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И ГОСУДАРСТВЕННОЙ СЛУЖБЫ
ПРИ ПРЕЗИДЕНТЕ РОССИЙСКОЙ ФЕДЕРАЦИИ

Finite Tight Frames in Walsh Analysis

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1. Introduction

Wavelet frames, from the works of Ron and Shen (1997), have been a productive research area, both in theory and applications.

[Ch16] Christensen O., An Introduction to Frames and Riesz Bases, 2nd edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Basel, 2016. (The bibliography lists 645 items.)

[W18] Waldron S., An Introduction to Finite Tight Frames, Applied and Numerical Harmonic Analysis, Birkhäuser, New York, 2018.

From the preface of Waldron's book:

"This book gives a unified introduction to the rapidly developing area of finite tight frames. Fifteen years ago, the existence of equal–norm tight frames of $n > d$ vectors for $\mathbb{R}^d$ and $\mathbb{C}^d$ was not widely known. Now equal–norm tight frames are known to be common, and those with optimal cross–correlation and symmetry properties are being constructed and classified. The impetus behind these rapid developments are applications to areas as diverse as signal processing, quantum information theory, multivariate orthogonal polynomials and splines, and compressed sensing. It can be thought of as an extension of the first chapter of Ole Christensen's book, which deals mostly with the infinite dimensional case. For finite dimensional Hilbert spaces the technicalities of Riesz bases disappear (though infinite frames are still of interest), and, with some work, usually a nice tight frame can be constructed explicitly. Hence the focus is on finite tight frames, which are the most intuitive generalisation of orthonormal bases."

The theory of p-adic wavelet frames is presented by Kozyrev, Khrennikov, and Shelkovich (2014).
It is known that the frames of p-adic wavelets for $\mathbb{Q}_p$ are the orbits of p-adic transformation groups (systems of coherent states).

Evdokimov and Skopina (2018) proved that any orthogonal wavelet basis for $L^2(\mathbb{Q}_p)$ that consists of band-limited (periodic) functions, is a modification of the Haar basis.

J. J. Benedetto and R. L. Benedetto [BB11] developed a wavelet theory for a locally compact abelian group $\mathcal{G}$ with compact open subgroup $\mathcal{H}$. The best known example of such a group is $\mathcal{G} = \mathbb{Q}_p$, the field of p-adic numbers (as a group under addition), which has compact open subgroup $\mathcal{H} = \mathbb{Z}_p$, the ring of p-adic integers. Wavelet bases for $L^2(\mathcal{G})$ are constructed by means of an iterative method giving rise to so-called wavelet sets in the dual group $\widehat{\mathcal{G}}$. For the Haar wavelets this construction is broader than those of Lang (1996) and Kozyrev (2002). A characterization of MRA-based wavelets on local fields of positive characteristic including the Vilenkin/Cantor groups is suggested by Behera and Jahan [BJ15].

The Walsh functions form a complete orthonormal set in the Hilbert space $L^2[0;1]$. They were introduced by Joseph Walsh (1923) and have since found wide application in digital signal processing. In this regard a fundamental requirement was the development of efficient computation routines for Walsh matrices and the associated function representations using Walsh series and Walsh transforms. For a detailed account see [SWS], [GES] and

[SBSW] Stanković R.S., Butzer P. L., Schipp F., Wade W. R. (eds.) *Dyadic Walsh analysis from 1924 onwards Walsh-Gibbs-Butzer dyadic differentiation in science.* Vol. 1, 2. Amsterdam: Atlantis Press, 2015.

As noted in [SBSW], interest in Walsh analysis can be explained by two reasons: "First engineering practice required fast implementation of various algorithms for signal and information processing based on spectral (Walsh-Fourier) analysis, and second the limited computing power of the technology at the time ultimatively demanded something that took less memory than older, more established classical Fourier analysis. For these reasons, regarding the present situation, we believe that the way of thinking that led to this concept can be equally useful in defining new concepts and related methods in computing with applications in signal and information processing now and in future". In this context, wavelets and frames defined through Walsh functions and their applications for signal and information processing are of considerable interest.

It is well known that the Walsh functions can be identified with characters of the Cantor group. The fruitful relationship between Walsh analysis and wavelet theory is reflected in the book

[F19a] Farkov Yu. A. et al., "Construction of Wavelets through Walsh Functions", Industrial and Applied Mathematics. Singapore: Springer, 2019.

Chapter 5 of this book is devoted to orthogonal and periodic wavelets on the p-adic Vilenkin group G_p (the case $p = 2$ corresponds to the Cantor group). The first section deals with multiresolution analysis for the space $L^2(G_p)$, section 2 introduces compactly supported orthogonal p-wavelets, section 3 presents periodic wavelets on G_p, and section 4 introduces periodic wavelets related to the Vilenkin-Christenson transform.

A review of discrete wavelet transforms defined through Walsh functions and used for image processing, compression of fractal signals, analysis of financial time series, and analysis of geophysical data is presented in [F19b].

Let $p, n \in \mathbb{N}$, $p \geq 2$. Denote by $\mathbf{G}(p, n)$ the set of all complex vectors $\mathbf{b} = (b_0, b_1, \dots, b_{p^n-1})$ such that

$$|b_l|^2 + |b_{l+N_1}|^2 + \cdots + |b_{l+(p-1)p^{n-1}}|^2 = 1, \quad 0 \leqslant l \leqslant p^{n-1} - 1.$$

For each $\mathbf{b} \in \mathbf{G}(p, n)$ the orthogonal discrete wavelet transform $O(p, n)$ associated with the generalized Walsh functions is defined. Due to the freedom in the choice of the parameter vector $\mathbf{b}$, for some fractal functions the transform $O(p, n)$ has an advantage over the method of zone coding.

It is known that classical methods of analysis based on the use of correlations between neighboring samples are ineffective in the processing of financial time series. In [LF17], it is proposed to go from analyzing the initial signals to examining the sequences of their non-linear properties (such that the entropy of the wavelet coefficients, the multi-fractal parameters and the autoregressive measure of signal nonstationarity) calculated in time intervals of small length. In [F19b] this approach is discussed in connection with the transform $O(p,n)$ along with the problem of measuring the distance between time series. It is also possible to use certain generalizations and modifications of the transform $O(p,n)$ (biorthogonal, nonstationary, periodic, frames etc. cases), while the discrete Haar transform remains the "starting point"for all these constructions. In [F19c], several parametric sets are indicated in addition to $\mathbf{G}(p,n)$, each of which corresponds to a certain class of discrete wavelet transforms.

A localization of functions defined on Vilenkin groups are studied in [KL15, KL18]. To measure the localization, the authors introduce uncertainty principles that are similar to the Heisenberg uncertainty principle. In particular, the dyadic uncertainty constant for Lang's wavelet is calculated. The lowest values of localization in [KL15] are attained for the dyadic wavelet frames which were constructed in [F12].

There are three types of compactly supported wavelet frames on the p-adic Vilenkin group G_p: (1) MRA-based tight frames, (2) frames obtained from the Daubechies-type "admissible condition", and (3) frames based on the Walsh-Dirichlet type kernels; see [F19b],[F21].

As recently noted in [F21], finite Walsh frames can be useful for detecting hidden regular structures of uniformly distributed point sets. Recall that a sequence $s_1, s_2, \ldots$ in $[0;1)$ is said to be *uniformly distributed* if, in the limit, the number of s_j falling in any given subinterval is proportional to its length. Equivalently, $s_1, s_2, \ldots$ is uniformly distributed if the sequence of equiweighted atomic probability measures $\mu_N(s_j) = 1/N$, supported by the initial N-segments $s_1, s_2, \ldots, s_N$, converges weakly to Lebesgue measure on $[0;1)$. This notion immediately generalizes to any topological space with a corresponding probability measure on the Borel sets. The concept of uniform distribution has important applications in a number of branches of mathematics such as number theory, combinatorics, ergodic theory, discrete geometry, statistics, numerical analysis, etc. (see, e.g., [ABC17] and the references therein).

The talk is mainly devoted to applications of Walsh functions for constructing finite tight frames. The construction of Parseval frames on the group G_p is briefly discussed in final section.

Some relationships of finite-dimensional frame constructions with several well-known problems and results will be noted, including the Thomson problem on minimizing energy of electric charges on a sphere, the finite-dimensional version of Naimark's theorem on representing tight frames as orthogonal projections of orthonormal bases, the problem of the existence of Hadamard matrices, the problem of the existence of Steiner systems, and the problem of calculating the maximum number of equiangular lines.

2. Discrete Walsh functions

This section contains definitions of the Walsh functions, Walsh matrices, and discrete Walsh systems.

Let r_0 be the function defined on $[0, 1)$ by

$$r_0(x) = \begin{cases} 1, & x \in [0, 1/2), \\ -1, & x \in [1/2, 1) \end{cases}$$

extended to $\mathbb{R}_+ = [0, +\infty)$ by periodicity of period 1.

The Walsh system $\{w_l : l \in \mathbb{Z}_+\}$ is defined by

$$w_0(x) \equiv 1, \quad w_l(x) = \prod_{j=1}^{k} (r_0(2^{j-1}x))^{\nu_j}, \quad l \in \mathbb{N}, \; x \in \mathbb{R}_+,$$

where

$$l = \sum_{j=1}^{k} \nu_j 2^{j-1}, \quad \nu_j \in \{0,1\}, \quad \nu_k = 1, \quad k = k(l).$$

The Walsh functions $w_l(x)$ with $0 \le l \le 2^n - 1$ are constant on binary intervals $[k/2^n, (k+1)/2^n)$ for all $k \in \mathbb{Z}_+$.

For $0 \le l, k \le 2^n - 1$, we let $w_{lk} = w_l(k/2^n)$.

