark 2.1. In what follows, we always assume that a norm in non-Archimedean valued field is nontrivial.

From the strong triangle inequality we get the following useful property of non-Archimedean value: If $|\lambda| \neq |\mu|$ then $|\lambda \pm \mu| = \max\{|\lambda|, |\mu|\}$. We frequently use this property, and call it as *the non-Archimedean norm's property*. A non-Archimedean valued field $\mathbb{K}$ is a metric space and it is called *ultrametric space*.

Let $a \in \mathbb{K}$ and $r > 0$. The set

$$B(a, r) := \{x \in \mathbb{K} : |x - a| \leq r\}$$

is called the *closed ball with radius r about a*. (Indeed, $B(a, r)$ is closed in the induced topology). Similarly,

$$B(a, r^-) := \{x \in \mathbb{K} : |x - a| < r\}$$

is called the *open ball with radius r about a*.

We set $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$ and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$, the *multiplicative group* of $\mathbb{K}$. Also, $|\mathbb{K}^\times| := \{|\lambda| : \lambda \in \mathbb{K}^\times\}$ is a multiplicative group of positive real numbers, the *value group* of $\mathbb{K}$.

Lemma 2.2. (*Lemma 1.4* [19]) *Let $\mathbb{K}$ be a non-Archimedean valued field. Then the value group of $\mathbb{K}$ either is dense or is discrete; in the latter case there is a real number $0 < r < 1$ such that $|\mathbb{K}^\times| = \{r^s : s \in \mathbb{Z}\}$.*

Definition 2.3. *A pair $(E, \|\cdot\|)$ is called a $\mathbb{K}$-normed space over $\mathbb{K}$, if E is a $\mathbb{K}$-vector space and $\|\cdot\|: E \to [0, +\infty)$ is a non-Archimedean norm, i.e.*
(i) $\| \boldsymbol{x} \| = \boldsymbol{0}$ if and only if $\boldsymbol{x} = \boldsymbol{0}$,
(ii) $\| \lambda \boldsymbol{x} \| = |\lambda| \| \boldsymbol{x} \|$,
(iii) $\| \boldsymbol{x} + \boldsymbol{y} \| \leq \max\{\| \boldsymbol{x} \|, \| \boldsymbol{y} \|\}$, for all $\boldsymbol{x}, \boldsymbol{y} \in E$, $\lambda \in \mathbb{K}$.

We frequently write E instead of $(E, \|\cdot\|)$. Moreover, E is called a $\mathbb{K}$-Banach space or a Banach space over $\mathbb{K}$ if it is complete with respect to the induced ultrametric $d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|$.

Example 2.1. Let $\mathbb{K}$ be a non-Archimedean valued field; then

$$l_\infty := \text{all bounded sequences on } \mathbb{K}$$

with pointwise addition and scalar multiplication and the norm

$$\| \mathbf{x} \|_\infty := \sup_n |x_n|$$

is a $\mathbb{K}$-Banach space.

Remark 2.2. From now on we often drop the prefix "$\mathbb{K}$"- and write vector space, normed space, Banach space instead of $\mathbb{K}$-vector space, $\mathbb{K}$-normed space, $\mathbb{K}$-Banach space, respectively.

In what follows, we need the following auxiliary fact.

Lemma 2.4. *Let E be a normed space over a non-Archimedean valued field $\mathbb{K}$. Then for each pair of sequences $(\boldsymbol{x}_n)$ and $(\boldsymbol{y}_n)$ in E such that $\| \boldsymbol{x}_n \| \cdot \| \boldsymbol{y}_n \| \to 0$ as $n \to \infty$ there exists a sequence $(\lambda_n) \subset \mathbb{K}^\times$ such that*

$$(2.1) \qquad \lambda_n \boldsymbol{x}_n \to \boldsymbol{0} \quad and \quad \lambda_n^{-1} \boldsymbol{y}_n \to \boldsymbol{0}, \qquad as \quad n \to \infty.$$

