Non-Archimedean Statistical Field Theory

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INTRODUCTION

In arXiv:2006.05559, Non-Archimedean Statistical Field Theory
We construct (in a rigorous mathematical way) interacting quantum field theories over a p-adic spacetime in an arbitrary dimension.

We provide a large family of energy functionals $E(\varphi, J)$ admitting natural discretizations in finite-dimensional vector spaces such that the partition function

$$Z^{\text{phys}}(J) = \int D(\varphi) e^{-\frac{1}{K_B T} E(\varphi, J)} \tag{1}$$

can be defined rigorously as the limit of the mentioned discretizations.

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- **Key fact**: $\mathcal{D}_{\mathbb{R}} = \varinjlim \mathcal{D}_{\mathbb{R}}^{l} = \cup_{l=1}^{\infty} \mathcal{D}_{\mathbb{R}}^{l}$, $\mathcal{D}_{\mathbb{R}}^{l} \hookrightarrow \mathcal{D}_{\mathbb{R}}^{l+1}$. Here $\mathcal{D}_{\mathbb{R}}^{l}$ is **finite-dimensional** real vector space. Given a test function $\varphi \in \mathcal{D}_{\mathbb{R}}$, there is l such that $\varphi \in \mathcal{D}_{\mathbb{R}}^{l}$, and thus φ has a **natural discretization**.

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- **Discrete** means that $J \in \mathcal{D}^{l}_{\mathbb{R}}$, and $Z^{\text{phys}}_{l}(J) = \int_{\mathcal{D}^{l}_{\mathbb{R}}} D(\varphi) e^{-\frac{1}{K_B T} E(\varphi, J)}$

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- The goal of the work is to understand the limit:

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- Our results include φ^4-theories. In this case, $E(\varphi, J)$ can be interpreted as a Landau-Ginzburg functional of a continuous Ising model (i.e. $\varphi \in \mathbb{R}$) with external magnetic field J.

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- Our results include φ^4-theories. In this case, $E(\varphi, J)$ can be interpreted as a Landau-Ginzburg functional of a continuous Ising model (i.e. $\varphi \in \mathbb{R}$) with external magnetic field J.

- If $J = 0$, then $E(\varphi, 0)$ is invariant under $\varphi \rightarrow -\varphi$. We show that the systems attached to discrete versions of $E(\varphi, 0)$ have spontaneous breaking symmetry when the temperature T is less than the critical temperature.

INTRODUCTION

$$E(\varphi,J)=\frac{\gamma}{2}\int\limits_{\mathbb{Q}_p^N}\varphi\left(x\right)\mathbf{W}\left(\partial,\delta\right)\varphi\left(x\right)d^Nx-\int\limits_{\mathbb{Q}_p^N}J\left(x\right)\varphi\left(x\right)d^Nx$$
$$+\frac{\alpha_2}{2}\int\limits_{\mathbb{Q}_p^N}\varphi^2\left(x\right)d^Nx+\frac{\alpha_4}{2}\int\limits_{\mathbb{Q}_p^N}\varphi^4\left(x\right)d^Nx,$$

$\mathbf{W}\left(\partial,\delta\right)\varphi\left(x\right)=\mathcal{F}^{-1}_{\kappa\rightarrow x}(A_{w_\delta}(\|\kappa\|)\mathcal{F}_{x\rightarrow\kappa}\varphi)$ is pseudodifferential operator, whose symbol has a singularity at the origin.

The operator $\int\limits_{\mathbb{Q}_p^N}\varphi\left(x\right)\mathbf{W}\left(\partial,\delta\right)\varphi\left(x\right)d^Nx$ is **non local**. Then $E(\varphi,J)$ is **a non local action**.

INTRODUCTION

An important example of a $\mathbf{W}(\partial, \delta)$ operator is the Taibleson-Vladimirov operator, which is defined as

$$\mathbf{D}^{\beta}\phi(x) = \frac{1-p^{\beta}}{1-p^{-\beta-N}} \int\limits_{\mathbb{Q}_p^N} \frac{\phi(x-y)-\phi(x)}{\|y\|_p^{\beta+N}} d^N y = \mathcal{F}^{-1}_{\kappa \to x}\left(\|\kappa\|_p^{\beta} \mathcal{F}_{x \to \kappa}\phi \right),$$

where $\beta > 0$ and $\phi \in \mathcal{D}\left(\mathbb{Q}_p^N\right)$.
If $N = \beta = 1$, the energy functional

$$S(\varphi) = C \iint\limits_{\mathbb{Q}_p \times \mathbb{Q}_p} \left\{ \frac{\varphi(x) - \varphi(y)}{|x-y|_p} \right\}^2 dxdy$$

appears in p-adic string theory.

Spokoiny, Boris L.: Quantum geometry of non-Archimedean particles and strings. Phys. Lett. B **208**(3-4), 401–406 (1988).
All the results presented in the article are valid if $\mathbb{Q}_p$ is replaced by any non-Archimedean local field.

Discretization of Energy Functionals

The W operators

Take $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$, and fix a function $w_\delta : \mathbb{Q}_p^N \rightarrow \mathbb{R}_+$ satisfying: (i) $w_\delta(y)$ is a radial i.e. $w_\delta(y) = w_\delta(\|y\|_p)$; (ii) there exist constants $C_0, C_1 > 0$ and $\delta > N$ such that

$$C_0 \|y\|_p^\delta \leq w_\delta(\|y\|_p) \leq C_1 \|y\|_p^\delta \text{, for } y \in \mathbb{Q}_p^N.$$

We now define the operator

$$\mathbf{W}_\delta \varphi(x) = \int_{\mathbb{Q}_p^N} \frac{\varphi(x-y) - \varphi(x)}{w_\delta(\|y\|_p)} d^N y, \text{ for } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^N\right).$$

The operator $\mathbf{W}_\delta$ is pseudodifferential, more precisely, if

$$A_{w_\delta}(\kappa) := \int_{\mathbb{Q}_p^N} \frac{1 - \chi_p(y \cdot \kappa)}{w_\delta(\|y\|_p)} d^N y,$$

then

$$\mathbf{W}_\delta \varphi(x) = -\mathcal{F}^{-1}_{\kappa \rightarrow x}\left[A_{w_\delta}(\kappa) \mathcal{F}_{x \rightarrow \kappa} \varphi\right] =: -\mathbf{W}(\partial, \delta) \varphi(x), \text{ for } \varphi \in \mathcal{D}\left(\mathbb{Q}_p^N\right).$$

Discretization of Energy Functionals

- A discretization is obtained by considering truncations of p-adic numbers of the form $a_{-l}p^{-l} + a_{-l+1}p^{-l+1} + \ldots + a_0 + \ldots + a_{l-1}p^{l-1}$, for some $l \geq 1$, i.e. elements from $G_l := p^{-l}\mathbb{Z}_p^N / p^l\mathbb{Z}_p^N$.