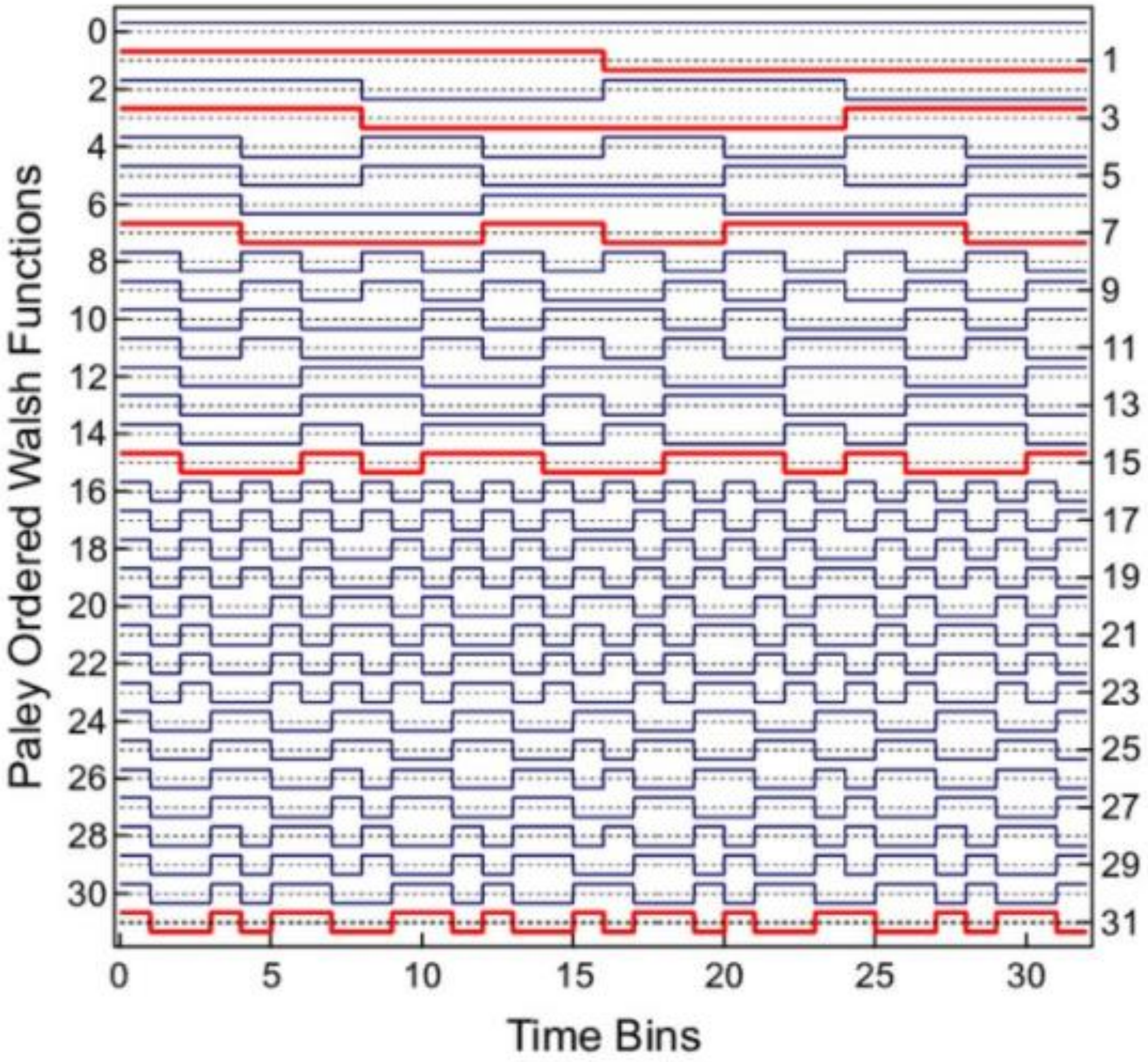
The *Walsh matrix* $W_n := [w_{lk}]_{l,k=0}^{2^n-1}$ is symmetric and satisfies the following orthogonality relations

$$\sum_{j=0}^{2^n-1} w_{lj} w_{kj} = \sum_{i=0}^{2^n-1} w_{il} w_{ik} = 2^n \delta_{lk}, \quad l, k \in \{0, 1, \dots, 2^n - 1\},$$

where δ_{lk} is the Kronecker delta. The rows of matrices W_n are called the *discrete Walsh functions in Paley ordering*.

In particular, we have

$$W_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$



Figure

Caption

FIG. 5. The first 32 Walsh functions in Paley ordering.

https://www.researchgate.net/publication/262948796_Experimental_noise_filtering_by_quantum_control/figures?lo=1

Given an $m \times n$ matrix A and a $p \times q$ matrix B, their Kronecker product is an $mp \times nq$ matrix defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{11}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

By definition, $H_1 := W_1$, and the matrix $H_n = [h_{lk}]_{l,k=0}^{2^n-1}$ coincides with the n-th Kronecker power of the matrix H_1 The rows of matrices H_n are called the *discrete Walsh functions in Hadamard ordering*.

The Hadamard matrices H_n of order 2^n were constructed by Sylvester in 1867 using the following two properties.

1. If X is a Hadamard matrix, then

$$\begin{bmatrix} X & X \\ X & -X \end{bmatrix}$$

is also a Hadamard matrix.

2. If X_1 and X_2 are Hadamard matrices of orders m_1 and m_2, then their Kronecker product $X_1 \otimes X_2$ is a Hadamard matrix of order $m_1 m_2$.

For example, we have

$$H_2 = H_1 \otimes H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Sylvester's recursive method for generating H_n matrices in Matlab is based on the identity $H_{n+1} = H_1 \otimes H_n$.

For $0 \leq l, k \leq 2^n - 1$, the values w_{lk} and h_{lk} can be calculated from the digits of the binary expansions

$$l = \sum_{j=1}^{n} l_j 2^{j-1}, \quad k = \sum_{j=1}^{n} k_j 2^{j-1}, \quad l_j, k_j \in \{0, 1\},$$

as follows:

$$w_{lk} = (-1)^{w(l,k)}, \quad h_{lk} = (-1)^{h(l,k)},$$

where

$$w(l, k) = \sum_{j=1}^{n} l_j k_{n-j+1}, \quad h(l, k) = \sum_{j=1}^{n} l_j k_j.$$

3. Properties of finite tight frames

A family $\{v_j : j \in J\}$ is a *frame* for a Hilbert space $\mathcal{H}$ if there exist positive constants A and B such that, for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, v_j \rangle|^2 \leq B\|f\|^2.$$

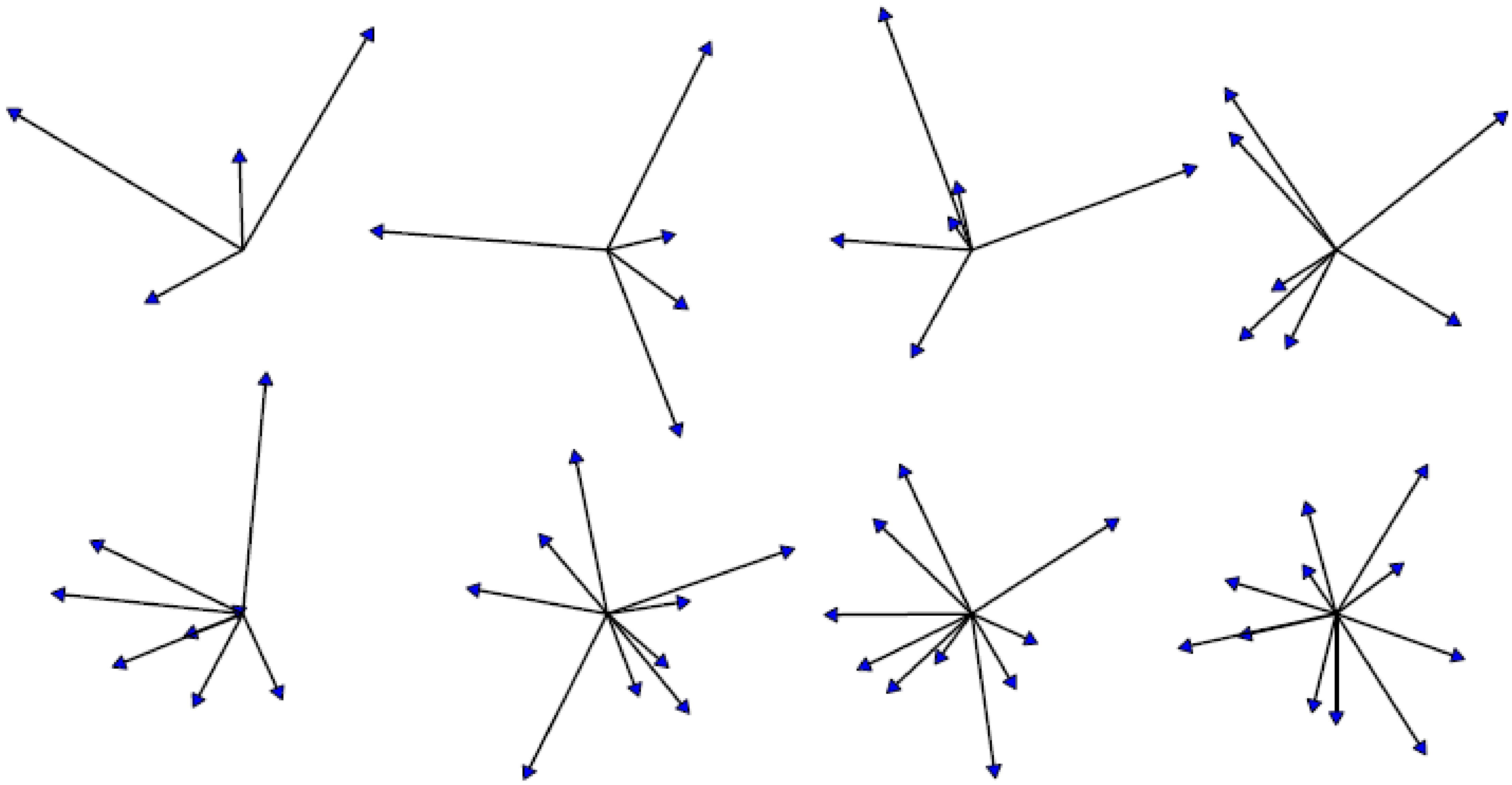
The constants A and B are known respectively as lower and upper frame bounds. A frame is called a *tight frame* if the lower and upper frame bounds are equal; $A = B$. A frame is a *Parseval frame* (or a *normalised tight frame*) if $A = B = 1$.

A sequence $\{v_j\}$ is a Parseval frame for a Hilbert space $\mathcal{H}$ if and only if $f = \sum_{j\in J}\langle f, v_j\rangle v_j$ for every $f \in \mathcal{H}$.

The concept of a Parseval frame generalizes the concept of an orthonormal basis to systems that do not have the minimal property.