Let X and Y be topological vector spaces over non-Archimedean valued field $\mathbb{K}$. By $L(X, Y)$ we denote the set of all continuous linear operators from X to Y. If $X = Y$ then $L(X, Y)$ is denoted by $L(X)$. In what follows, we use the following terminology: T is a linear continuous operator on X means that $T \in L(X)$. The T-*orbit* of a vector $\mathbf{x} \in X$, for some operator $T \in L(X)$, is the set

$$O(\mathbf{x}, T) := \{T^n(\mathbf{x}); n \in \mathbb{Z}_+\}.$$

An operator $T \in L(X)$ is called *hypercyclic* if there exists some vector $\mathbf{x} \in X$ such that its T-orbit is dense in X. The corresponding vector $\mathbf{x}$ is called T-*hypercyclic*, and the set of all T-hypercyclic vectors is denoted by $HC(T)$. Similarly, T is called *supercyclic* if there exists a vector $\mathbf{x} \in X$ such that whose projective orbit

$$\mathbb{K} \cdot O(\mathbf{x}, T) := \{\lambda T^n(\mathbf{x}); n \in \mathbb{Z}_+, \ \lambda \in \mathbb{K}\}$$

is dense in X. The set of all T-supercyclic vectors is denoted by $SC(T)$. Finally, we recall that T is called *cyclic* if there exists $\mathbf{x} \in X$ such that

$$\mathbb{K}[T]\mathbf{x} := \mathrm{span}O(\mathbf{x}, T) = \{P(T)x; P \text{ polynomial}\}$$

is dense in X. The set of all T-cyclic vectors is denoted by $CC(T)$.

Remark 2.3. We stress that the notion of hypercyclicity makes sense only if the space X is separable. Note that one has

$$HC(T) \subset SC(T) \subset CC(T).$$

Remark 2.4. Note that if T is a hypercyclic operator on a Banach space then $\|T\| > 1$.

3. HYPERCYCLICITY AND SUPERCYCLICITY OF LINEAR OPERATORS

In this section we find sufficient and necessary conditions to hypercyclicity of linear operators on F-spaces. In what follows, by *F-space* we mean a topological vector space X which is metrizable and complete over a non-Archimedean field.

Now we show that hypercyclicity turns out to be a purely infinite-dimensional phenomenon.

Proposition 3.1. *Let $X \neq \{0\}$ be a finite-dimensional space. Then each operator $T \in L(X)$ is not hypercyclic.*

Proof. Without loss of generality, we may assume that $X = \mathbb{K}^m$ for some $m \geq 1$. Now we are going to prove that each operator $T \in L(\mathbb{K}^m)$ is not hypercyclic. Suppose that a linear operator T on $\mathbb{K}^m$ is hypercyclic. Take $\mathbf{x} \in HC(T)$. The density of $O(\mathbf{x}, T)$ in $\mathbb{K}^m$ implies that the family $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^{m-1}(\mathbf{x})\}$ forms a linearly independent system. Hence, this collection is a basis of $\mathbb{K}^m$. For any $\alpha \in \mathbb{K} \setminus \{0\}$, one can find a sequence of integers (n_k) such that $T^{n_k}(\mathbf{x}) \to \alpha\mathbf{x}$. Then $T^{n_k}(T^i\mathbf{x}) = T^i(T^{n_k}\mathbf{x}) \to \alpha T^i(\mathbf{x})$ for each $i < m$. Hence for any $\mathbf{y} \in \mathbb{K}^m$ we obtain $T^{n_k}(\mathbf{y}) \to \alpha\mathbf{y}$ which yields that $\det(T^{n_k}) \to \alpha^m$, i.e. $\det(T)^{n_k} \to \alpha^m$. Thus, putting $a := \det(T)$, we have the set $\{a^n; n \in \mathbb{N}\}$ is dense in $\mathbb{K} \setminus \{0\}$, but it is impossible. Indeed, it is clear that $|a^n - z| > 1$ for any $z \in \mathbb{K} \setminus B(0, 1)$ if $|a| \leq 1$ and $|a^n - w| > 1$ for any $w \in B(0, 1)$ if $|a| > 1$. $\square$

Our first characterization of hypercyclicity is a direct application of the Baire category theorem.

