p-adic Black Holes, Quantum Gravity and Motives

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Outlook

1. Rational numbers as observables
2. Number theory as the ultimate physical theory
3. Invariance under change of number fields
4. p-adic gravity and black holes
5. Quantum dynamics and its complete integrability. Categories
6. Quantum gravity and black hole thermodynamics
7. Motives

Planck Units

- Fundamental physical constants constants $c, G, \hbar, k_b$

  Planck considered only the units based on the universal constants $G, h, c,$ and k_{B} to arrive at natural units for length, time, mass, and temperature

- Four Planck's quantities

Name	Dimension	Expression	Value (SI units)
Planck length	Length (L)	$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}}$	1.62×10^{-35} m
Planck mass	Mass (M)	$m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}}$	2.18×10^{-8} kg
Planck time	Time (T)	$t_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^5}}$	5.39×10^{-44} s
Planck temperature	Temperature (Θ)	$T_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G k_{\mathrm{B}}^2}}$	1.41×10^{32} K

Rational numbers

I.V. *p-adic string*, Class.Quant. Gravity, 4, 1987, L83

- Only rational numbers can be observed.
- Irrational numbers (infinite decimals) are not observed.
- On the field of rational numbers $\mathbb{Q}$ there are two norms: ordinary real and p-adic (Ostrowsky theorem)
- One cannot measure distances less the Planck length,

$$\Delta x > \ell_{Planck},$$

  because of the black hole creation.
- Non-Archimedean geometry appears

Number theory as an ultimate physical theory

- In ultimate physical theory on can put the principle **according to which fundamental physical laws should be invariant with respect to change of number field (I.V. 1987)**.

- It is in some sense a further generalization of the Einstein principle of general relativity.

p-adic gravity

- The Einstein gravitational field equations

$$R_{\nu\mu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

  are so far so fundamental that it is quite reasonable to hope that they obey the principle **of invariant with respect to change of number field**.
- Having in mind the natural definitions of all ingredient presented in the left hand side of this equation on p-adics case, it is natural to conclude that the Einstein equation has the same form in the p-adic case as in the real one.
- Allowing the energy-momentum tensor $T_{\mu\nu}$, and cosmological constant λ to be also p-adic and requiring $\kappa = \frac{8\pi G}{c^4}$ to be a rational number, we can introduce the same gravitational field equations, where all quantities are valued in Q_p.
- That is what we mean by saying that equation (1) is invariant with respect to the change of number fields R and Q_p

p-adic gravity. p-adic differential geometry

I follow I. Aref'eva, B. Dragovich, P. Frampton, I.V. *"Wave function of the universe and p-adic gravity,"* Int. J. Mod. Phys. A **6**, 4341-4358 (1991)

- Let us consider a p-adic generalization of the usual differential geometry.
- The simplest p-adic manifold is the space Q_p^n.
- An arbitrary p-adic manifold can be obtained by means of a local coordinate system, i.e. it is locally homeomorphic to an open set in Q_p^n.
- We consider only analytic p-adic manifolds.
- To describe a p-adic manifold one can use a corresponding generalization of the implicit-function theorem.

p-adic gravity. p-adic manifold. Examples

- An example of a nontrivial p-adic manifold is a sphere S_ρ^{n-1} in Q_p^n, which is defined by the equation

$$x_1^2 + \ldots + x_n^2 = \rho^2, x_i \in Q_p$$

Here ρ is a fixed p-adic number which can be called the p-adic radius ,e.g. $\rho = 1$ or ρ is a rational number.

- Note that unlike the real case the p-adic sphere (6) can be a non-compact space.

p-adic gravity. p-adic differential geometry

- p-adic tensors have the usual transformation rules under analytical transformati variables.
- For example for the second rank tensor $g_{\mu\nu}(x)$ on a manifold $\mathcal{M}^{\mathcal{D}}$ one has

$$g'_{\mu\nu}(x') = \frac{\partial x^{\sigma}}{\partial x_{\mu}} \frac{\partial x^{\lambda}}{\partial x_{\nu}} g_{\sigma\lambda}$$

  Here $x^{\mu} \in Q_p, \mu = 1, \ldots D; g_{\mu\nu} \in Q_p; x^{\mu}(x')$ is series with respect to x'^{μ}, which is convergent in p-adic norm.
- Here we use a natural definition of the derivatives in p-adic case. If one has a function $f : Q_p \to Q_p$ and $\left| \frac{f(x+\epsilon)-f(x)}{\epsilon} - \frac{\partial f(x)}{\partial x} \right|_p \to 0, \quad |\epsilon| \to 0$ then $\frac{\partial f(x)}{\partial x}$ is the derivation of the $f(x)$.
- The p-adic analog of the interval will be $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$.

p-adic gravity. p-adic differential geometry

- We introduce on $\mathcal{M}^D$
  - a p-adic connection $\Gamma^{\lambda}_{\mu\nu}$ with the usual transformation rule,
  - the Riemannian tensor $R^{\sigma}_{\mu\nu\lambda}$,
  - Ricci tensor $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$,
  - scalar curvature $R = R_{\mu\nu}g^{\mu\nu}$.