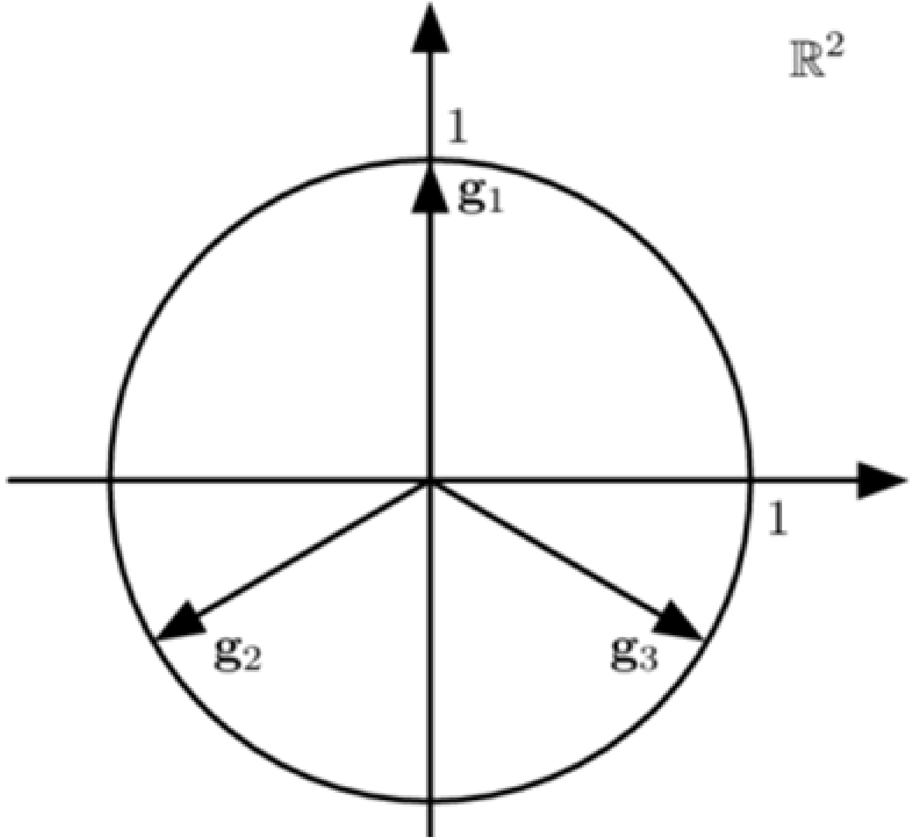
If a family $\{v_j\}$ is a tight frame for $\mathcal{H}$ with a frame bound A, then $\{v_j/\sqrt{A}\}$ is a Parseval frame for $\mathcal{H}$.

This section contains several properties of finite tight frames; for more detail see [H08] and [W18].



[W18,Fig.2.1]: Examples of normalised tight frames of $n = 4, 5, \ldots, 11$ vectors for $\mathbb{R}^2$.

The Mercedes-Benz frame



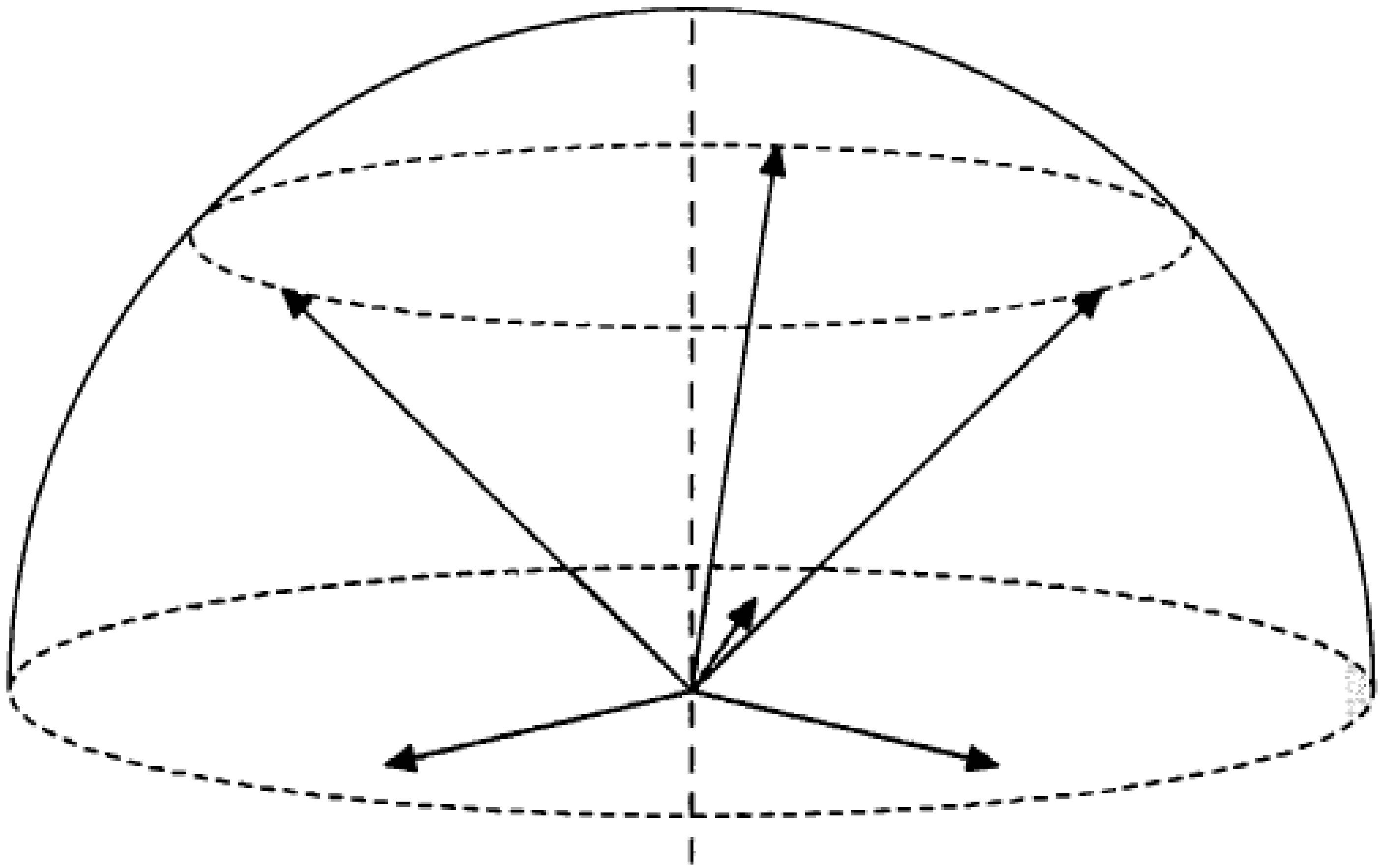
The prototypical example of a finite tight frame is three equally spaced unit vectors g_1, g_2, g_3 in $\mathbb{R}^2$, which provide the following redundant decomposition

$$f = \frac{2}{3} \sum_{k=1}^{3} \langle f, g_k \rangle \, g_k, \quad f \in \mathbb{R}^2.$$

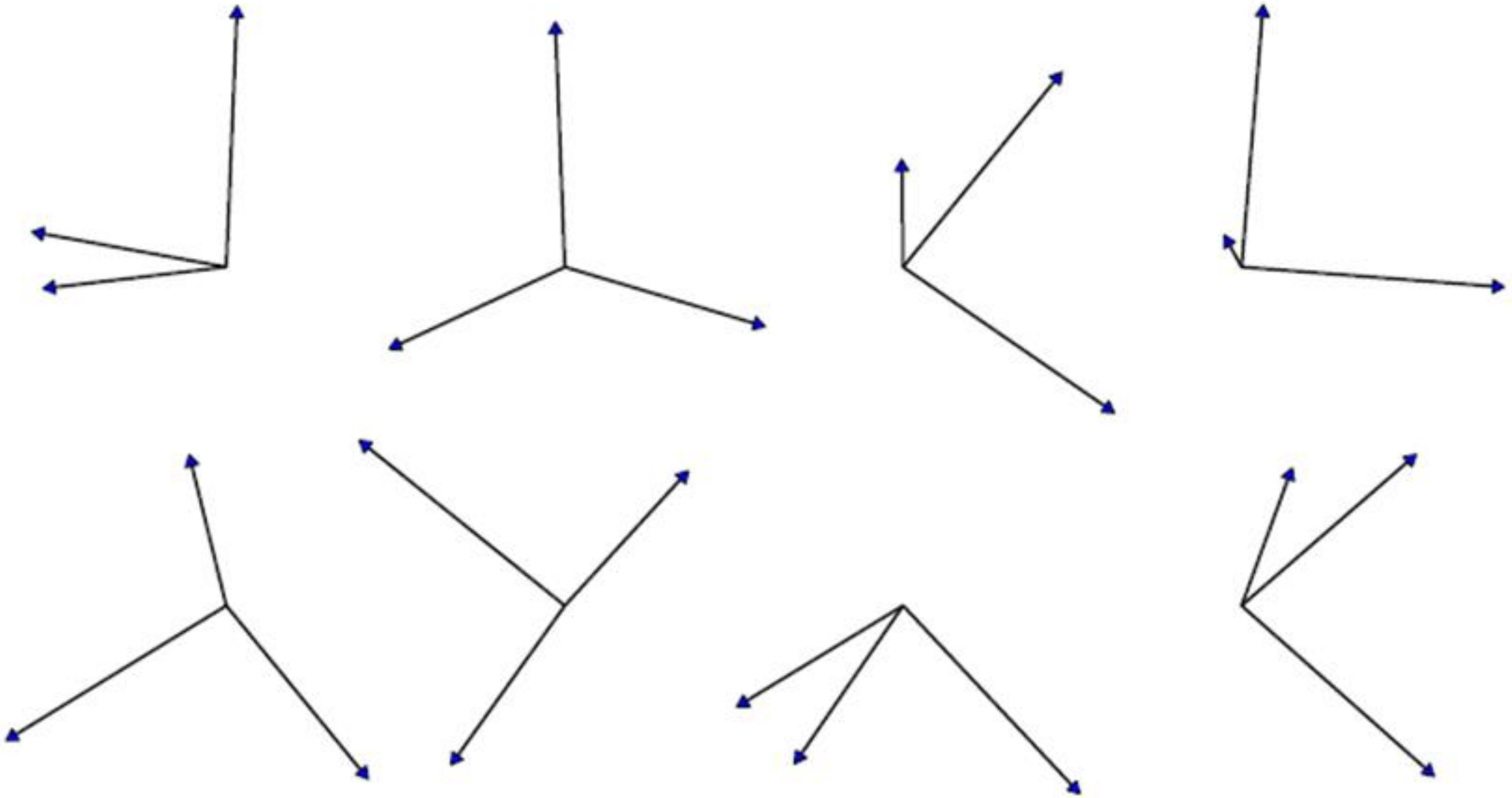
Finite-dimensional version of Naimark's theorem

All finite tight frames are projections of orthonormal bases from a larger space. More precisely, we have the following

Proposition 1. *Suppose that* $\{v_j\}_{j=1}^m$ *is a Parseval frame for a finite-dimensional space* $\mathcal{H}$. *Then there exists a space* $\mathcal{K} \supset \mathcal{H}$ *and an orthonormal basis* $\{u_j\}_{j=1}^m$ *for* $\mathcal{K}$ *such that* $v_j = Pu_j$, *where* P *is the orthogonal projection from* $\mathcal{K}$ *onto* $\mathcal{H}$.



