Effective quantum field theories and ultrametricity

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How quantum ultrametricity differs from quantum real?

My talk will be devoted to the investigation of the effective potentials in the quantum field theories (with quartic interaction) on the unramified extension of p-adic field.

Effective potential is the best way to visualize how quantum effects deforms the classical action and it really looks like at quantum level.

Field theories on ultrametric spaces are important and unusual, so it is very natural to take a look ‘how they looks like’. How quantum ultrametricity differs from the ordinary? How quantum real are related to quantum ultrametric?

Effective potential-1

$$\mathscr{L}(\phi, \partial_\mu \phi) \rightarrow \mathscr{L} + J(x)\phi(x)$$

Define a functional W(J) in terms of the probability amplitude for the vacuum state in the far past to go into the vacuum state in the far future in the presence of the external sources

$$e^{iW(J)} = \langle 0^+ \mid 0^- \rangle_J$$

Effective potential-2

$$W = \sum_n \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n G^{(n)}(x_1 \ldots x_n) J(x_1) \ldots J(x_n)$$

It is the generating functional for the connected Green's functions;

Effective potential-3

$$\phi_{\mathrm{c}}(x) = \frac{\delta W}{\delta J(x)} = \left[\frac{\langle 0^+ | \phi(x) | 0^- \rangle}{\langle 0^+ \mid 0^- \rangle} \right]_J$$

$$\Gamma(\phi_{\mathrm{c}}) = W(J) - \int \mathrm{d}^4 x J(x) \phi_{\mathrm{c}}(x)$$

$$\frac{\delta \Gamma}{\delta \phi_{\mathrm{c}}(x)} = -J(x)$$

Effective potential-4

$$\Gamma = \sum_n \frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \Gamma^{(n)}(x_1 \ldots x_n) \phi_\mathrm{c}(x_1) \ldots \phi_\mathrm{c}(x_n)$$

The sum of the IPI (one-particle-irreducible) Green's functions. They are defined as the sum of all connected Feynman diagrams which cannot be disconnected by cutting a single internal line; these are evaluated without propagators on the external lines

Effective potential-5

$$\Gamma(\Phi_c) = \int d^4x \left\{ -V(\varphi_c) + \frac{1}{2}\partial_\mu \Phi_c(x) \partial^\mu \Phi_c(x) Z(\varphi_c) + \cdots \right\}$$

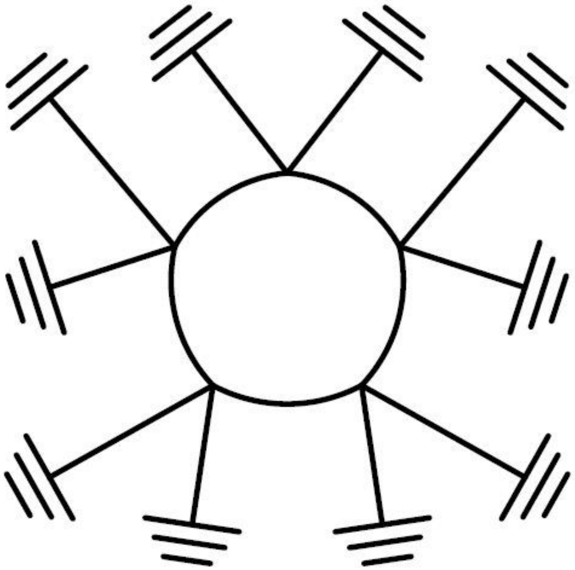
The first term, V, is called the effective potential. It is equal to the sum of all Feynman diagrams with only external scalar lines and with vanishing external momenta.