Theorem 3.2. *(cp. [3])* (TRANSITIVITY THEOREM) *Let X be a separable F-space and $T \in L(X)$. The following statements are equivalent:*

    (i) *T is hypercyclic;*

   (ii) *T is* **topologically transitive**; *that is, for each pair of nonempty open sets $(U, V) \subset X$ there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.*

Corollary 3.3. *Let X be a separable F-space and $T \in L(X)$. If T is hypercyclic then $HC(T)$ is a dense G_δ-set.*

Definition 3.4. [2] *Let X be a topological vector space, and let $T \in L(X)$. It is said that T satisfies the* **Hypercyclicity Criterion** *if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

    (1) $T^{n_k}(\boldsymbol{x}) \to \boldsymbol{0}$ for any $\boldsymbol{x} \in \mathcal{D}_1$;
    (2) $S_{n_k}(\boldsymbol{y}) \to \boldsymbol{0}$ for any $\boldsymbol{y} \in \mathcal{D}_2$;
    (3) $T^{n_k} S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ for any $\boldsymbol{y} \in \mathcal{D}_2$.

Note that in the above definition the maps S_{n_k} are not assumed to be continuous or linear. We will sometimes say that T satisfies the Hypercyclicity Criterion with respect to the sequence (n_k). When it is possible to take $n_k = k$ and $\mathcal{D}_1 = \mathcal{D}_2$, it is usually said that T satisfies *Kitai's Criterion* [10].

Theorem 3.5. *Let $T \in L(X)$, where X is a separable F-space. Assume that T satisfies the Hypercyclicity Criterion. Then the operator T is hypercyclic.*

Definition 3.6. *Let $T_0 : X_0 \to X_0$ and $T : X \to X$ be two continuous maps acting on topological spaces X_0 and X. The map T is said to be a* **quasi-factor** *of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$, we say that T_0 and T are* **topological conjugate**. *Finally, when $T_0 \in L(X_0)$ and $T \in L(X)$ and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a* **linear quasi-factor** *of T_0 (resp. that T_0 and T are* **linearly conjugate***).*

The usefulness and importance of these definitions can be seen in the following

Lemma 3.7. *Let $T_0 \in L(X_0)$ and $T \in L(X)$. Assume that there exists a continuous map with dense range $J : X_0 \to X$ such that $TJ = JT_0$. Then the following statements are satisfied:*

(1) *Hypercyclicity of T_0 implies hypercyclicity of T;*
(2) *Let J be a homeomorphism and T_0 satisfies Hypercyclicity Criterion then T satisfies Hypercyclicity Criterion;*
(3) *Let J be a linear homeomorphism then T is hypercyclic iff T_0 is hypercyclic.*

Remark 3.1. Note that if $T \in L(X)$ is hypercyclic and if $J \in L(X)$ has a dense range and $JT = TJ$ then $HC(T)$ is invariant under J.

We have already observed that if T is a hypercyclic operator on some F-space X then $HC(T)$ is a dense G_δ-set in X. It shows that the set $HC(T)$ is large in a topological sense. This implies largeness in an algebraic sense.

Proposition 3.8. *Let $T \in L(X)$ be hypercyclic on the separable F-space X. Then for every $\boldsymbol{x} \in X$ there exist $\boldsymbol{y}, \boldsymbol{z} \in HC(T)$ such that $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$.*

We say that a linear subspace $E \subset X$ is a *hypercyclic manifold* for T if $E \setminus \{\boldsymbol{0}\}$ consists entirely of hypercyclic vectors.

Theorem 3.9. [4, 7] *Let X be a topological vector space, and $T \in L(X)$ be hypercyclic. If $\boldsymbol{x} \in HC(T)$, then $\mathbb{K}[T]\boldsymbol{x}$ is a hypercyclic manifold for T. In particular, T admits a dense hypercyclic manifold.*

We now turn to the supercyclic analogues of Theorems 3.2 and 3.5.

Theorem 3.10. *Let X be a separable F-space, and $T \in L(X)$. The following statements are equivalent:*

(i) *T is supercyclic;*

(ii) *For each pair of non-empty open sets $(U, V) \subset X$ there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{K}$ such that $\lambda T^n(U) \cap V \neq \varnothing$.*

The proof is similar to the proof of Theorem 3.2.