[H08,Fig.5.1]: A projection of the standard orthonormal basis for $\mathbb{R}^3$ which yields the uniform 3-vector Parseval frame for the plane.



[W18,Fig.2.5]: Examples of normalised tight frames of three vectors for $\mathbb{R}^2$ obtained as the orthogonal projection of an orthonormal basis for $\mathbb{R}^3$.

Proposition 2. *Let* $\{v_j\}_{j=1}^m$ *be a Parseval frame for a finite-dimensional space* $\mathcal{H}$*. Then*

(i) $\dim \mathcal{H} = \sum_{j=1}^{m} \|v_j\|^2$*;*

(ii) $\{v_j\}_{j=1}^m$ *is an orthonormal basis if and only if each* v_j *is a unit vector.*

We recall that if $d = \dim \mathcal{H}$, then the ratio m/d is called the **redundancy** of a frame $\{v_j\}_{j=1}^m$ in $\mathcal{H}$.

Proposition 3. *Suppose that* $a_j > 0$ *for* $j = 1, \ldots, m$. *There is a tight frame for an n-dimensional space* $\mathcal{H}$ *with* m *vectors having norms* $\|v_j\| = a_j$, $j = 1, \ldots, m$, *if and only if the following inequality is satisfied*:

$$\max\{a_1^2, a_2^2, \ldots, a_m^2\} \leq \frac{1}{n} \sum_{j=1}^{m} a_j^2.$$

It is clear from Proposition 3 that there is no tight frame of four vectors for $\mathbb{R}^2$ having respective norms 3,2,1,1.

The *Mercedes-Benz frame*:

$$f_1 = (-\sqrt{3}/2, -1/2), \quad f_2 = (\sqrt{3}/2, -1/2), \quad f_3 = (0, 1).$$

We set $f_j^2 = f_j$, $j = 1, 2, 3$. For any $d \geq 3$ the *generalized frame Mercedes-Benz* for $\mathbb{R}^d$ is determined inductively:

$$f_j^d = \left(\frac{\sqrt{d^2 - 1}}{d} f_j^{d-1}, -\frac{1}{d} \right), \; 1 \leq j \leq d, \;\; f_{d+1}^d = (0, \ldots, 0, 1).$$

The constant of this frame coincides with its redundancy: $A = 1 + 1/d$.

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$.

The frame $\{f_j^d\}_{j=1}^{d+1}$ is uniquely determined by the conditions:

1) $f_j \in \mathbb{S}^{d-1}$, 2) $\langle f_i, f_j \rangle = -\frac{1}{d}$ for $i \neq j$, 3) $\sum_{j=1}^{d+1} f_j = 0$.

The points of this frame are the vertices of the regular d-dimensional simplex and for $N = d+1$ form a solution to the following problem

$$\sum_{i \neq j} \frac{1}{|x_i - x_j|} \to \min, \quad x_1, \ldots, x_N \in \mathbb{S}^{d-1}.$$

This property is related to Thomson's problem of the location of charges on a sphere (see, e.g., [IP07]). Related problems include the study of the geometry of the minimum energy configuration and the study of the large N behavior of the minimum energy.

For simplicity of notation, we write $\mathbb{F}$ for $\mathbb{R}$ or $\mathbb{C}$.

Proposition 4. *Let* $V = [v_1, v_2, \ldots, v_m]$ *be an* $d \times m$ *matrix with* $v_j \in \mathbb{F}^d$ *being the column vectors of* V.

(i) $\{v_j\}_{j=1}^m$ *is a frame for* $\mathbb{F}^d$ *if and only if* V *has rank* d.

(ii) $\{v_j\}_{j=1}^m$ *is a tight frame for* $\mathbb{F}^d$ *with frame bound* A *if and only if* $VV^* = AI_d$ *where* I_d *is the identity matrix of order* d.

According to Proposition 4, a family $\{v_j\}_{j=1}^m$ is a tight frame for $\mathbb{F}^d$ if and only if the set of row vectors of V is a pairwise orthogonal collection of vectors all having the same norm.

We say that $\{v_j\}_{j=1}^m$ is an *equal–norm* tight frame if $\|v_i\| = \|v_j\|$, $i, j \in \{1, \ldots, m\}$.

Equal–norm tight frames of d vectors for $\mathbb{F}^d$ can be obtained from an $m \times m$ unitary matrix U with entries of constant modulus, by taking the columns of any $d \times m$ submatrix.

The **harmonic tight frames** for $\mathbb{C}^d$ are constructed in this way from the Fourier matrix

$$F = \frac{1}{\sqrt{m}}[\omega^{jk}]_{0 \leq j,k \leq m}, \quad \omega = e^{2\pi i/m}; \quad \text{see [W18, Chapter 11].}$$

Equal-norm tight frames associated with Walsh matrices play a key role in [AHV18], where the relationship between frame theory and the Kadison-Singer conjecture is discussed.

It is well-known that if $m \leq N$ and $F = [\mathbf{f}_1, \ldots, \mathbf{f}_N] \in \mathbb{C}^{m \times N}$ defines a Parseval frame for $\mathbb{C}^m$ with $FF^* = I_m$ and $\|\mathbf{f}_j\|^2 = \alpha$ for each $j \in \{1, \ldots, m\}$, then $\alpha = m/N$. Conversely, as shown in [AHV18, Sect. 3], it is always possible to use the columns of a normalized Walsh matrix to represent the vectors of an equal-norm tight frame.

4. Tight frames for $\ell^2(\mathbb{Z}_N)$

Suppose that $N = 2^n$ and $N_1 = 2^{n-1}$, where n is a natural number. Denote by $\ell^2(\mathbb{Z}_N)$ the space of complex N-periodic sequences with standard inner product. For any N-dimensional complex nonzero vector $(b_0, b_1, \dots, b_{N-1})$ satisfying the condition

$$|b_l|^2 + |b_{l+N_1}|^2 \leq \frac{2}{N^2}, \quad l = 0, 1, \dots, N_1 - 1,$$

we find sequences $u_0, u_1, \dots, u_r$ such that their binary shifts form a tight frame for $\ell^2(\mathbb{Z}_N)$. The vector $(b_0, b_1, \dots, b_{N-1})$ specifies the discrete Walsh transform of the sequence u_0, and the choice of this vector makes it possible to adapt the proposed construction to signal processing according to the entropy, mean-square, or some other criterion.

Consider the set

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$$

being an abelian group under the operation of coordinate-wise addition modulo 2:

$$a \oplus b := \sum_{\nu=0}^{n-1} |a_\nu - b_\nu|\, 2^\nu, \quad a, b \in \mathbb{Z}_N,$$

where

$$a = \sum_{\nu=0}^{n-1} a_\nu 2^\nu, \quad b = \sum_{\nu=0}^{n-1} b_\nu 2^\nu, \quad a_\nu, b_\nu \in \{0, 1\}.$$

The space $\ell^2(\mathbb{Z}_N)$ consists of complex sequences

$$x = (\dots, x(-1), x(0), x(1), x(2), \dots),$$

$$x(j) \in \mathbb{C}, \quad x(j+N) = x(j), \quad j \in \mathbb{Z}.$$

An arbitrary sequence x from $\ell^2(\mathbb{Z}_N)$ is identified with the vector

$$(x(0), x(1), \dots, x(N-1)).$$

Linear operations on the space $\ell^2(\mathbb{Z}_N)$ are defined component-wise. As usual, for $x, y \in \ell^2(\mathbb{Z}_N)$ we set

$$\langle x, y \rangle := \sum_{j=0}^{N-1} x(j)\overline{y(j)}, \quad \|x\| := \langle x, x \rangle^{1/2}.$$

The Walsh basis $w_0^{(N)}, w_1^{(N)}, \ldots, w_{N-1}^{(N)}$ for the space $\ell^2(\mathbb{Z}_N)$ consists of the discrete Walsh functions in Paley ordering; see Section 2.

The *discrete Walsh transform* takes an arbitrary x from $\ell^2(\mathbb{Z}_N)$ to the sequence $\widehat{x}$ of its Fourier coefficients with respect to the system $\{w_k^{(N)}\}_{k=0}^{N-1}$:

$$\widehat{x}(k) := \frac{1}{N} \sum_{j=0}^{N-1} x(j) w_k^{(N)}(j), \quad k \in \mathbb{Z}_N.$$

The Walsh expansion of x can be written as

$$x(j) = \sum_{k=0}^{N-1} \widehat{x}(k) w_k^{(N)}(j), \quad j \in \mathbb{Z}_N.$$

For each $k \in \mathbb{Z}_N$, the binary shift operator T_k is defined by

$$(T_k x)(j) := x(j \oplus k), \quad x \in \ell^2(\mathbb{Z}_N), \quad j \in \mathbb{Z}_N.$$

For $x, y \in \ell^2(\mathbb{Z}_N)$, $k, l \in \mathbb{Z}_N$, we have

$$\widehat{(T_k x)}(l) = w_k^{(N)}(l)\,\widehat{x}(l), \quad \langle T_k x, T_l y \rangle = \langle x, T_{l \oplus k} y \rangle.$$

Recall that $N = 2^n$ and $N_1 = 2^{n-1}$.