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- We denote by $\mathcal{D}_{\mathbb{R}}^l(\mathbb{Q}_p^N) := \mathcal{D}_{\mathbb{R}}^l$ the $\mathbb{R}$-vector space of all test functions of the form $\varphi(x)=\sum_{\mathbf{i}\in G_l}\varphi(\mathbf{i})\,\Omega\left(p^l\left\|x-\mathbf{i}\right\|_p\right)$, $\varphi(\mathbf{i})\in\mathbb{R}$, where $\mathbf{i}$ runs through a fixed system of representatives of G_l, and $\Omega\left(p^l\left\|x-\mathbf{i}\right\|_p\right)$ is the characteristic function of the ball $\mathbf{i}+p^l\mathbb{Z}_p^N$.

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- Notice that φ is supported on $p^{-l}\mathbb{Z}_p^N$ and that $\mathcal{D}_{\mathbb{R}}^l$ is a finite dimensional vector space spanned by the basis$\left\{\Omega\left(p^l\left\|x-\mathbf{i}\right\|_p\right)\right\}_{\mathbf{i}\in G_l}$. We identify $\varphi\in\mathcal{D}_{\mathbb{R}}^l$ with the column vector $\left[\varphi(\mathbf{i})\right]_{\mathbf{i}\in G_l}$.

Discretization of Energy Functionals

- If m is positive integer then $\varphi^{m}(x) = \left\{ \sum_{\mathbf{i} \in G_l} \varphi(\mathbf{i}) \Omega\left(p^{l} \|x - \mathbf{i}\|_{p}\right) \right\}^{m} = \sum_{\mathbf{i} \in G_l} \varphi^{m}(\mathbf{i}) \Omega\left(p^{l} \|x - \mathbf{i}\|_{p}\right)$.

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- The functional $E_m^l(\varphi) := \int_{\mathbb{Q}_p^N} \varphi^m(x)\, d^N x$ for $m \in \mathbb{N} \smallsetminus \{0\}$, $\varphi \in \mathcal{D}_{\mathbb{R}}^l$, discretizes as $E_m^l(\varphi) = p^{-lN} \sum_{\mathbf{i}\in G_l} \varphi^m(\mathbf{i})$.

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- $E_{0}(\varphi) := \frac{\gamma}{4}\iint_{\mathbb{Q}_{p}^{N}\times\mathbb{Q}_{p}^{N}}\frac{\left\{\varphi(x)-\varphi(y)\right\}^{2}}{w_{\delta}\left(\left\|x-y\right\|_{p}\right)}d^{N}xd^{N}y + \frac{\alpha_{2}}{2}\int_{\mathbb{Q}_{p}^{N}}\varphi^{2}\left(x\right)d^{N}x \geq 0.$

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- If m is positive integer then $\varphi^{m}(x) = \left\{ \sum_{\mathbf{i}\in G_l} \varphi(\mathbf{i})\,\Omega\left(p^{l}\|x-\mathbf{i}\|_{p}\right)\right\}^{m} = \sum_{\mathbf{i}\in G_l} \varphi^{m}(\mathbf{i})\,\Omega\left(p^{l}\|x-\mathbf{i}\|_{p}\right)$.

- The functional $E'_{m}(\varphi) := \int_{\mathbb{Q}_p^N} \varphi^{m}(x)\, d^N x$ for $m \in \mathbb{N} \smallsetminus \{0\}$, $\varphi \in \mathcal{D}^{l}_{\mathbb{R}}$, discretizes as $E'_{m}(\varphi) = p^{-lN} \sum_{\mathbf{i}\in G_l} \varphi^{m}(\mathbf{i})$.

- $E_0(\varphi) := \frac{\gamma}{4} \iint_{\mathbb{Q}_p^N\times\mathbb{Q}_p^N} \frac{\{\varphi(x)-\varphi(y)\}^2}{w_{\delta}\left(\|x-y\|_p\right)} d^N x d^N y + \frac{\alpha_2}{2} \int_{\mathbb{Q}_p^N} \varphi^{2}(x)\, d^N x \geq 0.$

- The restriction of E_0 to $\mathcal{D}^{l}_{\mathbb{R}}$ (denoted as $E_0^{(l)}$) provides a natural discretization of E_0.

Discretization of Energy Functionals

We set $U(l) := U = [U_{\mathbf{i},\mathbf{j}}(l)]_{\mathbf{i},\mathbf{j}\in G_l}$, where

$$U_{\mathbf{i},\mathbf{j}}(l) := \left(\frac{\gamma}{2} d(l, w_\delta) + \frac{\alpha_2}{2}\right)\delta_{\mathbf{i},\mathbf{j}} - \frac{\gamma}{2} A_{\mathbf{i},\mathbf{j}}(l),$$

$$d(l, w_\delta) := \int_{\mathbb{Q}_p^N \setminus B_{-l}^N} \frac{d^N y}{w_\delta(\|y\|_p)} \text{ and } A_{\mathbf{i},\mathbf{j}}(l) := \begin{cases} \frac{p^{-lN}}{w_\delta(\|\mathbf{i}-\mathbf{j}\|_p)} & \text{if } \mathbf{i} \neq \mathbf{j} \\ 0 & \text{if } \mathbf{i} = \mathbf{j}. \end{cases}.$$

Lemma

With the above notation the following formula holds true:

$$E_0^{(l)}(\varphi) = [\varphi(\mathbf{i})]^T_{\mathbf{i}\in G_l} p^{-lN} U(l) [\varphi(\mathbf{i})]_{\mathbf{i}\in G_l} = \sum_{\mathbf{i},\mathbf{j}\in G_l} p^{-lN} U_{\mathbf{i},\mathbf{j}}(l)\varphi(\mathbf{i})\varphi(\mathbf{j}) \geq 0,$$

for $\varphi \in \mathcal{D}^l_{\mathbb{R}}$, where U is a symmetric, positive definite matrix. Consequently $p^{-lN}U(l)$ is a diagonalizable and invertible matrix.

Lizorkin spaces of second kind

The p-adic Lizorkin space of second kind;

$$\mathcal{L} := \mathcal{L}(\mathbb{Q}_p^N) = \left\{ \varphi \in \mathcal{D}(\mathbb{Q}_p^N); \int_{\mathbb{Q}_p^N} \varphi(x)\, d^N x = 0 \right\}$$

$\mathcal{L}_{\mathbb{R}} := \mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p^N) = \mathcal{L}(\mathbb{Q}_p^N) \cap \mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$, the real version.

$$\mathcal{F}\mathcal{L} := \mathcal{F}\mathcal{L}(\mathbb{Q}_p^N) = \left\{ \widehat{\varphi} \in \mathcal{D}(\mathbb{Q}_p^N); \widehat{\varphi}(0) = 0 \right\},$$

The Fourier transform gives rise to an isomorphism of $\mathbb{C}$-vector spaces from $\mathcal{L}$ into $\mathcal{F}\mathcal{L}$.