Definition 3.11. [17] *Let X be a topological vector space, and let $T \in L(X)$. We say that T satisfies the **Supercyclic Criterion** if there exist an increasing sequence of integers (n_k), two dense sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$ and a sequence of maps $S_{n_k} : \mathcal{D}_2 \to X$ such that:*

(1) *$\| T^{n_k}(\boldsymbol{x}) \| \| S_{n_k}(\boldsymbol{y}) \| \to 0$ for any $\boldsymbol{x} \in \mathcal{D}_1$ and any $\boldsymbol{y} \in \mathcal{D}_2$;*

(2) *$T^{n_k} S_{n_k}(\boldsymbol{y}) \to \boldsymbol{y}$ for any $\boldsymbol{y} \in \mathcal{D}_2$.*

Theorem 3.12. *Let $T \in L(X)$, where X is a separable Banach space. Assume that T satisfies the Supercyclic Criterion. Then T is supercyclic.*

Proposition 3.13. *Let X be a separable F-space over non-Archimedean valued field $\mathbb{K}$ and $T \in L(X)$. Then the following statements hold:*

(i) T is cyclic iff λT is cyclic for every $\lambda \in \mathbb{K}^{\times}$;

(ii) T is supercyclic iff λT is supercyclic for every $\lambda \in \mathbb{K}^{\times}$

Remark 3.2. We notice that hypercyclicity of T does not implies hypercyclicity of λT in general. Let us consideran operator $\alpha I + \beta B$ and proved that the operator is hypercyclic iff and only if $\max\{|\alpha|, 1\} < |\beta|$. One can see that if $|\lambda| \leq \frac{1}{|\beta|}$ then $\lambda(\alpha I + \beta B)$ can not be hypercyclic.

4. Backward shifts on c_0

In the present section, we are going to discuss the backward shifts on c_0. We notice that similar to results in the archimedean case have been investigated in [6, 16, 17]. Here, as usual, c_0 stands for the set of all sequences which tend to zero equipped with a norm

$$\| \mathbf{x} \| := \sup_n \{|x_n|\}, \quad \mathbf{x} \in c_0.$$

It is clear that c_0 is a Banach space. For convenience, we denote

$$c_0(\mathbb{Z}) := \{(x_n)_{n \in \mathbb{Z}} : x_n \in \mathbb{K}, |x_{\pm n}| \to 0 \text{ as } n \to +\infty\}$$

and

$$c_0(\mathbb{N}) := \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{K}, |x_n| \to 0 \text{ as } n \to +\infty\}$$

In what follows, we always assume that c_0 is a separable space. Note that the separability of c_0 is equivalent to the separability of $\mathbb{K}$. Let K be a countable dense subset of $\mathbb{K}$. Then the countable set

$$c_{00}(\mathbb{Z}) := \{\lambda_{-n}\mathbf{e}_{-n} + \lambda_{-n+1}\mathbf{e}_{-n+1} + \cdots + \lambda_n\mathbf{e}_n : \lambda_{\pm j} \in K,\ 0 \le j \le n, \forall n \in \mathbb{N}\}$$

is dense in $c_0(\mathbb{Z})$, where $\mathbf{e}_n$ is an unit vector such that only n-th coordinate equals to 1 and others are zero.

Let $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ be a bounded sequence of non-zero numbers of $\mathbb{K}$. An operator $B_{\mathbf{a}}$ on $c_0(\mathbb{Z})$ defined by $B_{\mathbf{a}}(\mathbf{e}_n) = a_n\mathbf{e}_{n-1}$ is called *bilateral weighted backward shift* if $a_i \neq 1$ for some $i \in \mathbb{Z}$, otherwise it is called *bilateral unweighted backward shift* and we denote it by B. In general, the (unweighted) backward shift B is considered as a weighted shift and is thus not excluded from the family of weighted shifts. The operator B is an example of weighted shifts where each weight is equal to 1.

Theorem 4.1. *Let $B_{\mathbf{a}}$ be a bilateral weighted backward shift operator on $c_0(\mathbb{Z})$. Then the following statements hold:*

(i) $B_{\mathbf{a}}$ *is hypercyclic if and only if, for any $q \in \mathbb{N}$,*

$$(4.1) \qquad \liminf_{n\to+\infty} \max\left\{ \prod_{i=1}^{n+q} |a_i^{-1}|, \prod_{j=1}^{n-q} |a_{-j+1}| \right\} = 0.$$

(ii) $B_{\mathbf{a}}$ *is supercyclic if and only if, for any $q \in \mathbb{N}$,*

$$(4.2) \qquad \liminf_{n\to+\infty} \prod_{i=1}^{n+q} |a_i^{-1}| \times \prod_{j=1}^{n-q} |a_{-j+1}| = 0.$$

From this theorem we immediately find the following facts.