Definition 1. Let $u_0, u_1, \ldots, u_r \in \ell^2(\mathbb{Z}_N)$. If the system

$$B(u_0, u_1, \ldots, u_r) := \{T_{2k}u_0\}_{k=0}^{N_1-1} \cup \{T_{2k}u_1\}_{k=0}^{N_1-1} \cup \cdots \cup \{T_{2k}u_r\}_{k=0}^{N_1-1}$$

is a Parseval frame for $\ell^2(\mathbb{Z}_N)$, then $B(u_0, u_1, \ldots, u_r)$ called the *Parseval frame of the first stage for* $\ell^2(\mathbb{Z}_N)$*, generated a set of vectors* $u_0, u_1, \ldots, u_r$.

Theorem 1 [FR19]. *Let $u_0, u_1, \ldots, u_r \in \ell^2(\mathbb{Z}_N)$, $r \geq 1$, be vectors such that, for each $l = 0, 1, \ldots, N_1 - 1$, the matrix*

$$M(l) := \frac{N}{\sqrt{2}} \begin{pmatrix} \widehat{u_0}(l) & \widehat{u_1}(l) & \ldots & \widehat{u_r}(l) \\ \widehat{u_0}(l+N_1) & \widehat{u_1}(l+N_1) & \ldots & \widehat{u_r}(l+N_1) \end{pmatrix}$$

satisfies the condition

$$M(l)M^*(l) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Then the system $B(u_0, u_1, \ldots, u_r)$ is a Parseval frame of the first stage for $\ell^2(\mathbb{Z}_N)$, generated a set of vectors $u_0, u_1, \ldots, u_r$.

To prove:

Using the identity $\langle x, y \rangle = N \langle \widehat{x}, \widehat{y} \rangle$ for any $x \in \ell^2(\mathbb{Z}_N)$ the following equality verified

$$\|\widehat{x}\|^2 = N \sum_{s=0}^{r} \sum_{k=0}^{N_1-1} |\langle \widehat{x}, \widehat{T_{2k} u_s} \rangle|^2,$$

where

$$|\langle \widehat{x}, \widehat{T_{2k} u_s} \rangle|^2 = |\langle \widehat{x}, w_{2k}^{(N)} \widehat{u_s} \rangle|^2 = \left| \sum_{l=0}^{N-1} \widehat{x}(l)\, w_k^{(N_1)}(l)\, \overline{\widehat{u_s}(l)} \right|^2$$

$$= \sum_{l=0}^{N-1} |\widehat{x}(l)|^2\, |\widehat{u_s}(l)|^2 + 2\,\mathrm{Re}\,(A_k^{(N)}(\widehat{x}, \widehat{u_s})).$$

It is noted that

$$
\begin{aligned}
A_k^{(N)}(\widehat{x}, \widehat{u_s}) := {}& \widehat{x}(0)\,\overline{\widehat{u_s}(0)} \sum_{l'=1}^{N-1} w_k^{(N_1)}(l')\,\overline{\widehat{x}(l')}\,\widehat{u_s}(l') \\
&+ \widehat{x}(1)\,\overline{\widehat{u_s}(1)} \sum_{l'=2}^{N-1} w_k^{(N_1)}(l' \oplus 1)\,\overline{\widehat{x}(l')}\,\widehat{u_s}(l') \\
&+ \widehat{x}(2)\,\overline{\widehat{u_s}(2)} \sum_{l'=3}^{N-1} w_k^{(N_1)}(l' \oplus 2)\,\overline{\widehat{x}(l')}\,\widehat{u_s}(l') + \dots \\
&+ \widehat{x}(N-3)\,\overline{\widehat{u_s}(N-3)}(w_k^{(N_1)}(3)\,\overline{\widehat{x}(N-2)}\,\widehat{u_s}(N-2) \\
&+ w_k^{(N_1)}(2)\,\overline{\widehat{x}(N-1)}\,\widehat{u_s}(N-1)) \\
&+ \widehat{x}(N-2)\,\overline{\widehat{u_s}(N-2)}\,w_k^{(N_1)}(1)\,\overline{\widehat{x}(N-1)}\,\widehat{u_s}(N-1).
\end{aligned}
$$

Hence, since

$$\sum_{s=0}^{r} |\widehat{u}_s(l)|^2 = \sum_{s=0}^{r} |\widehat{u}_s(l+N_1)|^2 = \frac{2}{N^2},$$

we have

$$\sum_{s=0}^{r} \sum_{k=0}^{N_1-1} |\langle \widehat{x}, \widehat{T_{2k}u_s} \rangle|^2 = \frac{2N_1}{N^2} \sum_{l=0}^{N-1} |\widehat{x}(l)|^2 + 2\,\mathrm{Re}\left(\sum_{s=0}^{r} \sum_{k=0}^{N_1-1} A_k^{(N)}(\widehat{x}, \widehat{u}_s) \right).$$

In conclusion, the following equalities apply

$$\sum_{k=0}^{N_1-1} w_k^{(N_1)}(l) = \begin{cases} N_1 & \text{if } l \text{ is divisible by } N_1, \\ 0 & \text{if } l \text{ is not divisible by } N_1, \end{cases}$$

$$\sum_{s=0}^{r} \widehat{u}_s(l)\overline{\widehat{u}_s(l+N_1)} = 0, \quad 2N_1 = N, \quad \sum_{l=0}^{N-1} |\widehat{x}(l)|^2 = \|\widehat{x}\|^2.$$

For the case $r = 1$, the system $B(u_0, u_1)$ from Theorem 1 is an orthogonal bases in $\ell^2(\mathbb{Z}_N)$; see [FS11].

Example 1 (cf [L98]). Let $r = 1$, $n = 2$. Choose complex numbers a and b such that $|a|^2 + |b|^2 = 1$ and define $u_0, u_1 \in \ell^2(\mathbb{Z}_4)$ by

$$u_0(0) = \frac{1+a+b}{2\sqrt{2}}, \quad u_0(1) = \frac{1+a-b}{2\sqrt{2}},$$

$$u_0(2) = \frac{1-a-b}{2\sqrt{2}}, \quad u_0(3) = \frac{1-a+b}{2\sqrt{2}},$$

$$u_1(0) = \frac{1+a-b}{2\sqrt{2}}, \quad u_1(1) = -\frac{1+a+b}{2\sqrt{2}},$$

$$u_1(2) = \frac{1-a+b}{2\sqrt{2}}, \quad u_1(3) = -\frac{1-a-b}{2\sqrt{2}}.$$

Then $B(u_0, u_1)$ is an orthonormal basis in $\ell^2(\mathbb{Z}_4)$. In particular, when $a = 1$, $b = 0$ we obtain a Haar-type basis:

$$\frac{1}{\sqrt{2}}(1,1,0,0), \quad \frac{1}{\sqrt{2}}(1,-1,0,0),$$

$$\frac{1}{\sqrt{2}}(0,0,1,1), \quad \frac{1}{\sqrt{2}}(0,0,1,-1).$$

Example 2. Let $r = 2$, $n = 1$. Then

$$M(0) = \sqrt{2}\begin{bmatrix} \widehat{u}_0(0) & \widehat{u}_1(0) & \widehat{u}_2(0) \\ \widehat{u}_0(1) & \widehat{u}_1(1) & \widehat{u}_2(1) \end{bmatrix}.$$

Let us take orthogonal vectors (x_0, x_1, x_2) and (y_0, y_1, y_2) with unit lengths:

$$x_0\bar{y}_0 + x_1\bar{y}_1 + x_2\bar{y}_2 = 0,$$

$$|x_0|^2 + |x_1|^2 + |x_2|^2 = 1, \quad |y_0|^2 + |y_1|^2 + |y_2|^2 = 1.$$

The condition (1) will be satisfied if we set

$$\widehat{u}_i(0) = \frac{\sqrt{2}}{2}\,x_i, \quad \widehat{u}_i(1) = \frac{\sqrt{2}}{2}\,y_i, \quad i = 0, 1, 2.$$

In particular, if $x_0 = a$, $y_0 = b$, $|a|^2 + |b|^2 \leq 1$, then we can take

$$x_1 = 0, \quad x_2 = \sqrt{1 - |a|^2},$$

$$y_2 = -\frac{a\bar{b}}{\sqrt{1 - |a|^2}}, \quad y_1 = \sqrt{1 - |b|^2 - |y_2|^2}.$$

As a result, for each pair of complex numbers (a, b), satisfying the condition $0 < |a|^2 + |b|^2 \leq 1$, we get the Parseval frame $\{u_0, u_1, u_2\}$ for $\ell^2(\mathbb{Z}_2)$.

Example 3. Let $r = 2$ and $N = 4$. Choose u_0, u_1, u_2 in $\ell^2(\mathbb{Z}_4)$ such that

$$\sum_{s=0}^{2} \widehat{u}_s(l)\overline{\widehat{u}_s(l+2)} = 0, \quad \sum_{s=0}^{2} |\widehat{u}_s(l)|^2 = \sum_{s=0}^{2} |\widehat{u}_s(l+2)|^2 = \frac{1}{8}, \quad l = 0, 1.$$

Then $\{u_0, u_1, u_2, T_2 u_0, T_2 u_1, T_2 u_2\}$ is a Parseval frame for $\ell^2(\mathbb{Z}_4)$. Indeed, in this case

$$M(l) = \frac{4}{\sqrt{2}} \begin{pmatrix} \widehat{u_0}(l) & \widehat{u_1}(l) & \widehat{u_2}(l) \\ \widehat{u_0}(l+2) & \widehat{u_1}(l+2) & \widehat{u_2}(l+2) \end{pmatrix}, \quad l = 0, 1,$$

and (1) holds.

Example 4. In the case where $N = 4$ and $r = 3$, we have

$$M(l) = \frac{4}{\sqrt{2}} \begin{pmatrix} \widehat{u}_0(l) & \widehat{u}_1(l) & \widehat{u}_2(l) & \widehat{u}_3(l) \\ \widehat{u}_0(l+2) & \widehat{u}_1(l+2) & \widehat{u}_2(l+2) & \widehat{u}_3(l+2) \end{pmatrix},$$

where $l = 0$ or $l = 1$. To satisfy condition (1), we can choose the matrices $M(0)$ and $M(1)$ so that the matrix $(\widehat{u}_j(l))_{l,j=0}^3$ is proportional to the matrix $(w_j^{(4)}(l))_{l,j=0}^3$. Then the system

$$\{u_0, u_1, u_2, u_3, T_2 u_0, T_2 u_1, T_2 u_2, T_2 u_3\}$$

will be a Parseval frame for $\ell^2(\mathbb{Z}_4)$. Note that Example 4 can be extended to the case $r = N - 1$ for an arbitrary N.