Effective potential-6

$$V = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$

$$V_{\text{zero-loop}} = \frac{\lambda}{4!}\varphi_c^4$$

Effective potential-massless



$$\frac{1}{2m}\left(\frac{-\lambda\phi_b^2}{2|k|^2}\right)^m$$

Effective potential-massless-2

euclidean!

$$\Delta\Gamma = VT\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{-\frac{1}{2}\lambda\phi_b^2}{k^2}\right)^n = -VT\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\lambda\phi_b^2}{2k^2}\right)$$

$$\Delta V = \frac{1}{2}\frac{\Omega}{(2\pi)^n}\int_0^{\Lambda} dk_E k_E^{(n-1)}\log\left(1+\frac{\alpha}{k_E^2}\right)$$

$$\Omega = \Omega_n = 2\pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}\right) \text{ for } n > 1$$

Effective potential-QM

If $n = 1$, the integral converges as $\Lambda \to \infty$, and

$$\Delta V = \frac{\sqrt{\lambda}|\phi_b|}{4\sqrt{2}}$$

Expanding for large cut-off we get the Coleman-Weinberg potential

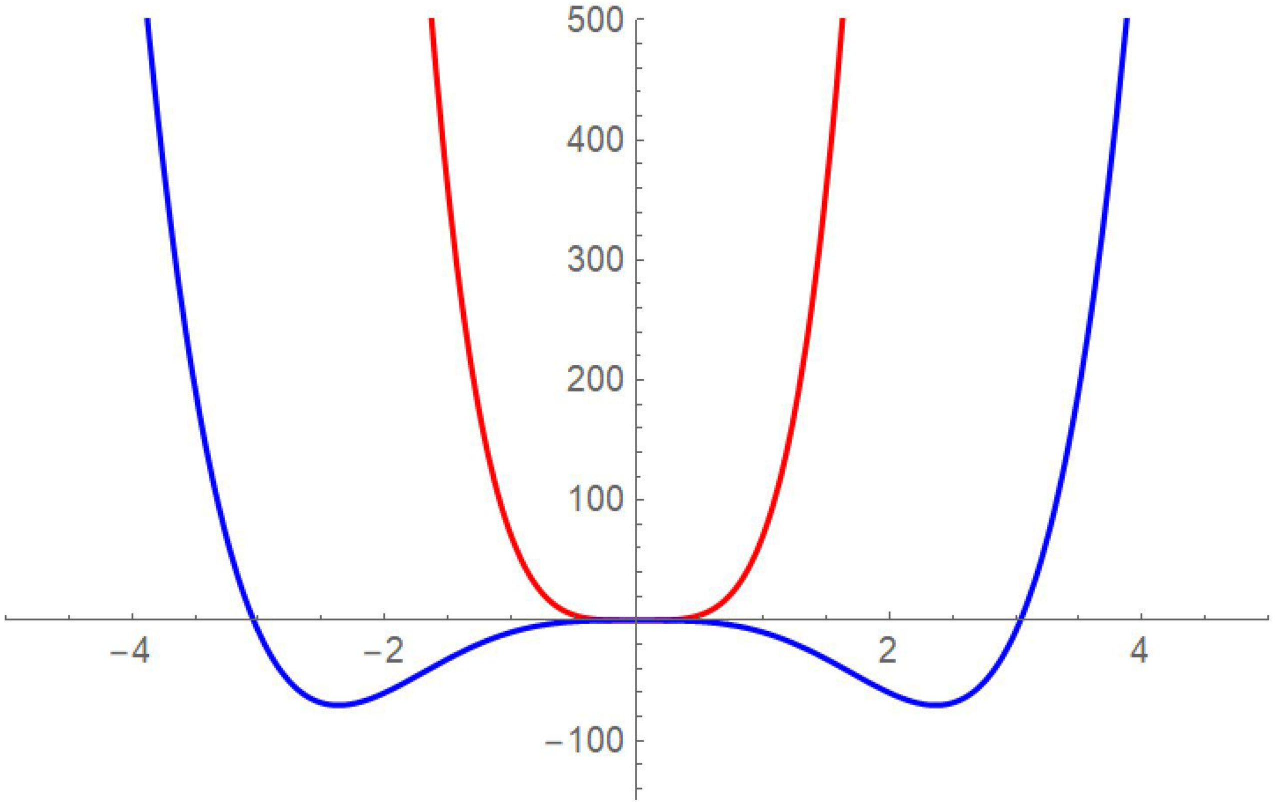
$$\Delta V = \frac{\lambda^2 \phi_b^4}{256\pi^2}\left(\log \frac{\phi_b^2}{\phi_0^2} - \frac{25}{6}\right)$$