Corollary 4.2. *Let $B_{\mathbf{a}}$ be a bilateral weighted backward shift on $c_0(\mathbb{Z})$. Then the following statements hold:*

  (i) *if $B_{\mathbf{a}}$ is supercyclic then $\lambda B_{\mathbf{a}}$ is supercyclic for any $\lambda \in \mathbb{K}^{\times}$;*

  (ii) *if the weight sequence $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ is symmetrical to the norm, i.e. $|a_n| = |a_{-n}|$, $n = 1, 2, \ldots$ then $B_{\mathbf{a}}$ is not supercyclic.*

Corollary 4.3. *Let B be the bilateral unweighted backward shift on $c_0(\mathbb{Z})$. Then B is not supercyclic. Moreover, λB is not supercyclic for any $\lambda \in \mathbb{K}$.*

Corollary 4.4. *Let $\mathbf{a}$ and $\mathbf{b}$ be weighted sequences such that $|a_n| > |b_n|$ for any $n \in \mathbb{Z}$. Then $B_{\mathbf{a}+\mathbf{b}}$ is hypercyclic (resp. supercyclic) if and only if $B_{\mathbf{a}}$ is hypercyclic (resp. supercyclic).*

Remark 4.1. We first notice that Theorem 4.1 remains the same in the real setting, but the valuation should be replaced with the usual absolute value. However, in the real case, Corollary 4.4 is not true. Indeed, for the weights $\mathbf{a}$ and $\mathbf{b}$ defined by

$$a_n = \begin{cases} n, & \text{if } n \geq 1, \\ -\frac{1}{n-1}, & \text{if } n < 1. \end{cases}$$

$$b_n = \begin{cases} -n + \frac{1}{n+1}, & \text{if } n \geq 1, \\ \frac{1}{n-1} - \frac{1}{n-2}, & \text{if } n < 1. \end{cases}$$

the operators $B_{\mathbf{a}}$ and $B_{\mathbf{b}}$ are hypercyclic. But, the weight $\mathbf{a} + \mathbf{b}$ does not satisfy (4.1). Consequently, according to Theorem 4.1 the operator $B_{\mathbf{a}+\mathbf{b}}$ can not be hypercyclic.

Due to Remark 2.3 from the last theorem we can formulate the following fact.

Theorem 4.5. *Let $\boldsymbol{a} = (b_n)_{n \in \mathbb{N}} \in \ell^\infty$ such that $b_n \neq 0$ for all $n \geq 1$. Then weighted backward shift $B_{\boldsymbol{a}}$ on $c_0(\mathbb{N})$ is cyclic. In particular, the backward shift operator B on $c_0(\mathbb{N})$ is cyclic.*

Corollary 4.6. *Let B be the backward shift on $c_0(\mathbb{N})$. Then the λB is cyclic for any $\lambda \in \mathbb{K}^\times$*

Now let us consider a unilateral weighted backward shifts on $c_0(\mathbb{N})$. Recall that the operator defined as $B_{\mathbf{a}}(\mathbf{e}_1) = 0$ and $B_{\mathbf{a}}(\mathbf{e}_n) = a_{n-1}\mathbf{e}_{n-1}$ if $n \geq 2$, is called *unilateral* weighted backward shift. Here $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ be a bounded sequence of non-zero numbers of $\mathbb{K}$. The operator $B_{\mathbf{a}}$ is called unilateral unweighted backward shift if $a_n = 1$ for all $n \geq 1$. We denote by B a unilateral unweighted backward shift operator.