Under condition (1), the matrix $U(l)$ can be complemented to a unitary matrix of order $r+1$. Therefore, we have

$$|\widehat{u}_s(l)|^2+|\widehat{u}_s(l+N_1)|^2 \leq \frac{2}{N^2}, \quad s=0,1,\dots,r,\ l=0,1,\dots,N_1-1.$$

Using Theorem 1, we obtain the following algorithm for constructing a Parseval frame of the first stage for $\ell^2(\mathbb{Z}_N)$.

Algorithm A.

Step 1. Choose a natural number $m \leq n$ and complex numbers b_l, $0 \leq l \leq 2^m - 1$, such that

$$|b_l|^2 + |b_{l+2^{m-1}}|^2 \leq 1, \quad l = 0, 1, \ldots, 2^{m-1} - 1.$$

Step 2. Calculate $a_0, \ldots, a_{2^m-1}$ by formulas

$$a_j = 2^{-m+1/2} \sum_{l=0}^{2^m-1} b_l w_l^{(2^m)}(j), \quad j = 0, 1, \ldots, 2^m - 1.$$

Step 3. Define a vector $u_0 \in \ell^2(\mathbb{Z}_N)$ such that

$$u_0(j) = \begin{cases} a_j, & \text{if} \quad 0 \le j \le 2^m - 1, \\ 0, & \text{if} \quad 2^m \le j \le 2^n - 1. \end{cases}$$

Step 4. For the vector u_0 obtained at Step 3, find vectors $u_1, \ldots, u_r \in \ell^2(\mathbb{Z}_N)$ such that the matrix $M(l)$, for each $l = 0, 1, \ldots, N_1 - 1$, satisfies condition (1).

Step 5. Define $B(u_0, u_1, \ldots, u_r)$ by the formula

$$B(u_0, u_1, \ldots, u_r) = \{T_{2k} u_0\}_{k=0}^{N_1-1} \cup \{T_{2k} u_1\}_{k=0}^{N_1-1} \cup \cdots \cup \{T_{2k} u_r\}_{k=0}^{N_1-1}.$$

Remark 1. Step 2 is implemented by using fast algorithms for the discrete Walsh transform, and to implement Step 4, we can use the same methods as in the construction of MRA-frames on the *p*-adic Vilenkin group.

Remark 2. To select the values of the parameters b_l in Algorithm A, it is natural to follow two approaches:

1) adaptation to the signal as in [Mallat, § 9.4],

2) approximation to a "good"(equal–norm, symmetric, ...) tight frame.

For image processing, some examples of the implementation of the first approach are given in [FS11].

For $\nu = 0, 1, \ldots, n$, we set $N_\nu := N/2^\nu$.

Definition 2. Let $m \in \mathbb{N}$, $m \leq n$. By a *sequence of dyadic Parseval frames of the mth stage* we mean the sequence of vectors

$$u_0^{(1)}, u_1^{(1)}, \ldots, u_r^{(1)}, \quad u_0^{(2)}, u_1^{(2)}, \ldots, u_r^{(2)}, \quad \ldots, \quad u_0^{(m)}, u_1^{(m)}, \ldots, u_r^{(m)},$$

such that $u_0^{(\nu)}, u_1^{(\nu)}, \ldots, u_r^{(\nu)} \in \mathbb{C}_{N_{\nu-1}}$, and, for the matrices

$$M_\nu(l) := \frac{N}{\sqrt{2}} \begin{pmatrix} \widehat{u}_0^{(\nu)}(l) & \widehat{u}_1^{(\nu)}(l) & \ldots & \widehat{u}_r^{(\nu)}(l) \\ \widehat{u}_0^{(\nu)}(l+N_\nu) & \widehat{u}_1^{(\nu)}(l+N_\nu) & \ldots & \widehat{u}_r^{(\nu)}(l+N_\nu) \end{pmatrix}$$

for each $l = 0, 1, \ldots, N_\nu - 1$, the following equality holds:

$$M_\nu(l) M_\nu^*(l) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 2. *Suppose that a vector $u_0 \in \ell^2(\mathbb{Z}_N)$ satisfies condition*

$$|\widehat{u}_0(l)|^2 + |\widehat{u}_0(l+N_1)|^2 \leq \frac{2}{N^2}, \quad l = 0, 1, \dots, N_1 - 1,$$

and vectors $u_1, \dots, u_r \in \ell^2(\mathbb{Z}_N)$ satisfy the conditions of Theorem 1. Let

$$u_0^{(1)} = u_0, u_1^{(1)} = u_1, \dots, u_r^{(1)} = u_r$$

and let the vectors $u_0^{(\nu)}, u_1^{(\nu)}, \dots, u_r^{(\nu)}$, $\nu = 2, \dots, m$, be defined by the formulas

$$u_s^{(\nu)}(j) = 2^{1-\nu} \sum_{k=0}^{2^{\nu-1}-1} u_s(j + kN_{\nu-1}), \quad j \in \mathbb{Z}_{N_{\nu-1}}, \quad s = 0, 1, \dots, r.$$

Then the sequence

$$u_0^{(1)}, u_1^{(1)}, \dots, u_r^{(1)}, \quad u_0^{(2)}, u_1^{(2)}, \dots, u_r^{(2)}, \quad \dots, \quad u_0^{(m)}, u_1^{(m)}, \dots, u_r^{(m)},$$

is a sequence of dyadic Parseval frames of the mth stage.

The constructions from Theorems 1 and 2 can be completed using the convolution

$$(x * y)(k) := \sum_{j=0}^{N-1} x(k \oplus j) y(j), \quad x, y \in \ell^2(\mathbb{Z}_N), \quad k \in \mathbb{Z}_N.$$

The operator $U : \ell^2(\mathbb{Z}_{N_1}) \to \ell^2(\mathbb{Z}_N)$ given by the equality

$$(Uy)(j) := \begin{cases} y(j/2) & \text{if } j \text{ even}, \\ 0 & \text{if } j \text{ odd}, \end{cases}$$

where $y \in \ell^2(\mathbb{Z}_{N_1})$, is called the *upsampling operator*.

Example 4. Let $m = n = 2$. Suppose that $u_0^{(1)}, u_1^{(1)}, u_2^{(1)} \in \mathbb{C}_4$, $u_0^{(2)}, u_1^{(2)}, u_2^{(2)} \in \mathbb{C}_2$, and the sequence $u_0^{(1)}, u_1^{(1)}, u_2^{(1)}, u_0^{(2)}, u_1^{(2)}, u_2^{(2)}$ is a second-stage sequence of Parseval frames. We set

$$\varphi^{(1)} = u_0^{(1)}, \quad \psi_1^{(1)} = u_1^{(1)}, \quad \psi_2^{(1)} = u_2^{(1)},$$

$$\varphi^{(2)} = \varphi^{(1)} * \ Uu_0^{(2)}, \quad \psi_1^{(2)} = \varphi^{(1)} * \ Uu_1^{(2)}, \quad \psi_2^{(2)} = \varphi^{(1)} * \ Uu_2^{(2)}.$$

Then the system

$$\{T_2\psi_1^{(1)},\ T_2\psi_2^{(1)},\ \psi_1^{(1)},\ \psi_2^{(1)},\ \psi_1^{(2)},\ \psi_2^{(2)},\ \varphi^{(2)}\}$$

is a Parseval frame for $\ell^2(\mathbb{Z}_4)$.

5. Finite frames related to Hadamard matrices

A **Hadamard matrix** is a square matrix with entries equal to ± 1 whose rows (and hence columns) are mutually orthogonal.

In special cases, the matlab function *hadamard*(m) (defined for m, $m/12$ or $m/20$ a power of 2) can be used to construct equal–norm tight frames of m vectors in $\mathbb{R}^d$. The Walsh matrices correspond to the case $m = 2^n$.

It is well known that if H is a Hadamard matrix of order $m \geq 4$, then m is a multiple of 4.

The **Hadamard conjecture** is that there exists a Hadamard matrix of size $m = 4k$, for every k. The smallest open case (in 2010) is $m = 668$.

Theorem (J.S. Hadamard, 1893). Let $H = [h_{ij}]$ be a real matrix of order n whose entries satisfy the condition $|h_{ij}| \leq 1$ for all $1 \leq i, j \leq m$. Then

$$| \det H|^2 \leq \prod_{i=1}^{m} \sum_{j=1}^{m} |h_{ij}|^2 \leq m^m.$$

equality holds if and only if H is a Hadamard matrix.

The volume of an n-dimensional parallelepiped does not exceed the product of the lengths of its edges; it is equal to this product if and only if, when its edges are orthogonal.

Professor Jennifer Seberry:

The Hadamard Matrix Lady – Mother of Cryptology in Australia – The Grandmother of Computer Security

https://documents.uow.edu.au/ jennie/

Sloane, N.J.A.: A library of Hadamard matrices

http://neilsloane.com/hadamard/

[SWW05] Seberry J., Wysocki B., Wysocki T. On some applications of Hadamard matrices, *Metrika* **62** (2005) (over 300 links in zbMATH).

In the recent paper [HT20], it is shown that for binary measurements (modelled with Walsh functions and Hadamard matrices) and wavelet reconstruction the stable sampling rate is linear. This implies that binary measurements are as efficient as Fourier samples when using wavelets as the reconstruction space. Powerful techniques for reconstructions include generalized sampling and its compressed versions, as well as recent methods based on data assimilation. Common to these methods is that the reconstruction quality depends highly on the subspace angle between the sampling and the reconstruction space, which is dictated by the stable sampling rate.

A frame $\{v_j\}_{j=1}^m$ is called *equiangular* if $\|v_j\| = 1$ for all j and there is a constant $\gamma \in [0, 1)$ such that

$$|\langle v_i, v_j \rangle| = \gamma \quad \text{for} \quad i \neq j.$$

For an arbitrary equiangular frame $\{v_j\}_{j=1}^m$ in d-dimensional Hilbert space we have

$$\gamma \geq \sqrt{\frac{m-d}{d(m-1)}}.$$

Equality is achieved if and only if the frame $\{v_j\}_{j=1}^m$ is tight. In communications and coding theory equiangular tight frames with small γ are used.