The topological dual $\mathcal{L}' := \mathcal{L}'(\mathbb{Q}_p^N)$ of the space $\mathcal{L}$ is called *the p-adic Lizorkin space of distributions of second kind.* The real version is denoted as $\mathcal{L}'_{\mathbb{R}} := \mathcal{L}'_{\mathbb{R}}(\mathbb{Q}_p^N)$.

Lizorkin spaces of second kind

The discrete p-adic Lizorkin space of second kind:

$$\mathcal{L}^{l} := \mathcal{L}^{l}(\mathbb{Q}_p^N)$$
$$= \left\{ \varphi(x) = \sum_{\mathbf{i} \in G_l} \varphi(\mathbf{i})\, \Omega\left(p^{l}\left\|x - \mathbf{i}\right\|_p\right), \varphi(\mathbf{i}) \in \mathbb{C};\, p^{-lN} \sum_{\mathbf{i} \in G_l} \varphi(\mathbf{i}) = 0 \right\},$$

$l \in \mathbb{N} \smallsetminus \{0\}$. The real version $\mathcal{L}_{\mathbb{R}}^{l} := \mathcal{L}_{\mathbb{R}}^{l}(\mathbb{Q}_p^N) = \mathcal{L}^{l} \cap \mathcal{D}_{\mathbb{R}}^{l}$.

$$\mathcal{F}\mathcal{L}^{l} := \mathcal{F}\mathcal{L}^{l}(\mathbb{Q}_p^N) =$$
$$\left\{ \widehat{\varphi}(\kappa) = \sum_{\mathbf{i} \in G_l} \widehat{\varphi}(\mathbf{i})\, \Omega\left(p^{l}\left\|\kappa - \mathbf{i}\right\|_p\right), \widehat{\varphi}(\mathbf{i}) \in \mathbb{C};\, \widehat{\varphi}(\mathbf{0}) = 0 \right\},$$

The Fourier transform $\mathcal{F} : \mathcal{L}^{l} \rightarrow \mathcal{F}\mathcal{L}^{l}$ is an automorphism of $\mathbb{C}$-vector spaces.

Energy functionals in the momenta space

$$E_0(\varphi) = \frac{\gamma}{2} \int_{\mathbb{Q}_p^N} \varphi(x) \left(-\mathbf{W}_\delta\right) \varphi(x)\, d^N x + \frac{\alpha_2}{2} \int_{\mathbb{Q}_p^N} \varphi^2(x)\, d^N x$$

$$= \frac{\gamma}{2} \int_{\mathbb{Q}_p^N} \varphi(x)\, \mathbf{W}(\partial, \delta)\, \varphi(x)\, d^N x + \frac{\alpha_2}{2} \int_{\mathbb{Q}_p^N} \varphi^2(x)\, d^N x$$

$$= \int_{\mathbb{Q}_p^N} \left(\frac{\gamma}{2} A_{w_\delta}(\|\kappa\|_p) + \frac{\alpha_2}{2} \right) |\widehat{\varphi}(\kappa)|^2\, d^N \kappa.$$

For $\varphi \in \mathcal{D}_{\mathbb{R}}^l$, we have

$$E_0(\varphi) = p^{-lN} \sum_{\mathbf{j} \in G_l \smallsetminus \{\mathbf{0}\}} \left(\frac{\gamma}{2} A_{w_\delta}(\|\mathbf{j}\|_p) + \frac{\alpha_2}{2} \right) |\widehat{\varphi}(\mathbf{j})|^2$$

$$+ |\widehat{\varphi}(\mathbf{0})|^2 \left\{ \int_{p^l \mathbb{Z}_p^N} \left(\frac{\gamma}{2} A_{w_\delta}(\|z\|_p) + \frac{\alpha_2}{2} \right) d^N z \right\},$$

where $\widehat{\varphi}(\mathbf{j}) = \widehat{\varphi}_1(\mathbf{j}) + \sqrt{-1}\,\widehat{\varphi}_2(\mathbf{j}) \in \mathbb{C}$.

Energy functionals in the momenta space

We use the alternative notation $\widehat{\varphi_1}(\mathbf{j}) = \mathrm{Re}\left(\widehat{\varphi}(\mathbf{j})\right)$, $\widehat{\varphi}_2(\mathbf{j}) = \mathrm{Im}\left(\widehat{\varphi}(\mathbf{j})\right)$. Notice that

$$\mathcal{FL}_{\mathbb{R}}^{l} =$$

$$\left\{\widehat{\varphi}(\kappa) = \sum_{\mathbf{i}\in G_l} \widehat{\varphi}(\mathbf{i})\,\Omega\left(p^{l}\left\|\kappa - \mathbf{i}\right\|_p\right), \widehat{\varphi}(\mathbf{i}) \in \mathbb{C}; \widehat{\varphi}(0) = 0,\ \overline{\widehat{\varphi}(\kappa)} = \widehat{\varphi}(-\kappa)\right\}$$

and that the condition $\overline{\widehat{\varphi}(\kappa)} = \widehat{\varphi}(-\kappa)$ implies that $\widehat{\varphi}_1(-\mathbf{i}) = \widehat{\varphi}_1(\mathbf{i})$ and $\widehat{\varphi}_2(-\mathbf{i}) = -\widehat{\varphi}_2(\mathbf{i})$ for any $\mathbf{i} \in G_l$. This implies that $\mathcal{FL}_{\mathbb{R}}^{l}$ is $\mathbb{R}$-vector space of dimension $\# G_l - 1$.

$$E_0^{(l)}(\varphi) = 2p^{-lN} \sum_{r\in\{1,2\}} \sum_{\mathbf{j}\in G_l^{+}} \left(\frac{\gamma}{2} A_{w_\delta}(\|j\|_p) + \frac{\alpha_2}{2}\right) \widehat{\varphi_r}^{2}(\mathbf{j}).$$

We now define the diagonal matrix $B^{(r)} = \left[B^{(r)}_{\mathbf{i},\mathbf{j}} \right]_{\mathbf{i},\mathbf{j} \in G_l^+}$, $r = 1, 2$, where

$$B^{(r)}_{\mathbf{i},\mathbf{j}} := \begin{cases} \frac{\gamma}{2} A_{w_\delta}(\|\mathbf{j}\|_p) + \frac{\alpha_2}{2} & \text{if} \quad \mathbf{i} = \mathbf{j} \\ 0 & \text{if} \quad \mathbf{i} \neq \mathbf{j}. \end{cases}$$

Notice that $B^{(1)}_{\mathbf{i},\mathbf{j}} = B^{(2)}_{\mathbf{i},\mathbf{j}}$.