Theorem 4.7. *Any unilateral weighted backward shift $B_{\mathbf{a}}$ on $c_0(\mathbb{N})$ is supercyclic. Moreover, $B_{\mathbf{a}}$ is hypercyclic iff*

$$(4.3) \qquad \limsup_{n \to \infty} \prod_{i=1}^{n} |a_i| = \infty.$$

Corollary 4.8. *Let B be an unilateral unweighted backward shift on $c_0(\mathbb{N})$. Then the following assertions hold:*

  (i) *The operator λB is supercyclic for any $\lambda \in \mathbb{K}^{\times}$;*

  (ii) *λB is hypercyclic iff $|\lambda| > 1$.*

5. $\lambda \mathbf{I} + \mu \mathbf{B}$ OPERATORS ON c_0

In this section, we are going to consider the following operator

$$T_{\lambda,\mu} = \lambda I + \mu B,$$

where I is a identity and B is the unweighted backward shift.

Theorem 5.1. *The operator $T_{\lambda,\mu}$ on $c_0(\mathbb{Z})$ is not supercyclic for all $\lambda, \mu \in \mathbb{K}$.*

From Remark 2.3 we obtain the following

Corollary 5.2. *The operator $T_{\lambda,\mu}$ on $c_0(\mathbb{Z})$ is not hypercyclic for all $\lambda, \mu \in \mathbb{K}$.*

Now we consider the operator $T_{\lambda,\mu}$ on $c_0(\mathbb{N})$. We will show that hypercyclicity of $T_{\lambda,\mu}$ is equivalent to the Hypercyclicity Criterion.

Theorem 5.3. *For the the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ the following statements are equivalent:*

    (i) *$T_{\lambda,\mu}$ satisfies Hypercyclicity Criterion;*
    (ii) *$T_{\lambda,\mu}$ is hypercyclic;*
    (iii) *$|\lambda| < |\mu|$ and $|\mu| > 1$.*

To prove the theorem we first prove three auxiliary lemmas.

Lemma 5.4. *If the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ is hypercyclic then $|\mu| > |\lambda|$ and $|\mu| > 1$.*

Lemma 5.5. *Let $|\mu| > 1$. If $|\lambda| < 1$, then $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ is hypercyclic.*

Proposition 5.6. *If $1 \leq |\lambda| < |\mu|$ then $\bigcup_{n \geq 1}^{\infty} T_{\lambda,\mu}^{-n}(\boldsymbol{0})$ is a dense set in $c_0(\mathbb{N})$.*

Lemma 5.7. *Let $1 \leq |\lambda| < |\mu|$. Then the operator $T_{\lambda,\mu}$ satisfies the Hypercyclicity Criterion.*

Remark 5.1. According to Theorem 5.3 an operator $I + \mu B$ on $c_0(\mathbb{N})$ can not be hypercyclic for any $\mu \in \mathbb{K}$. But, in real case [18], it is hypercyclic for $\mu \neq 0$.

Now we will study supercyclicity of $T_{\lambda,\mu}$. Similarly to the hypercyclic case we have the following

Theorem 5.8. *For the operator $T_{\lambda,\mu}$ acting on $c_0(\mathbb{N})$ the following statements are equivalent:*

(i) $T_{\lambda,\mu}$ *satisfies Supercyclicity Criterion;*
(ii) $T_{\lambda,\mu}$ *is supercyclic;*
(iii) $|\lambda| < |\mu|$.

Remark 5.2. We stress that all operators on c_0 considered above are hypercyclic (resp. supercyclic) if they satisfy Hypercyclic (reps. Supercyclic) Criterion. It is natural to ask: does there exist a hypercyclic (resp. supercyclic) linear operator on c_0 which does not satisfy HC (SC)? We conjecture that such kind of linear operators on c_0 do not exist.

6. $I + B_{\mathbf{b}}$ OPERATOR ON $c_0(\mathbb{N})$

Now let us consider weighted backward shifts on $c_0(\mathbb{N})$. Recall that an operator defined as $B_{\mathbf{b}}(\mathbf{e}_1) = 0$ and $B_{\mathbf{b}}(\mathbf{e}_n) = b_{n-1}\mathbf{e}_{n-1}$ if $n \geq 2$, is called *weighted backward shift*. Here $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ is taken to be a bounded sequence on $\mathbb{K}$. The operator $B_{\mathbf{b}}$ is called *backward shift* if $b_n = 1$ for all $n \geq 1$ such shift is denoted by B.