Coleman, Weinberg 1973

$$V_{\phi_b}^{(4)}(\phi_0) = \lambda$$

$$V_{\phi_b}''(0) = m_R^2 = 0$$

Symmetry breaking



Quantum fields on unramified extension

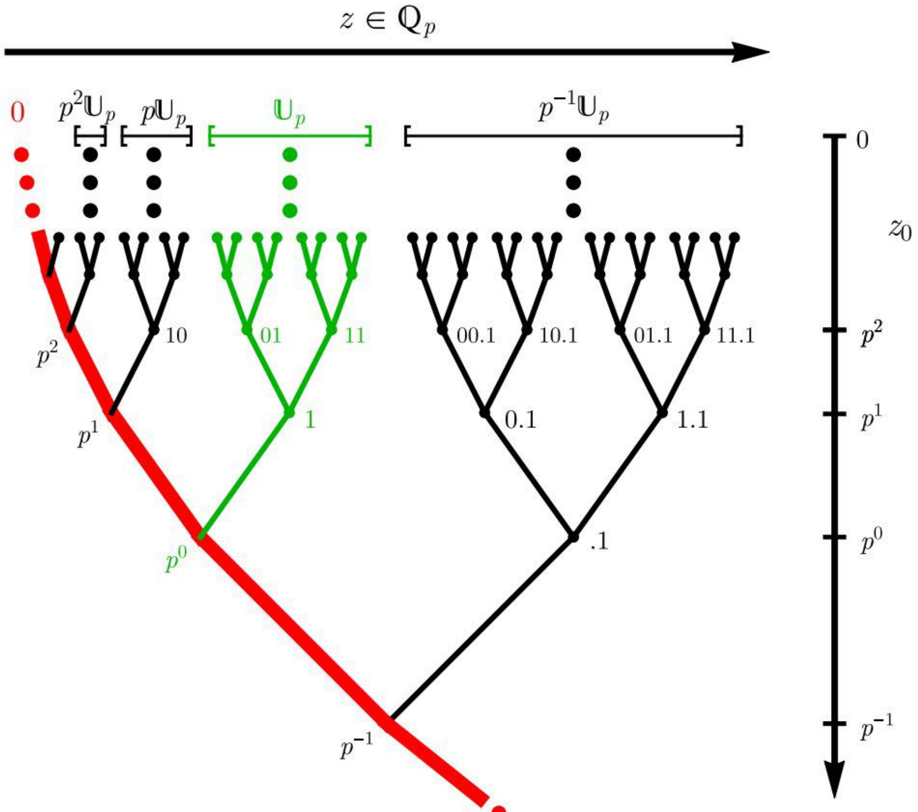
Aref'eva, Dragovich, Frampton, Freund, Khrennikov, Kozyrev, Lerner, Missarov, Okada, Parisi, Smirnov, Vladimirov, Volovich, Witten, Zelenov, Zuniga-Galindo….