We set

$$B(l) := B(l, \delta, \gamma, \alpha_2) = \begin{bmatrix} B^{(1)} & \mathbf{0} \\ \mathbf{0} & B^{(2)}. \end{bmatrix}$$

The matrix $B = [B_{\mathbf{i},\mathbf{j}}]$ is a diagonal of size $2\left(\#G_l^+\right) \times 2\left(\#G_l^+\right)$. In addition, the indices $\mathbf{i}, \mathbf{j}$ run through two disjoint copies of G_l^+.

Energy functionals in the momenta space

Lemma

Assume that $\alpha_2 > 0$. With the above notation the following formula holds true:

$$E_0^{(l)}(\varphi) := E_0^{(l)}\left(\widehat{\varphi_1}(\mathbf{j}), \widehat{\varphi_2}(\mathbf{j}); \mathbf{j} \in G_l^+\right)$$

$$= \begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix}^T 2p^{-lN} B(l) \begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix} \geq 0,$$

for $\varphi \in \mathcal{L}_{\mathbb{R}}^l \simeq \mathcal{F}\mathcal{L}_{\mathbb{R}}^l \simeq \mathbb{R}^{(\# G_l\ -1)}$, where $2p^{-lN}B(l)$ is a diagonal, positive definite, invertible matrix.

Gaussian measures

Gaussian measures

$$\mathcal{Z} := \mathcal{Z}(\delta, \gamma, \alpha_2) = \int\limits_{\mathcal{FL}_{\mathbb{R}}(\mathbb{Q}_p^N)} D(\varphi) e^{-E_0(\varphi)}.$$

We set

$$\mathcal{Z}^{(l)} = \mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2) = \int\limits_{\mathcal{FL}_{\mathbb{R}}^{l}(\mathbb{Q}_p^N)} D_l(\varphi)\, e^{-E_0(\varphi)}$$

$$=: \mathcal{N}_l \int\limits_{\mathbb{R}^{\left(p^{2lN}-1\right)}} \exp\left(- \begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix}^T 2p^{-lN} B(l) \begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix} \right)$$

$$\times \prod_{\mathbf{i}\in G_l^+} d\widehat{\varphi_1}(\mathbf{i})\, d\widehat{\varphi_2}(\mathbf{i}),$$

where $\mathcal{N}_l$ is a normalization constant, and $\prod_{\mathbf{i}\in G_l^+} d\widehat{\varphi_1}(\mathbf{i})\, d\widehat{\varphi_2}(\mathbf{i})$ is the Lebesgue measure of $\mathbb{R}^{\left(p^{2lN}-1\right)}$.

$\mathcal{Z}^{(l)}$ is a Gaussian integral, then

$$\mathcal{Z}^{(l)} = \mathcal{N}_l \frac{(2\pi)^{\frac{\left(p^{2lN}-1\right)}{2}}}{\sqrt{\det 4p^{-lN}B(l)}} = \mathcal{N}_l \left(\frac{\pi}{2}\right)^{\frac{\left(p^{2lN}-1\right)}{2}} \frac{p^{\frac{lN\left(p^{2lN}-1\right)}{2}}}{\sqrt{\det B}}.$$

We set

$$\mathcal{N}_l = \frac{\left(\frac{2}{\pi}\right)^{\frac{\left(p^{2lN}-1\right)}{2}} \sqrt{\det B}}{p^{\frac{lN\left(p^{2lN}-1\right)}{2}}}.$$

We define the following family of Gaussian measures:

$$d\mathbb{P}_l\left(\left[\begin{array}{c}[\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+}\\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+}\end{array}\right]\right)$$

$$=\mathcal{N}_l\exp(-\left[\begin{array}{c}[\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+}\\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+}\end{array}\right]^T 2p^{-lN}B(l)\left[\begin{array}{c}[\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+}\\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G+_l}\end{array}\right])$$

$$\times\prod_{\mathbf{i}\in G_l^+} d\widehat{\varphi_1}(\mathbf{i})\, d\widehat{\varphi_2}(\mathbf{i})$$

in $\mathcal{FL}_{\mathbb{R}}^l \simeq \mathbb{R}^{\left(p^{2lN}-1\right)}$, for $l\in\mathbb{N}\smallsetminus\{0\}$.

Gaussian measures

Thus for any Borel subset A of $\mathbb{R}^{\left(p^{2lN}-1\right)} \simeq \mathcal{FL}_{\mathbb{R}}^{l}$ and any continuous and bounded function $f : \mathcal{FL}_{\mathbb{R}}^{l} \rightarrow \mathbb{R}$ the integral

$$\int_A f\left(\begin{bmatrix} \left[\widehat{\varphi_1}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+} \\ \left[\widehat{\varphi_2}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+} \end{bmatrix}\right) d\mathbb{P}_l\left(\begin{bmatrix} \left[\widehat{\varphi_1}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+} \\ \left[\widehat{\varphi_2}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+} \end{bmatrix}\right) =: \int_A f\left(\widehat{\varphi}\right) d\mathbb{P}_l\left(\widehat{\varphi}\right)$$

is well-defined.

Lemma

There exists a probability measure space $(X, \mathcal{F}, \mathbb{P})$ *and random variables*

$$\begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix}, \text{ for } l \in \mathbb{N} \smallsetminus \{0\},$$

such that $\mathbb{P}_l$ *is the joint probability distribution of* $\begin{bmatrix} [\widehat{\varphi_1}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \\ [\widehat{\varphi_2}(\mathbf{j})]_{\mathbf{j}\in G_l^+} \end{bmatrix}$. *The space* $(X, \mathcal{F}, \mathbb{P})$ *is unique up to isomorphisms of probability measure spaces. Furthermore, for any bounded continuous function* f *supported in* $\mathcal{F}\mathcal{L}_{\mathbb{R}}^l$*, we have*

$$\int_{\mathcal{F}\mathcal{L}_{\mathbb{R}}^l} f(\widehat{\varphi})\, d\mathbb{P}_l(\widehat{\varphi}) = \int_{\mathcal{F}\mathcal{L}_{\mathbb{R}}^l} f(\widehat{\varphi})\, d\mathbb{P}(\widehat{\varphi}).$$

Gaussian measures

For $\delta > N$, γ, $\alpha_2 > 0$, we define the operator

$$\begin{array}{ccc} \mathcal{D}\left(\mathbb{Q}_p^N\right) & \rightarrow & L^2\left(\mathbb{Q}_p^N\right) \\ \varphi & \rightarrow & \left(\frac{\gamma}{2}\mathbf{W}\left(\partial,\delta\right)+\frac{\alpha_2}{2}\right)^{-1}\varphi, \end{array}$$

where $\left(\frac{\gamma}{2}\mathbf{W}\left(\partial,\delta\right)+\frac{\alpha_2}{2}\right)^{-1}\varphi\left(x\right) := \mathcal{F}_{\kappa\rightarrow x}^{-1}\left(\frac{\mathcal{F}_{x\rightarrow\kappa}\varphi}{\frac{\gamma}{2}A_{w_\delta}(\|\kappa\|_p)+\frac{\alpha_2}{2}}\right)$.