Theorem 6.1. *Let $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell^\infty$ such that $b_n \neq 0$ for all $n \geq 1$. Then the following statements hold true:*

(i) *the weighted backward shift $B_{\boldsymbol{b}}$ on $c_0(\mathbb{N})$ is supercyclic. In particular, the backward shift operator B on $c_0(\mathbb{N})$ is supercyclic.*

(ii) *the weighted backward shift $B_{\boldsymbol{b}}$ on $c_0(\mathbb{N})$ is hypercyclic iff*

$$(6.1) \qquad \limsup_{n\to\infty} \prod_{j=1}^{n} |b_j| = \infty.$$

Theorem 6.2. *Let $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$ with $b_1 = 0$. Then $T_{\boldsymbol{b}}$ is supercyclic iff $T_{\boldsymbol{b}'}$ is hypercyclic, where $\boldsymbol{b}' = B(\boldsymbol{b})$, here as before, B is the backward shift.*

To establish this result we have used the following facts.

Lemma 6.3. *Let $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$ and $T_{\boldsymbol{b}}$ be a cyclic operator. Then*

$$card(\{n \in \mathbb{N} : b_n = 0\}) \leq 1.$$

Here $card(\cdot)$ stands for a cardinality of a set.

Lemma 6.4. *Let $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$ and $T_{\boldsymbol{b}}$ be a supercyclic operator. Then*

$$b_k \neq 0, \qquad \forall k \geq 2.$$

Lemma 6.5. *Let $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$ and $T_{\boldsymbol{b}}$ be a hypercyclic operator. Then*

$$b_k \neq 0, \qquad \forall k \geq 1.$$

Now, we are going to provide an example of a supercyclic operator $T_{\mathbf{b}}$ which satisfies all conditions of Theorem 6.2.

Example 6.1. Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$ with $b_1 = 0$ and $b_k = \mu$, for all $k \geq 1$, where $|\mu| > 1$. Then for $\mathbf{b}' = B(\mathbf{b})$ we have $T_{\mathbf{b}'} = I + \mu B$. We know that $I + \mu B$ is hypercyclic if and only if $|\mu| > 1$. So, $T_{\mathbf{b}'}$ is hypercyclic. Then, due to Theorem 6.2 the operator $T_{\mathbf{b}}$ is supercyclic.

Remark 6.1. For given $\mathbf{b} \in \ell^\infty(\mathbb{N})$ with $card(\{n \in \mathbb{N} : b_n = 0\}) \geq 1$ Lemma 6.3 and Lemma 6.4 show a difference between supercyclicity and cyclicity of $T_{\mathbf{b}}$. In order to be sure that we need to give an example for the existence of cyclic operator $T_{\mathbf{b}}$ with $b_k = 0$ where $k \neq 1$.

Proposition 6.6. *Let* $\mathbb{K} = \mathbb{Q}_p$ *and* $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$ *with* $b_2 = 0$ *and* $b_k \neq 0$ *for all* $k \neq 2$. *If* $T_{\boldsymbol{b}''}$ *is hypercyclic then* $T_{\boldsymbol{b}}$ *is cyclic, where* $\boldsymbol{b}'' = B^2(\boldsymbol{b})$.

Now, we find sufficiency conditions of the hypercyclicity of $T_{\mathbf{b}}$.

Theorem 6.7. *Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$. If*

$$(6.2) \qquad \lim_{n\to\infty} \prod_{j=1}^{n} |b_j| = \infty,$$

then the operator $T_{\mathbf{b}}$ on $c_0(\mathbb{N})$ is hypercyclic.

According to Remark 2.3, as a corollary of Theorems 6.2 and 6.7 we can formulate the following result.

Theorem 6.8. *Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$. If*

$$(6.3) \qquad \lim_{n\to\infty} \prod_{j=2}^{n} |b_j| = \infty,$$

then the operator $T_{\mathbf{b}}$ on $c_0(\mathbb{N})$ is supercyclic.

Using Proposition 3.13 and Theorem 6.8, we have the next corollary.