Frame Research Center:

Equiangular frames first appeared in discrete geometry but today have applications to signal processing, communications, coding theory and quantum physics. The main problem hindering their application is the fact that we know very few of them. This lacking is due to the fact that the starting point for constructing equiangular tight frames is the existence of maximal equiangular line sets. The construction of equiangular line sets has proved exceptionally difficult and in sixty years of research in the real case, the maximal number of equiangular lines is known only for 35 dimensions. Our goal is to develop completely new tools for constructing equiangular line sets and applying these methods to the construction of equiangular tight frames.

https://framerc.missouri.edu/projects

A method for constructing equiangular frames using Hadamard matrices is proposed in

[FMT12] Fickus M., Mixon D.G., Tremain J.C. Steiner equiangular tight frames, *Linear Algebra Appl.* **436** (2012).

Recall that a *Steiner system* $S(t, k, \nu)$ is a collection of k–element subsets (called *blocks*) of an ν-element set S, with the property that each t–element subset of S is contained in exactly one block.

Example 5 [FMT12]. Let $S = \{1, 2, 3, 4\}$. Then the blocks for $S(2, 2, 4)$ are 2-element sets:

$$\{1, 2\},\ \{1, 3\},\ \{1, 4\},\ \{2, 3\},\ \{2, 4\},\ \{3, 4\}.$$

Let's compose the incidence matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

For each $a \in S$ there are three blocks containing the number a. Then we denote:

$$\beta_1(1) = \{1,2\},\ \beta_2(1) = \{1,3\},\ \beta_3(1) = \{1,4\},$$

$$\beta_1(2) = \{1,2\},\ \beta_2(2) = \{2,3\},\ \beta_3(2) = \{2,4\},$$

$$\beta_1(3) = \{1,3\},\ \beta_2(3) = \{2,3\},\ \beta_3(3) = \{3,4\},$$

$$\beta_1(4) = \{1,4\},\ \beta_2(4) = \{2,4\},\ \beta_3(4) = \{3,4\}.$$

Using the Hadamard matrix

$$H_2 = H_1 \otimes H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

for each $a \in S$ we define row vectors

$$h^{(a)}_{\beta_1(a)} = [1, -1, 1, -1], \quad h^{(a)}_{\beta_2(a)} = [1, 1, -1, -1],$$

$$h^{(a)}_{\beta_3(a)} = [1, -1, -1, 1].$$

Replacing the nonzero elements of the matrix C by these row vectors, we get a block matrix

$$V = \begin{bmatrix} h^{(1)}_{\{1,2\}} & h^{(2)}_{\{1,2\}} & 0 & 0 \\ h^{(1)}_{\{1,3\}} & 0 & h^{(3)}_{\{1,3\}} & 0 \\ h^{(1)}_{\{1,4\}} & 0 & 0 & h^{(4)}_{\{1,4\}} \\ 0 & h^{(2)}_{\{2,3\}} & h^{(3)}_{\{2,3\}} & 0 \\ 0 & h^{(2)}_{\{2,4\}} & 0 & h^{(4)}_{\{2,4\}} \\ 0 & 0 & h^{(3)}_{\{3,4\}} & h^{(4)}_{\{3,4\}} \end{bmatrix}.$$

Columns of this block matrix are 16 vectors that form an equiangular tight frame for $\mathbb{R}^6$.

Remarks:

1. Any three rows of the matrix H_2 can be used to convert C to V.

2. To construct a frame of 16 vectors in $\mathbb{C}^6$, instead of H_2, we can take the complex Hadamard matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Example 6. In the case $k = 3$, $\nu = 7$, the Steiner system $S(2, 3, 7)$ is known as the *Fano plane*. The blocks for $S(2, 3, 7)$ are 3-element sets:

$\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$, $\{3, 5, 6\}$.

Each point of the set $S = \{1, 2, \ldots, 7\}$ belongs to three of these seven blocks:

$$\beta_1(1) = \{1, 2, 3\}, \ \beta_2(1) = \{1, 4, 5\}, \ \beta_3(1) = \{1, 6, 7\},$$
$$\beta_1(2) = \{1, 2, 3\}, \ \beta_2(2) = \{2, 4, 6\}, \ \beta_3(2) = \{2, 5, 7\},$$
$$\beta_1(3) = \{1, 2, 3\}, \ \beta_2(3) = \{3, 4, 7\}, \ \beta_3(3) = \{3, 5, 6\},$$
$$\beta_1(4) = \{1, 4, 5\}, \ \beta_2(4) = \{2, 4, 6\}, \ \beta_3(4) = \{3, 4, 7\}.$$
$$\beta_1(5) = \{1, 4, 5\}, \ \beta_2(5) = \{2, 5, 7\}, \ \beta_3(5) = \{3, 5, 6\},$$
$$\beta_1(6) = \{1, 6, 7\}, \ \beta_2(6) = \{2, 4, 6\}, \ \beta_3(6) = \{3, 5, 6\},$$
$$\beta_1(7) = \{1, 6, 7\}, \ \beta_2(7) = \{2, 5, 7\}, \ \beta_3(7) = \{3, 4, 7\}.$$

Further, as in Example 5, we obtain an equiangular tight frame of 28 vectors in the space $\mathbb{R}^7$ (the main properties of this frame are given in [FJM18]).

In [FMT12], each $a \in S(2, k, \nu)$ is associated with its own (possibly complex) Hadamard matrix H^a of order $1 + \frac{\nu-1}{k-1}$ and an equiangular tight frame of m vectors in the space of dimension d, where

$$m = \nu \left(1 + \frac{\nu - 1}{k - 1}\right), \quad d = \frac{\nu(\nu - 1)}{k(k - 1)}, \quad \frac{m}{d} = k \left(1 + \frac{k - 1}{\nu - 1}\right).$$

As a result, eight infinite families of Steiner equiangular tight frames are obtained.

In 2014 Adam Marcus, Daniel Spielman and Nikhil Srivastava used random vectors to prove a key discrepancy theorem and in so doing gave a positive answer to the long-standing Kadison-Singer Problem (KSP).

Problem (KSP). *Does every pure state on the algebra of bounded diagonal operators acting on the Hilbert space of square summable complex-valued sequences have a unique extension to a regular state on the algebra of all bounded operators?*

The KSP problem connects with an unusually large number of research areas including: operator algebras (pure states), set theory (ultrafilters), operator theory (paving), random matrix theory, linear and multilinear algebra, algebraic combinatorics (real stable polynomials), algebraic curves, frame theory, harmonic analysis (Fourier frames), and functional analysis; see, e.g., [S16, B18].

In [AHV18], a class of frames defined by Walsh matrices is constructed and discuss how these frames relate to the key discrepancy theorem. This paper is motivated by the following Marcus-Spielman-Srivistava Discrepancy Theorem:

Theorem (MSSDT). *If* $v_1, \ldots, v_m \in \mathbb{C}^d$*,* $m \geq d$*, are such that* $\|v_j\| \leq \alpha$ *for all* $j = 1, \ldots, m$ *and*

$$\sum_{j=1}^{m} v_j v_j^* = I_m,$$

then there exists a partition of the set of indices $\{1, \ldots, m\}$ *into two disjoint sets* J_1 *and* J_2 *such that*

$$\|\sum_{j \in J_k} v_j v_j^*\|_2 \leq \left(\frac{1}{\sqrt{2}} + \sqrt{\alpha}\right)^2, \quad k = 1, 2,$$

where $\|\cdot\|_2$ *is the spectral matrix norm.*

Suppose that $N = 2^n$ for some $n \in \mathbb{N}$. Let us choose $m < N$ and for an arbitrary orthogonal $N \times N$ matrix G we let

$$G = [\,\mathbf{g}_1, \dots, \mathbf{g}_m \,|\, \mathbf{g}_{m+1}, \dots, \mathbf{g}_N\,].$$

Then we define the matrix $G_1 \in \mathbb{R}^{N \times m}$ by the equality $G_1 = [\,\mathbf{g}_1, \dots, \mathbf{g}_m\,]$ and denote by $V = [\,\mathbf{v}_1, \dots, \mathbf{v}_N\,]$ matrix transposed to G_1. By Proposition 4, the vectors $\mathbf{v}_1, \dots, \mathbf{v}_N$ form a Parseval frame for $\mathbb{C}^m$. In the case of $G = H_n/\sqrt{N}$, the frame $\{\mathbf{v}_j\}_{j=1}^N$ will be called a (m, N)-*Walsh frame*. Note that $\|\mathbf{v}_j\| = \sqrt{m/N}$ for $j = 1, \dots, N$.

The (m, N)-Walsh frame $\{\mathbf{v}_1, \ldots, \mathbf{v}_N\}$ for $m \leq N/2$ is split into two identical tight frames for $\mathbb{C}^m$, which correspond to matrices

$$V_1 = [\mathbf{v}_1, \ldots, \mathbf{v}_{N/2}], \quad V_2 = [\mathbf{v}_{N/2+1}, \ldots, \mathbf{v}_N],$$

and $V_1 V_1^* = V_2 V_2^* = I_m/2$ (that is the frame constant $A = 1/2$). This property easily follows from the equality

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix},$$

For $N/2 < m \leq N$ it is shown in [AHV18] that the frames can no longer be evenly split but there is an explicit expression for the discrepancy in a best possible split.

Example 7. Let $m = 3$ and $N = 8$. Then the construction presented in [AHV18] leads to the matrix

$$V = \frac{1}{2\sqrt{2}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{array} \right] = \left[\begin{array}{c|c} V_a & V_b \end{array} \right],$$

for which $VV^* = I_3$. The Parseval frame determined by this matrix is split into two identical frames such that $V_a V_a^* = V_b V_b^* = I_3/2$.

At the next step, after changing the normalization, we obtain the matrix

$$V = \frac{1}{2} \left[\begin{array}{rr|rr} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{array} \right] = \left[\begin{array}{c|c} V_a & V_b \end{array} \right],$$

for which $VV^* = I_3$. However, now $V_a \neq V_b$ and splitting into identical frames is impossible. Moreover, we have

$$\| V_a V_a^* - I_3/2 \|_2 = \| V_b V_b^* - I_3/2 \|_2 = 1/2,$$

where $\| \cdot \|_2$ is the spectral matrix norm.