Gaussian measures

We define the distribution

$$G(x) := G(x;\delta,\gamma,\alpha_2) = \mathcal{F}^{-1}_{\kappa\to x}\left(\frac{1}{\frac{\gamma}{2}A_{w_\delta}(\|\kappa\|_p) + \frac{\alpha_2}{2}}\right) \in \mathcal{D}'\left(\mathbb{Q}_p^N\right).$$

By using the fact that $\frac{1}{\frac{\gamma}{2}A_{w_\delta}(\|\kappa\|_p)+\frac{\alpha_2}{2}}$ is radial and $(\mathcal{F}(\mathcal{F}\varphi))(\kappa) = \varphi(-\kappa)$ one verifies that

$$G(x) \in \mathcal{D}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right).$$

Gaussian measures

$$\mathbb{B}:\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)\times\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \rightarrow \mathbb{R}$$

$$(\varphi,\theta) \rightarrow \left\langle \varphi,\left(\frac{\gamma}{2}\mathbf{W}\left(\partial,\delta\right)+\frac{\alpha_2}{2}\right)^{-1}\theta\right\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2\left(\mathbb{Q}_p^N\right)$.

Lemma

$\mathbb{B}$ is a positive, continuous bilinear form from $\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)\times\mathcal{D}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ into $\mathbb{R}$.

Lemma

For $\varphi\in\mathcal{L}_{\mathbb{R}}^{l}\simeq\mathcal{F}\mathcal{L}_{\mathbb{R}}^{l}$,

$$\mathbb{B}_l(\varphi,\varphi):=\mathbb{B}(\varphi,\varphi)=\begin{bmatrix}\left[\widehat{\varphi_1}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+}\\ \left[\widehat{\varphi_2}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+}\end{bmatrix}^T 2p^{-lN}B^{-1}(l)\begin{bmatrix}\left[\widehat{\varphi_1}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+}\\ \left[\widehat{\varphi_2}\left(\mathbf{j}\right)\right]_{\mathbf{j}\in G_l^+}\end{bmatrix}.$$

Corollary

The collection $\{\mathbb{B}_{\mathcal{Y}}; \mathcal{Y}$ *finite dimensional subspace of* $\mathcal{L}_{\mathbb{R}}\}$ *is completely determined by the collection* $\{\mathbb{B}_l; l \in \mathbb{N} \smallsetminus \{0\}\}$*. In the sense that given any* $\mathbb{B}_{\mathcal{Y}}$ *there is an integer* l *and a subset* $J \subset G_l^+$*, the case* $J = \emptyset$ *is included, such that* $\mathbb{B}_{\mathcal{Y}} = \mathbb{B}_l \mid_{\{\widehat{\varphi}_1(\mathbf{j})=0, \widehat{\varphi}_2(\mathbf{j})=0; \mathbf{j} \notin J\}}$.

The spaces

$$\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \hookrightarrow L^2_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \hookrightarrow \mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$$

form a Gel'fand triple, that is, $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ is a nuclear space which is densely and continuously embedded in $L^2_{\mathbb{R}}$ and $\|g\|_2^2 = \langle g, g\rangle$ for $g \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$.

The mapping

$$\begin{array}{cccc} \mathcal{C}: & \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) & \rightarrow & \mathbb{C} \\ & f & \rightarrow & e^{-\frac{1}{2}\mathbb{B}(f,f)} \end{array}$$

defines a characteristic functional, i.e. $\mathcal{C}$ is continuous, positive definite and $\mathcal{C}(0) = 1$.

Gaussian measures in the non-Archimedean framework

By the Bochner-Minlos theorem, there exists a probability measure $\mathbb{P} := \mathbb{P}(\delta, \gamma, \alpha_2)$ called *the canonical Gaussian measure* on $\left(\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right), \mathcal{B}\right)$, given by its characteristic functional as

$$\int\limits_{\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{\sqrt{-1}\langle W,f\rangle} d\mathbb{P}(W) = e^{-\frac{1}{2}\mathbb{B}(f,f)}, \;\; f \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right). \tag{2}$$

Gaussian measures in the non-Archimedean framework

The measure $\mathbb{P}$ is uniquely determined by the family of Gaussian measures

$$\left\{\mathbb{P}_{\mathcal{Y}};\mathcal{Y}\subset\mathcal{L}_{\mathbb{R}},\text{ finite dimensional space}\right\},$$

where

$$\mathbb{P}_{\mathcal{Y}}\left(A\right)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{A}e^{-\frac{1}{2}\mathbb{B}(\psi,\psi)}d\psi,$$

if $\mathcal{Y}$ has dimension n.

Equivalently, $\mathbb{P}$ is uniquely determined by the family of bilinear forms

$$\left\{\mathbb{B}_{\mathcal{Y}};\mathcal{Y}\subset\mathcal{L}_{\mathbb{R}},\text{ finite dimensional space}\right\},$$

where $\mathbb{B}_{\mathcal{Y}}$ denotes the restriction of the scalar product to $\mathbb{B}$ to $\mathcal{Y}$.

Equivalently, $\mathbb{P}$ is uniquely determined by the family of bilinear forms $\left\{\mathbb{B}_{l};l\in\mathbb{N}\smallsetminus\{0\}\right\}$.

Gaussian measures in the non-Archimedean framework

Theorem

Assume that $\delta > N$, $\gamma > 0$, $\alpha_2 > 0$. (i) The cylinder probability measure $\mathbb{P} = \mathbb{P}(\delta, \gamma, \alpha_2)$ is uniquely determined by the sequence $\mathbb{P}_l = \mathbb{P}_l(\delta, \gamma, \alpha_2)$, $l \in \mathbb{N} \smallsetminus \{0\}$, of Gaussian measures. (ii) Let $f : \mathcal{FL}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right) \to \mathbb{R}$ be a continuous and bounded function. Then

$$\lim_{l \to \infty} \int_{\mathcal{FL}_{\mathbb{R}}^{l}\left(\mathbb{Q}_p^N\right)} f\left(\widehat{\varphi}\right) d\mathbb{P}_l\left(\widehat{\varphi}\right) = \int_{\mathcal{FL}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} f\left(\widehat{\varphi}\right) d\mathbb{P}\left(\widehat{\varphi}\right).$$

The sequence of discretizations $E_0^{(l)}$ determines a probability measure P in $L_{\mathbb{R}}'\left(\mathbb{Q}_p^N\right)$.