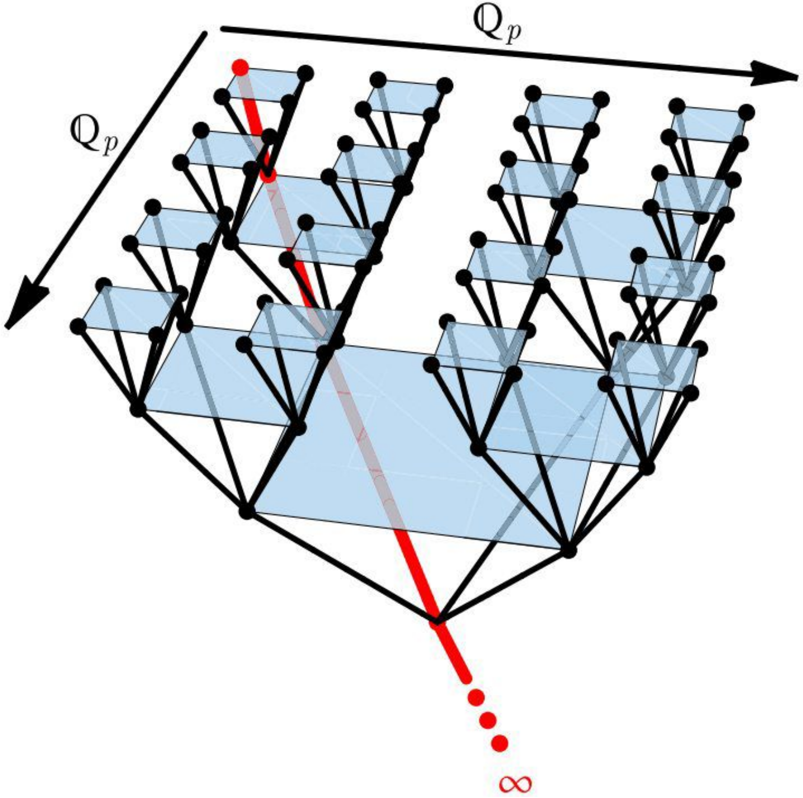
$$S = \int_{\mathbb{Q}_{p^n}} dk \tilde{\phi}(-k)(|k|^s)\tilde{\phi}(k) + \frac{\lambda}{4!}\int_{\mathbb{Q}_{p^n}} dx \phi(x)^4, \quad x \in \mathbb{Q}_{p^n}$$

Gubser, Trundy, Jepsen, Parikh-unramified

Why unramified extensions?(before formalities)

from 1605.01061





$$\int_{\mathbb{Q}_p^n} f(|x|)dx = \left(1 - p^{-n}\right) \sum_{j=-\infty}^{\infty} p^{jn} f(p^j)$$

In order to describe higher-dimensional structures in p-adic mathematical physics, one has to construct a non-Archimedean analogue of $\mathbb{R}^n$ space. A direct way to do that would be to simply take an external product $\mathbb{Q}_p^n$ of n copies of p-adic field and equip it with a structure of vector space. In many cases this would be sufficient. However, bearing in mind possible applications to the AdS/CFT correspondence, it is desirable to have a space that admits a natural holographic interpretation. $\mathbb{Q}_p^n$ is not a field *per se*, and thus does not possess a structure of the Bruhat-Tits tree which would play a role of dual bulk geometry.

This issue can be resolved by using instead of $\mathbb{Q}_p^n$ unramified extension of the p-adic number field $\mathbb{Q}_{p^n}$ of degree $[\mathbb{Q}_{p^n} : \mathbb{Q}_p] = n$. As a vector space, $\mathbb{Q}_{p^n}$ is isomorphic to $\mathbb{Q}_p^n$. To be an unramified extension, it must obey the following requirement. If L and K are two fields, and L is an extension of K, we can consider quotients of these fields by their maximal ideals $\ell = L/m_L$, $k = K/m_K$. Then k is a field extension of ℓ, and if its' degree is equal to the degree of L, so that $[\ell : k] = [L : K]$, L is an unramified extension. Explicitly, $\mathbb{Q}_{p^n}$ can be obtained from $\mathbb{Q}_p$ by adjoining a primitive $(p^n - 1)$-st root of unity [37].

We also need to equip $\mathbb{Q}_{p^n}$ with a norm that satisfies the requirement of ultra-metricity and becomes the standard p-adic norm for $n = 1$. It is also handy to assume that the norm takes values in integer powers of p, since it induces a branching structure that can serve as a skeleton of the Bruhat-Tits tree. The natural choice is:

$$|x| = |N(x)|_p^{1/n}, \tag{A.1}$$

where $N(x)$ is a determinant of a linear map induced by multiplication in $\mathbb{Q}_{p^n}$: $f(a) = xa$, $a \in \mathbb{Q}_{p^n}$, that can be seen as a linear operator acting on $\mathbb{Q}_p^n$.