6. Parseval frames on the p-adic Vilenkin group

For any integer $p \geq 2$, Vilenkin's group $G = G_p$ consists of sequences $x = (x_j)$, where $x_j \in \{0, 1, \dots, p-1\}$, $j \in \mathbb{Z}$, and there exists at most a finite number of negative j such that $x_j \neq 0$. The group operation $\oplus$ on G is defined as the coordinatewise addition modulo p while the topology on G is introduced via the complete system of neighbourhoods of the zero element of G:

$$U_l := \{(x_j) \in G \;:\; x_j = 0 \text{ for all } j \leq l\}, \quad U_{l+1} \subset U_l, \quad l \in \mathbb{Z}.$$

As usual, the equality $z = x \ominus y$ means that $z \oplus y = x$.

Let $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{Z}_+ := \{0, 1, \dots\}$. The mapping $\lambda : G \to \mathbb{R}_+$ is defined by

$$\lambda(x) = \sum_{j \in \mathbb{Z}} x_j p^{-j}, \qquad x = (x_j) \in G.$$

The image of the discrete subgroup

$$H := \{(x_j) \in G \ : \ x_j = 0 \text{ for all } j > 0\}$$

under λ is the set of nonnegative integers: $\lambda(H) = \mathbb{Z}_+$. For each $\alpha \in \mathbb{Z}_+$, let $h_{[\alpha]}$ denote the element of H such that $\lambda(h_{[\alpha]}) = \alpha$.

The group G is self-dual. The duality pairing on G takes $x = (x_j)$ and $\omega = (\omega_j)$ to

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j\in\mathbb{Z}} x_j \omega_{1-j}\right).$$

The *generalized Walsh functions* on G can be defined by

$$W_k(x) = \chi(x, h_{[k]}), \quad x \in G, \quad k \in \mathbb{Z}_+.$$

So, these functions are characters for G. Also, it is known that $\{W_k \,:\, k \in \mathbb{Z}_+\}$ is an orthonormal basis for $L^2(U)$.

We define an automorphism $A: G \to G$ by letting $(Ax)_j = x_{j+1}$ for all $x = (x_j) \in G$ and let $N = p^n$. The sets

$$U_{n,s} := A^{-n}(h_{[s]}) \oplus A^{-n}(U), \quad 0 \leq s \leq N-1,$$

are cosets of the subgroup $A^{-n}(U)$ in the group U. For every $0 \leq \alpha \leq N-1$ the Walsh function W_α is constant on each $U_{n,s}$.
Note: $A^{-n}(U) = U_n$, $\lambda(U_n) = [0, p^{-n}]$, $\mu(U_n) = \mu(U_{n,s}) = p^{-n}$, $\lambda(U_{n,s}) = [s/N, (s+1)/N]$.

The *Vilenkin-Chrestenson transform* translates a vector $\boldsymbol{b} = (b_0, b_1, \ldots, b_{N-1}) \in \mathbb{C}^N$ into a vector $\boldsymbol{a} = (a_0, a_1, \ldots, a_{N-1})$ with components

$$a_\alpha = \frac{1}{N} \sum_{s=0}^{N-1} b_s W_s(A^{-n} h_{[\alpha]}), \quad 0 \le \alpha \le N-1. \tag{1}$$

The inverse transform acts by the formula

$$b_s = \sum_{\alpha=0}^{N-1} a_\alpha \overline{W_\alpha(A^{-n} h_{[s]})}, \qquad 0 \le s \le N-1. \tag{2}$$

These transforms can be realized by the fast algorithms.

A mask $m_{\boldsymbol{b}}$ associated with a vector $\boldsymbol{b}$ has the form

$$m_{\boldsymbol{b}}(\omega) = \sum_{k=0}^{N-1} a_k \overline{W_k(\omega)}, \quad \omega \in G,$$

where the coefficients a_k are defined by (1). A compactly supported function $\varphi \in L^2(G)$ is a *refinable function* with the mask $m_{\boldsymbol{b}}$ if it satisfies the equation

$$\varphi(x) = p \sum_{k=0}^{N-1} a_k \varphi(Ax \ominus h_{[k]}), \tag{3}$$

or, in the Fourier domain, $\widehat{\varphi}(\omega) = m_{\boldsymbol{b}}(A^{-1}\omega)\widehat{\varphi}(A^{-1}\omega)$.

For any $f \in L^2(G)$ we let

$$f_{j,k}(x) := p^{j/2} f(A^j x \ominus h_{[k]}), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}_+.$$

Given $\Psi := \{\psi^{(1)}, \ldots, \psi^{(r)}\} \subset L^2(G)$ with $r \geq p$, we define the *wavelet system* as

$$X(\Psi) := \{\psi_{j,k}^{(\nu)} \ :\ 1 \leq \nu \leq r,\ j \in \mathbb{Z},\ k \in \mathbb{Z}_+\}.$$

The system $X(\Psi)$ is a Parseval frame (or a *wavelet tight frame*) for $L^2(G)$ if

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}_+} \sum_{\nu=1}^{r} |\langle f, \psi_{j,k}^{(\nu)} \rangle|^2 = \|f\|^2$$

for all $f \in L^2(G)$.

Now, we let $N_1 := p^{n-1}$ and denote by $\boldsymbol{F}(p,n)$ the set of all vectors $\boldsymbol{b} = (b_0, b_1, \dots, b_{N-1})$ of the space $\mathbb{C}^N$ such that

$$b_0 = 1, \quad |b_l|^2 + |b_{l+N_1}|^2 + \cdots + |b_{l+(p-1)N_1}|^2 \leq 1, \tag{4}$$

for all $0 \leq l \leq N_1 - 1$. The following algorithm allows us to construct a Parseval frame for $L^2(G)$ from any vector $\boldsymbol{b} \in \boldsymbol{F}(p,n)$.

Algorithm A.

- **Step 1**. Choose an arbitrary vector $\boldsymbol{b} = (b_0, b_1, \ldots, b_{N-1})$ in $\boldsymbol{F}(p, n)$.
- **Step 2**. Compute a_α, $0 \le \alpha \le N-1$, by (1) and define

$$m_0(\omega) = \sum_{\alpha=0}^{N-1} a_\alpha \overline{W_\alpha(\omega)}.$$

- **Step 3**. Define $\varphi \in L^2(G)$ such that

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} m_0(A^{-j}\omega), \quad \omega \in G. \tag{5}$$

Note that, according to Step 1 and Step 2,

$$\sum_{l=0}^{p-1} |m_0(\omega \oplus \delta_l)|^2 \leq 1, \qquad \omega \in G.$$

This implies that the function φ on Step 3 belongs to $L^2(G)$.

Application of the unitary extension principle gives the following

Theorem 1. *Let* $\Psi = \{\psi^{(1)}, \ldots, \psi^{(r)}\}$ *be the wavelet system determined in Algorithm A. Then* $X(\Psi)$ *is a Parseval frame for* $L^2(G)$.

We write $\boldsymbol{b} \in \boldsymbol{G}(p,n)$, if for a vector $\boldsymbol{b} \in \boldsymbol{F}(p,n)$ all inequalities in (4) become equalities:

$$b_0 = 1, \ |b_l|^2 + |b_{l+N_1}|^2 + \cdots + |b_{l+(p-1)N_1}|^2 = 1, \qquad (6)$$

where $0 \le l \le N_1 - 1$. Further, denote by $\boldsymbol{W}(p,n)$ the set of all vectors $\boldsymbol{b} \in \boldsymbol{G}(p,n)$ for which

$$V(\varphi_{\boldsymbol{b}}) := \{\varphi_{\boldsymbol{b}}(\cdot \ominus h) \ : \ h \in H\}$$

is an orthonormal system in $L^2(G)$. It is known that $\boldsymbol{W}(p,n) \subset \boldsymbol{G}(p,n)$. Also, if a vector $\boldsymbol{b} = (b_0, b_1, \ldots, b_{N-1})$ lies in $\boldsymbol{G}(p,n)$ and $b_l \neq 0$ for all $1 \le l \le N_1 - 1$, then $\boldsymbol{b} \in \boldsymbol{W}(p,n)$.

- **Step 4**. Given $r \geq p$ find the Walsh polynomials

$$m_\nu(\omega) = \sum_{\alpha=0}^{N-1} a_\alpha^{(\nu)} \overline{W_\alpha(\omega)}, \quad 1 \leq \nu \leq r,$$

  such that, for each $\omega \in G$, the rows of the matrix $M(\omega)$ form an orthonormal system, where

$$M(\omega) := \begin{bmatrix} m_0(\omega) & m_1(\omega) & \dots & m_r(\omega) \\ m_0(\omega \oplus \delta_1) & m_1(\omega \oplus \delta_1) & \dots & m_r(\omega \oplus \delta_1) \\ \dots & \dots & \dots & \dots \\ m_0(\omega \oplus \delta_{p-1}) & m_1(\omega \oplus \delta_{p-1}) & \dots & m_r(\omega \oplus \delta_{p-1}) \end{bmatrix}$$

  with $\delta_l \in U$, $\lambda(\delta_l) = l/p$, $l = 0, \dots, p-1$.

- **Step 5**. Define $\psi^{(1)}, \dots, \psi^{(r)}$ as follows:

$$\psi^{(\nu)}(x) = p \sum_{\alpha=0}^{N-1} a_\alpha^{(\nu)} \varphi(Ax \ominus h_{[\alpha]}), \quad 1 \leq \nu \leq r.$$

Algorithm A with $\boldsymbol{b} \in \boldsymbol{G}(p, n)$ in Step 1 can be applied for $r = p - 1$ and Theorem 1 is still valid in this case. Moreover, it is known that Algorithm A with $\boldsymbol{b} \in \boldsymbol{W}(p, n)$ leads to orthogonal MRA-based wavelets $\psi^{(1)}, \dots, \psi^{(p-1)}$ in $L^2(G)$. In this case, the system $X(\Psi)$ is an orthonormal basis for $L^2(G)$.

There are three ways to verify the orthogonality of $V(\varphi_{\boldsymbol{b}})$: (a) the modified Cohen criterion [F. (2007)], (b) the blocking sets criterion [Protasov, F. (2006), F. (2007)], and (c) the N-valid trees method [Lukomskii, Berdnikov (2015)].

Applications:

- for image processing: [F., Maksimov, and Stroganov 2011], [F. and Stroganov, 2011],
- for coding of fractal functions: [F., Borisov, 2012], [F. and Rodionov, 2015],
- for estimate of smoothness of low-frequency microseismic oscillations: [Stroganov, 2012].

The algorithms used in these applications with a detailed bibliography are given in a recent review article [F19c].

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Thank you for your attention!