- Note that the inverse tensor $g^{\mu\nu}$ is defined in the p-adic case by the usual formula $g^{\mu\nu}g_{\nu\lambda} = \delta^{\mu}_{\lambda}$.

p-adic gravity. p-adic differential geometry

- The sphere S_o^{n-1} is a space of constant curvature in the sense that

$$R_{\mu\nu\lambda\sigma} = k\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}\right) \tag{2}$$

where $k = \frac{1}{\rho}$ is the p-adic curvature of the sphere S_ρ^{n-1}.

- The formula (2) can be derived using the corresponding expression for the metric on the sphere, i.e.

$$ds^2 = \frac{4\delta_{\mu\nu}du^\mu du^\nu}{\left(\rho^2 + (u,u)\right)^2} \tag{3}$$

- The metric (3) is obtained from the flat metric in Q_p^D

$$ds^2 = dx_1^2 + \ldots + dx_D^2$$

using the parametrisation

$$x^i = \frac{\rho u^i}{\rho^2 + (u,u)}, i = 1, \ldots D-1; (u,u) = \sum_{i=1}^{D-1}\left(u^i\right)^2.$$

-

p-adic gravity. p-adic differential geometry

- Let us note an important property of p-adic quadratic forms. The equation $x_1^2 + x_2^2 = 0$ has a solution in Q_p for non-vanishing x_1 and x_2 if $p \equiv 1 (\text{mod} 4)$.
- According to the Minkowski-Hasse theorem an arbitrary quadratic form in Q_p

$$\sum_{i,j=1}^{n} a_{ij} x^i x^j$$

  has a non-trivial zero iff the dimension $n > 4$.
- Therefore $n = 4$ is the maximal dimensionality for which p-adic quadratic forms have no new singularities compared to real case. Note a remarkable coincidence of this dimenssionality with the dimensionality of space-time!

Singularities Theorems

- It is proved that in the raisonnable conditions in general relativity (Black holes, cosmology) always occur singularities (Penrose, Hawking,...)

Black Holes, Schwarzschild solution

- The Schwarzschild metric in the Schwarzschild coordinates

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2\,(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$r > 2M > 0,$$

- $r = 2M$ — coordinate singularity
- Geodesics

$$-dt = \frac{dr}{1 - \frac{2M}{r}}$$

- Solution with the boundary condition $r(0) = r_0$

$$r_0 - t = r + 2M \log\left(\frac{r - 2M}{r_0 - 2M}\right)$$

Black Holes, Schwarzschild solution

- The Kruskal coordinates are

$$U = -e^{-\frac{u}{4M}},\ V = e^{v/4M}, \qquad \text{where} \quad u = t - r_*, \qquad v = t + r_*$$

and is the tortoise coordinate r_* which solves the equation

$$dr_* = (1 - 2M/r)^{-1} dr$$

Solution $r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$

- The Schwarzschild metric in the Kruskal coordinates

$$ds^2 = -\frac{32M^3}{r}\, e^{-r/2M}\, dU dV \ + r^2 d\Omega^2,$$

where r is defined from equation

$$\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = -UV$$

p-adic Black Holes

- p-adic Schwarzschild metric. t, r, θ, φ, $M \in \mathbb{Q}_p$

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

- Geodesics

$$-dt = \frac{dr}{1 - \frac{2M}{r}}$$

- Solution with the boundary condition $r(0) = r_0$

$$r_0 - t = r + 2M \log_p \left(\frac{r - 2M}{r_0 - 2M} \right)$$

where $\log_p(x)$ is p-adic log.