The same story for ultrametric

$$\Delta\Gamma(\phi_b) = V_{p^n} \sum_m \int \frac{1}{2m} \left(\frac{-\lambda \phi_b^2}{2|k|^s} \right)^m dk = -\frac{V_{p^n}}{2} \int \ln\left(1 + \frac{\lambda \phi_b^2}{2|k|^s} \right) dk$$

The same story for ultrametric

$$\Delta V = -\frac{\Delta\Gamma(\phi_b)}{V_{p^n}} = \frac{1}{2}\int_{\mathbb{Q}_{p^n}} dk \log\left(1 + \frac{\lambda\phi_b^2}{2|k|^s}\right)$$

$$\int_{\mathbb{Q}_p n} f(|x|)dx = (1 - p^{-n}) \sum_{i=-\infty}^{\infty} p^{ni} f(p^i)$$

$$\Delta V(\alpha) = \frac{1}{2}(1 - p^{-n}) \sum_{i=-\infty}^{M} p^{ni} \ln(1 + \alpha p^{-si})$$

where $\alpha = \lambda\phi_b^2/2$. One can think of number M as of a logarithm of the corresponding ultraviolet momentum scale $|k|_{UV} = \Lambda = p^M$.

After summation...

After careful (but rather technical and cumbersome) estimation of different part of this divergent series we get for n=4

$$\Delta V = \frac{\lambda^2 \phi_b^4}{32 \log p} \left(1 - \frac{1}{p^4}\right) \left(\log \frac{\phi_b^2}{\phi_0^2} - \frac{25}{6}\right)$$

Integral formula

$$\sum_{j=-\infty}^{M} f(j) \simeq \int_{-\infty}^{M} f(x)\, dx. \tag{5.1}$$

That would be possible if the Euler–Maclaurin formula for infinitely differentiable functions was valid:

$$\sum_{i=m}^{M} f(i) = \int_{m}^{M} f(x)\, dx + \frac{f(M) + f(m)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(M) - f^{(2k-1)}(m)\right), \tag{5.2}$$

and the residual term was small enough.

Integral approximation gives the same answer

$$\Delta V = \frac{1}{2}\left(1 - p^{-4}\right)\int_{-\infty}^{M} p^{4x} \ln\left(1 + \frac{\lambda \phi_b^2}{2p^{2x}}\right) dx$$

$$\Delta V = \frac{\lambda^2 \phi_b^4}{32 \log p}\left(1 - \frac{1}{p^4}\right)\left(\log\frac{\phi_b^2}{\phi_0^2} - \frac{25}{6}\right)$$

What about 'quantum mechanical' example of Qp

$$\Delta V = \frac{1}{2}(1-p^{-1})\int_{-\infty}^{+\infty} p^x \ln\left(1+\frac{\lambda\phi_b^2}{2p^{2x}}\right)dx = (1-p^{-1})|\phi_b|\frac{\sqrt{|\lambda|}\pi}{2\sqrt{2}\ln p}$$

rersus the result of series summation (2.16) :

$$\Delta V = \frac{\sqrt{|\lambda|}}{2\sqrt{2}}|\phi_b|N(p)$$

$$N(p) = \frac{p^{-1}}{1-p^{-1}}2\ln p + (1-p^{-1})\sum_{l=1}^{\infty}\frac{(-1)^l}{l}\left(\frac{1}{p^{1-2l}-1} - \frac{1}{p^{1+2l}-1}\right).$$

There is a clear discrepancy between these two expressions for large values of p since:

$$\lim_{p\to\infty} N(p) = \ln 2$$

$$\lim_{p\to\infty} \frac{\pi(1-p^{-1})}{\ln p} = 0$$

On the other hand, for small p the Euler-Maclaurin estimate has surprisingly good accuracy. For example, for $p = 7$:

$$N(7) \simeq 1.387$$

$$\frac{\pi(1-7^{-1})}{\ln 7} \simeq 1.384$$

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$p \to 1$ limit

$$\lim_{p\to 1} \int_{\mathbb{Q}_p{}^n} f(|x|)dx = \frac{n\Gamma(n/2)}{2\pi^{n/2}} \cdot \int_{\mathbb{R}^n} f(|x|)dx, \;\; n > 1$$

$$\lim_{p\to 1} \int_{\mathbb{Q}_p} f(|x|)dx = \int_{\mathbb{R}} f(|x|)dx, \;\; n = 1,$$

Future directions

-mixed fields

-product vs unramified

-p-adic Liouville theory

Thank you for your attention