Corollary 6.9. *Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$ and $\lambda \in \mathbb{K}^\times$. Then $\lambda I + B_{\mathbf{b}}$ is supercyclic on $c_0(\mathbb{N})$ if*

$$\lim_{n\to\infty} \prod_{j=2}^{n} \left| \frac{b_j}{\lambda} \right| = \infty.$$

Now we give a necessity condition for the supercyclicity of $T_{\mathbf{b}}$.

Theorem 6.10. *Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$. If the operator $T_{\mathbf{b}}$ on $c_0(\mathbb{N})$ is supercyclic, then*

$$(6.4) \qquad \limsup_{n \to \infty} \prod_{j=2}^{n} |b_j| = \infty.$$

Corollary 6.11. *Let $\mathbf{b} \in \ell^\infty(\mathbb{N})$. If $T_{\mathbf{b}}$ is a supercyclic operator then $\|\mathbf{b}\| > 1$.*

Remark 6.2. Thanks to Theorem 6.10 we can say that if (6.4) does not hold then $T_{\mathbf{b}}$ can not be supercyclic. A natural question arises: can (6.4) in Theorem 6.10 be replaced with

$$(6.5) \qquad \limsup_{n \to \infty} \prod_{j=1}^{n} |b_j| = \infty.$$

In fact, due to Theorem 6.2 it is possible that $T_{\mathbf{b}}$ is supercyclic when $b_1 = 0$. So, in this case, we can not say that (6.5) would be necessary for the supercyclicity of $T_{\mathbf{b}}$. By the way, if $b_1 \neq 0$ one can replace (6.4) with (6.5) in Theorem 6.10.

As a corollary of Lemma 6.5 and Theorem 6.10 we can formulate the following result.

Theorem 6.12. *Let $\boldsymbol{b} \in \ell^\infty(\mathbb{N})$. If the operator $T_{\boldsymbol{b}}$ on $c_0(\mathbb{N})$ is hypercyclic then*

$$(6.6) \qquad \limsup_{n\to\infty} \prod_{j=1}^{n} |b_j| = \infty.$$

This result together with Theorem 6.1 (ii) yields that the hypercyclicity of $T_{\boldsymbol{b}}$ implies the hypercyclicity of $B_{\boldsymbol{b}}$ on $c_0(\mathbb{N})$. On the other hand, $B_{\boldsymbol{b}}$ is always supercyclic (see Theorem 6.1(i)), while $T_{\boldsymbol{b}}$ is supercyclic under some conditions.

7. CONCLUSIONS

In the previous sections, we found necessity conditions of the cyclicity, supercyclicity and hypercyclicity of $I + B_\mathbf{b}$. Moreover, we also gave sufficiency conditions to cyclicity, supercyclicity and hypercyclicity of that operator. Unfortunately, we were not able to show that our sufficiency condition would be necessary for the cyclicity (resp. supercyclicity and hypercyclicity). However, in the case that $(b_n)_{n \in \mathbb{N}}$ is a stationary sequence, we can get the following result.

Theorem 7.1. *Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of non-zero numbers. Assume that there exists an integer $n_0 \geq 1$ such that $b_n = \mu$ for all $n \geq n_0$. Then the following statements are equivalent:*

A. *$T_\mathbf{b}$ is hypercyclic;*
B. *$T_\mathbf{b}$ satisfies Hypercyclic Criterion;*
C. *$|\mu| > 1$;*
D. *$T_\mathbf{b}$ is supercyclic;*
E. *$T_\mathbf{b}$ satisfies Supercyclic Criterion;*
F. *$T_\mathbf{b}$ is cyclic.*

We would like to stress that the following problems remains open for $T_{\mathbf{b}}$.

Problem 7.2. Does there exist $\mathbf{b} \in \ell^{\infty}(\mathbb{N})$ such that $T_{\mathbf{b}}$ is hypercyclic under condition

$$\liminf_{n \to \infty} \prod_{j=1}^{n} |b_j| < \infty.$$

Problem 7.3. Does there exist a supercyclic operator $T_{\mathbf{b}}$ if

$$\liminf_{n \to \infty} \prod_{j=2}^{n} |b_j| < \infty.$$

Problem 7.4. Does there exist $\mathbf{b} \in \ell^{\infty}(\mathbb{N})$ such that $T_{\mathbf{b}}$ is hypercyclic and $T_{\mathbf{b}}^2$ is not cyclic?

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