Partition functions and generating functionals

Partition functions

- We consider interactions of the form:
$\mathcal{P}(X) = a_3X^3 + a_4X^4 + \ldots + a_{2k}X^{2D} \in \mathbb{R}[X]$, with $D \geq 2$, satisfying $\mathcal{P}(\alpha) \geq 0$ for any $\alpha \in \mathbb{R}$. Which implies that $\exp\left(-\frac{\alpha_4}{2}\int \mathcal{P}(\varphi) d^N x\right) \leq 1$.

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- Each of these theories corresponds to a thermally fluctuating field which is defined by means of a functional integral representation of the partition function.

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- Each of these theories corresponds to a thermally fluctuating field which is defined by means of a functional integral representation of the partition function.

- All the thermodynamic quantities and correlation functions of the system can be obtained by functional differentiation from a generating functional as in the classical case.

Partition functions

We assume that $\varphi \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ represents a field that performs thermal fluctuations. We also assume that in the normal phase the expectation value of the field φ is zero. Then the fluctuations take place around zero.

The size of these fluctuations is controlled by the energy functional:

$$E(\varphi) := E_0(\varphi) + E_{\text{int}}(\varphi),$$

where

$$E_{\text{int}}(\varphi) := \frac{\alpha_4}{4} \int\limits_{\mathbb{Q}_p^N} \mathcal{P}\left(\varphi\left(x\right)\right) d^N x, \ \ \alpha_4 \geq 0,$$

corresponds to the interaction energy.

Partition functions

Definition

Assume that $\delta > N$, and γ, $\alpha_2 > 0$. The free-partition function is defined as

$$\mathcal{Z}_0 = \mathcal{Z}_0(\delta, \gamma, \alpha_2) = \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} d\mathbb{P}\left(\varphi\right).$$

The discrete free-partition function is defined as

$$\mathcal{Z}_0^{(l)} = \mathcal{Z}_0^{(l)}(\delta, \gamma, \alpha_2) = \int\limits_{\mathcal{L}_{\mathbb{R}}^{l}\left(\mathbb{Q}_p^N\right)} d\mathbb{P}_l\left(\varphi\right)$$

for $l \in \mathbb{N} \smallsetminus \{0\}$.

$\lim_{l\to\infty} \mathcal{Z}_0^{(l)} = \mathcal{Z}_0$. Notice that the term $e^{-E_0(\varphi)}$ is used to construct the measure $\mathbb{P}\left(\varphi\right)$.

Partition functions

Definition

Assume that $\delta > N$, and γ, α_2, $\alpha_4 > 0$. The partition function is defined as

$$\mathcal{Z} = \mathcal{Z}(\delta, \gamma, \alpha_2, \alpha_4) = \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}\left(\varphi\right).$$

The discrete partition functions are defined as

$$\mathcal{Z}^{(l)} = \mathcal{Z}^{(l)}(\delta, \gamma, \alpha_2, \alpha_4) = \int\limits_{\mathcal{L}_{\mathbb{R}}^{l}\left(\mathbb{Q}_p^N\right)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}_l\left(\varphi\right),$$

for $l \in \mathbb{N} \smallsetminus \{0\}$.

From a mathematical perspective a $\mathcal{P}(\varphi)$-theory is given by a cylinder probability measure of the form

$$\frac{1_{\mathcal{L}_{\mathbb{R}}}(\varphi)\, e^{-E_{\text{int}}(\varphi)} d\mathbb{P}}{\int_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{-E_{\text{int}}(\varphi)} d\mathbb{P}} = \frac{1_{\mathcal{L}_{\mathbb{R}}}(\varphi)\, e^{-E_{\text{int}}(\varphi)} d\mathbb{P}}{\mathcal{Z}} \tag{3}$$

in the space of fields $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$. It is important to mention that we do not require the Wick regularization operation in $e^{-E_{\text{int}}(\varphi)}$ because we are restricting the fields to be test functions

Partition functions

The m-point correlation functions of a field $\varphi \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ are defined as

$$G^{(m)}\left(x_1, \ldots, x_m\right) = \frac{1}{\mathcal{Z}} \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} \left(\prod_{i=1}^{m} \varphi\left(x_i\right)\right) e^{-E_{\text{int}}(\varphi)} d\mathbb{P}.$$

The discrete m-point correlation functions of a field $\varphi \in \mathcal{L}_{\mathbb{R}}^{l}\left(\mathbb{Q}_p^N\right)$ are defined as

$$G_l^{(m)}\left(x_1, \ldots, x_m\right) = \frac{1}{\mathcal{Z}^{(l)}} \int\limits_{\mathcal{L}_{\mathbb{R}}^{l}\left(\mathbb{Q}_p^N\right)} \left(\prod_{i=1}^{m} \varphi\left(x_i\right)\right) e^{-E_{\text{int}}(\varphi)} d\mathbb{P}_l,$$

for $l \in \mathbb{N} \smallsetminus \{0\}$.

Generating functionals

We now introduce a current $J(x) \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ and add to the energy functional $E(\varphi)$ a linear interaction energy of this current with the field $\varphi(x)$,

$$E_{\text{source}}(\varphi, J) := -\int_{\mathbb{Q}_p^N} \varphi(x) J(x) d^N x,$$

in this way we get a new energy functional

$$E(\varphi, J) := E(\varphi) + E_{\text{source}}(\varphi, J).$$

Notice that $E_{\text{source}}(\varphi, J) = -\langle \varphi, J \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{Q}_p^N)$.

Generating functionals

Definition

Assume that $\delta > N$, and γ, α_2, $\alpha_4 > 0$. The partition function corresponding to the energy functional $E(\varphi, J)$ is defined as

$$\mathcal{Z}(J;\delta,\gamma,\alpha_2,\alpha_4) := \mathcal{Z}(J) = \frac{1}{\mathcal{Z}_0} \int\limits_{\mathcal{L}_{\mathbb{R}}(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)+\langle\varphi,J\rangle}\, d\mathbb{P},$$

and the discrete versions

$$\mathcal{Z}^{(l)}(J;\delta,\gamma,\alpha_2,\alpha_4) := \mathcal{Z}^{(l)}(J) = \frac{1}{\mathcal{Z}_0^{(l)}} \int\limits_{\mathcal{L}_{\mathbb{R}}^{l}(\mathbb{Q}_p^N)} e^{-E_{\text{int}}(\varphi)+\langle\varphi,J\rangle}\, d\mathbb{P}_l,$$

for $l \in \mathbb{N} \smallsetminus \{0\}$.

Generating functionals

Definition

For $\theta \in \mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$, the functional derivative $D_\theta \mathcal{Z}(J)$ of $\mathcal{Z}(J)$ is defined as

$$D_\theta \mathcal{Z}(J) = \lim_{\epsilon \to 0} \frac{\mathcal{Z}(J+\epsilon\theta) - \mathcal{Z}(J)}{\epsilon} = \left[\frac{d}{d\epsilon}\mathcal{Z}(J+\epsilon\theta)\right]_{\epsilon=0}.$$

Generating functionals

Lemma

Let $\theta_1,\ldots,\theta_m$ be test functions from $\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$. The functional derivative $D_{\theta_1}\cdots D_{\theta_m}\mathcal{Z}(J)$ exists, and the following formula holds true:

$$D_{\theta_1}\cdots D_{\theta_m}\mathcal{Z}(J)=\frac{1}{\mathcal{Z}_0}\int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{-E_{int}(\varphi)+\langle\varphi,J\rangle}\left(\prod_{i=1}^{m}\langle\varphi,\theta_i\rangle\right)d\mathbb{P}(\varphi).$$

Furthermore, the functional derivative $D_{\theta_1}\cdots D_{\theta_m}\mathcal{Z}(J)$ can be uniquely identified with the distribution from $\mathcal{L}'_{\mathbb{R}}\left(\left(\mathbb{Q}_p^N\right)^m\right)$:

$$\prod_{i=1}^{m}\theta_i(x_i)\to\frac{1}{\mathcal{Z}_0}\int\limits_{\mathbb{Q}_p^N\times\cdots\times\mathbb{Q}_p^N}\cdots\int\prod_{i=1}^{m}\theta_i(x_i)\times$$
$$\left\{\int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{-E_{int}(\varphi)+\langle\varphi,J\rangle}\prod_{i=1}^{m}\varphi(x_i)\,d\mathbb{P}(\varphi)\right\}\prod_{i=1}^{m}d^Nx_i.$$

Generating functionals

In an alternative way, one can define the functional derivative $\frac{\delta}{\delta J(y)}\mathcal{Z}(J)$ of $\mathcal{Z}(J)$ as the distribution from $\mathcal{L}'_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)$ satisfying

$$\int\limits_{\mathbb{Q}_p^N} \theta\left(y\right)\left(\frac{\delta}{\delta J\left(y\right)}\mathcal{Z}(J)\right)\left(y\right) d^N y = \left[\frac{d}{d\epsilon}\mathcal{Z}(J+\epsilon\theta)\right]_{\epsilon=0}.$$

Using this notation, we obtain that

$$\frac{\delta}{\delta J\left(x_1\right)}\cdots\frac{\delta}{\delta J\left(x_m\right)}\mathcal{Z}(J) = \frac{1}{\mathcal{Z}_0}\int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{-E_{\text{int}}(\varphi)+\langle\varphi,J\rangle}\left(\prod_{i=1}^{m}\varphi\left(x_i\right)\right) d\mathbb{P}(\varphi) \in \mathcal{L}'_{\mathbb{R}}\left(\left(\mathbb{Q}_p^N\right)^m\right).$$

Generating functionals

Proposition

The correlations functions $G^{(m)}(x_1, \ldots, x_m) \in \mathcal{L}'_{\mathbb{R}}\left(\left(\mathbb{Q}_p^N\right)^m\right)$ *are given by*

$$G^{(m)}(x_1, \ldots, x_m) = \frac{\mathcal{Z}_0}{\mathcal{Z}} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \mathcal{Z}(J) \mid_{J=0} .$$

Free-field theory

We set $\mathcal{Z}_0(J) := \mathcal{Z}(J; \delta, \gamma, \alpha_2, 0)$.

Proposition

$\mathcal{Z}_0(J) = \mathcal{N}_0' \exp\left\{\int_{\mathbb{Q}_p^N}\int_{\mathbb{Q}_p^N} J(x) G(\|x-y\|_p) J(y) d^N x \, d^N y\right\}$, where $\mathcal{N}_0'$ denotes a normalization constant

For $J \in \mathcal{L}_{\mathbb{R}}$, the equation

$$\left(\frac{\gamma}{2} W(\partial, \delta) + \frac{\alpha_2}{2}\right) \varphi_0 = J$$

has unique solution $\varphi_0 \in \mathcal{L}_{\mathbb{R}}$. Indeed, $\widehat{\varphi_0}(\kappa) = \frac{\widehat{J}(\kappa)}{\frac{\gamma}{2} A_{w_\delta}(\|\kappa\|_p) + \frac{\alpha_2}{2}}$ is a test function satisfying $\widehat{\varphi_0}(0) = 0$. Furthermore,

$$\varphi_0(x) = \mathcal{F}^{-1}_{\kappa \to x}\left(\frac{1}{\frac{\gamma}{2} A_{w_\delta}(\|\kappa\|_p) + \frac{\alpha_2}{2}}\right) * J(x) = G(\|x\|_p) * J(x) \text{ in } \mathcal{D}'_{\mathbb{R}}.$$

Free-field theory

Proof.

We now change variables in $\mathcal{Z}_0(J)$ as $\varphi = \varphi_0 + \varphi'$,

$$\mathcal{Z}_0(J) = \frac{1}{\mathcal{Z}_0} \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{\langle \varphi, J\rangle}\, d\mathbb{P} = \frac{e^{\langle \varphi_0, J\rangle}}{\mathcal{Z}_0} \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{\langle \varphi', J\rangle}\, d\mathbb{P}'\left(\varphi'\right)$$

$$= \left(\frac{1}{\mathcal{Z}_0} \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} e^{\left\langle \varphi', \left(\frac{\gamma}{2} W(\partial,\delta) + \frac{\alpha_2}{2}\right)\varphi_0 \right\rangle}\, d\mathbb{P}'\left(\varphi'\right) \right) e^{\langle G * J, J\rangle}$$

$$= \mathcal{N}_0' e^{\langle G*J,J\rangle} = \mathcal{N}_0' \exp\left\{ \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} J(x) G(\|x-y\|_p) J(y) d^N x\, d^N y \right\}.$$

□

Free-field theory

The correlation functions $G_0^{(m)}(x_1, \ldots, x_m)$ of the free-field theory are obtained from the functional derivatives of $\mathcal{Z}_0(J)$ at $J = 0$:

Proposition

$$G_0^{(m)}(x_1, \ldots, x_m) = \left[\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \mathcal{Z}_0(J) \right]_{J=0}$$

$$= \mathcal{N}_0' \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_m)} \exp \left\{ \int_{\mathbb{Q}_p^N} \int_{\mathbb{Q}_p^N} J(x) G(\|x - y\|_p) J(y) d^N x \, d^N y \right\} |_J$$

The existence of a convergent power series expansion for $Z(J)$ (*the perturbation expansion*) in the coupling parameter α_4 follows from the fact that $\exp\left(-E_{\text{int}}(\varphi) + \langle \varphi, J\rangle\right)$ is an integrable function, by using the dominated convergence theorem, more precisely, we have

$$\mathcal{Z}(J) = \mathcal{Z}_0(J) +$$

$$\frac{1}{\mathcal{Z}_0}\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{-\alpha_4}{4}\right)^m \int\limits_{\mathcal{L}_{\mathbb{R}}\left(\mathbb{Q}_p^N\right)} \left\{ \int\limits_{\left(\mathbb{Q}_p^N\right)^m} \left(\prod_{i=1}^{m}\varphi^4(z_i)\right) e^{\langle\varphi, J\rangle} \prod_{i=1}^{m} d^N z_i \right\} d\mathbb{P}(\varphi)$$

$$=: \mathcal{Z}_0(J) + \sum_{m=1}^{\infty}\mathcal{Z}_m(J).$$

Theorem

Assume that $\mathcal{P}(\varphi)=\varphi^4$. The n-point correlation function of the field φ admits the following convergent power series in the coupling constant:

$$G^{(n)}(x_1,\ldots,x_n)=\frac{\mathcal{Z}_0}{\mathcal{Z}}\left\{G_0^{(n)}(x_1,\ldots,x_n)+\sum_{m=1}^{\infty}G_m^{(n)}(x_1,\ldots,x_n)\right\},$$

where

$$G_m^{(n)}(x_1,\ldots,x_n):=\frac{1}{m!}\left(\frac{-\alpha_4}{4}\right)^m\times$$

$$\int_{\left(\mathbb{Q}_p^N\right)^m}G_0^{(n+4m)}(z_1,z_1,z_1,z_1,\ldots,z_m,z_m,z_m,z_m,x_1,\ldots,x_n)\prod_{i=1}^{m}d^Nz_i.$$

Ginzburg-Landau phenomenology

A non-Archimedean Ginzburg-Landau free energy:

$$E(\varphi, J) : E(\varphi, J; \delta, \gamma, \alpha_2, \alpha_4) = \frac{\gamma(T)}{2} \iint\limits_{\mathbb{Q}_p^N \times \mathbb{Q}_p^N} \frac{\{\varphi(x) - \varphi(y)\}^2}{w_\delta\left(\|x - y\|_p\right)} d^N x d^N y$$

$$+ \frac{\alpha_2(T)}{2} \int\limits_{\mathbb{Q}_p^N} \varphi^2(x)\, d^N x + \frac{\alpha_4(T)}{4} \int\limits_{\mathbb{Q}_p^N} \varphi^4(x)\, d^N x - \int\limits_{\mathbb{Q}_p^N} \varphi(x) J(x) d^N x,$$

where J, $\varphi \in \mathcal{D}_{\mathbb{R}}$, and

$$\begin{aligned} \gamma(T) &= \gamma + O((T - T_c)); \quad \alpha_2(T) = (T - T_c) + O((T - T_c)^2); \\ \alpha_4(T) &= \alpha_4 + O((T - T_c)), \end{aligned}$$

where T is temperature, T_C is the critical temperature and $\gamma > 0$, $\alpha_4 > 0$.

$\mathbf{Z}_2$ symmetry

If $J = 0$, then E is invariant under $\varphi \to -\varphi$.

Ginzburg-Landau phenomenology

- We consider that $\varphi \in \mathcal{D}'_{\mathbb{R}}$ is the local order parameter of a continuous Ising system with 'external magnetic field' $J \in \mathcal{D}'_{\mathbb{R}}$.

Ginzburg-Landau phenomenology

- We consider that $\varphi \in \mathcal{D}^l_{\mathbb{R}}$ is the local order parameter of a continuous Ising system with 'external magnetic field' $J \in \mathcal{D}^l_{\mathbb{R}}$.

- The system is contained in the ball B^N_l. We divide this ball into sub-balls (boxes) $B^N_{-l}(\mathbf{i})$, $\mathbf{i} \in G_l$. The volume of each of these balls is p^{-lN} and the radius is $a := p^{-l}$.

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- Each $\varphi(\mathbf{i}) \in \mathbb{R}$ represents the 'average magnetization' in the ball $B^N_{-l}(\mathbf{i})$. We take $\varphi(x) = \sum_{\mathbf{i} \in G_l} \varphi(\mathbf{i}) \, \Omega\left(p^l \|x - \mathbf{i}\|_p\right)$ which is a locally constant function.

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- Notice that the distance between two points in the ball $\mathbf{i} + p^l \mathbb{Z}^N_p$ is $\leq p^{-l}$. Then $\varphi(x)$ varies appreciable over distances larger than p^{-l}.

Ginzburg-Landau phenomenology

Then considering $\varphi(\mathbf{i}) \in \mathbb{R}$ as the continuous spin at the site $\mathbf{i} \in G_l$, the partition function of our continuos Ising model is

$$\mathcal{Z}^{(l)}(\beta) = \sum_{\{\varphi(\mathbf{i});\ \mathbf{i} \in G_l\}} e^{-\beta E(\varphi(\mathbf{i}), J(\mathbf{i}))}.$$

Theorem

The minimizers of the functional $E(\varphi, 0)$, $\varphi \in \mathcal{D}^l_{\mathbb{R}}$ are constant solutions of

$$\left(-\frac{\gamma}{2}\mathbf{W}^{(l)}_{\delta} + \alpha_2 - \frac{\gamma}{2} \int_{\mathbb{Q}_p^N \setminus B_l^N} \frac{d^N y}{w_\delta(\|y\|_p)} \right) \varphi(x) + \alpha_4 \varphi^3(x) = 0, \tag{4}$$

i.e. solutions of

$$\varphi\left(\alpha_4 \varphi^2 + \alpha_2\right) = 0. \tag{5}$$

Spontaneous symmetry breaking

If $J = 0$, the field $\varphi \in \mathcal{D}^{\prime}_{\mathbb{R}}$ is a minimum of the energy functional E, if it satisfies (5). When $T > T_C$ we have $\alpha_2 > 0$ and the ground state is $\varphi_0 = 0$. In contrast, when $T < T_C$, $\alpha_2 < 0$, there is a degenerate ground state $\pm\varphi_0$ with

$$\varphi_0 = \sqrt{-\frac{\alpha_2}{\alpha_4}}.$$

This implies that below T_C the systems must pick one of the two states $+\varphi_0$ or $-\varphi_0$, which means that there is a spontaneous symmetry breaking.


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