- $\log_p(1+z)$ is defined for $|z|_p < 1$. Here $z = \frac{r - r_0}{r_0 - 2M}$ and we get

$$\left| \frac{r_0 - r}{r_0 - 2M} \right|_p < 1$$

p-adic Kruskal coordinates

- The Kruskal coordinates are

$$U = -e^{-\frac{u}{4M}},\ V = e^{v/4M}, \qquad \text{where} \quad u = t - r_*, \qquad v = t + r_*$$

  and $r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$
- We have to impose

$$\left|\frac{u}{4M}\right|_p < p^{-\frac{1}{p-1}}, \quad \left|\frac{v}{4M}\right|_p < p^{-\frac{1}{p-1}}$$

- p-adic Kruskal metric

$$ds^2 = -\frac{32M^3}{r}\, e^{-r/2M}\, dU dV\, + r^2 d\Omega^2,$$

  where r is defined from equation $\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} = -UV$
- Maximal extension?

Quantum explosions of black holes

I. Aref'eva, I.V. *"Quantum explosions of black holes and thermal coordinates,"* arXiv:2104.12724"

- Hawking temperature $T_H = 1/8\pi M$ is singular as $M \to 0$
- Kruskal coordinates are singular as $M \to 0$
- New thermal coordinates

$$\mathcal{U} = -e^{-\frac{u}{4M+b}}, \quad \mathcal{V} = e^{\frac{v}{4M+b}}, \tag{4}$$

- New temperature

$$T = \frac{1}{2\pi(4M+b)}.$$

Quantum explosions of p-adic black holes

- Improving the convergence

$$\left|\frac{u}{4M+b}\right|_p < p^{-\frac{1}{p-1}}, \quad \left|\frac{v}{4M+b}\right|_p < p^{-\frac{1}{p-1}}$$

Quantum explosions of black holes

- Conformal invariant wave equations

$$\partial_u \partial_v \phi = 0, \qquad \partial_{\mathcal{U}} \partial_{\mathcal{V}} \Phi = 0$$

- Bogolubov's coefficient

$$\beta_{\omega\,\mu} = \int_{\mathbb{R}} du \, \exp\{i(\omega u + \mu\, e^{-\frac{u}{4M+b}})\},$$

- We get the Planck distribution

$$f(\omega) = \frac{1}{e^{2\pi(4M+b)\omega} - 1},$$

Quantum explosions of p-adic black holes

- Conformal invariant p-adic wave equations

$$\partial_u \partial_v \phi = 0, \qquad \partial_{\mathcal{U}} \partial_{\mathcal{V}} \Phi = 0$$

- p-adic Bogolubov's coefficient

$$\beta_{\omega\,\mu} = \int_{D_p} du\, \chi(\omega u + \mu\, e^{-\frac{u}{4M+b}})\}$$

- Planck distribution ?

Integrability. Introduction

I.V. *"Remarks on the complete integrability of quantum and classical dynamical systems,"* P-Adic Numbers Ultrametric Anal. Appl., 11:4 (2019), 328-334, arXiv:1911.01335 .

- Any quantum dynamical system is completely integrable.
- Moreover, it is unitary equivalent to a set of classical noninteracting harmonic oscillators
- Any classical dynamical system with an invariant measure is also completely integrable
- Explicit integrability wide classes of classical and quantum multidimensional PDE by using wave operators.

Quantum dynamical systems

- Quantum dynamical system: $(\mathcal{H},\ H)$ or $(\mathcal{H},\ U_t)$,

$$U_t = e^{i\,t\,H}$$

- Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi$$

- $(\mathcal{H},\ U_t)$, $t \in \mathbb{Q}_p$, or locally compact commutative group G.
- Systems of harmonic oscillators

$$(L^2(X,\mu), V_t), \qquad V_t\phi(x) = e^{-it\omega_x}\phi(x)$$

Integrability. Theorem

- **Theorem. Let $\mathcal{H}$ be a separable Hilbert space and H a self-adjoint operator with a dense domain $D(H) \subset \mathcal{H}$. Then the Schrödinger equation**

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \;\; \psi(t) \in D(H), \; t \in \mathbb{R}$$

completely integrable in the sense that this equation is unitary equivalent to the complexified system of equations for a family of classical non-interacting harmonic oscillators.

Equivalently, quantum dynamical systems $(\mathcal{H}, \; U_t)$ is unitary equivalent to a system of harmonics oscillators $(L^2(X,\mu), V_t)$.

There exists a set of non-trivial integrals of motion for the arbitrary Schrödinger equation.

Proof of Theorem; 1/3

The solution of the Cauchy problem for the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \;\; \psi(0) = \psi_0 \in D(H)$$

by the Stone theorem has the form $\psi(t) = U_t\psi_0$,

where $U_t = e^{-itH}$ is the group of unitary operators, $t \in \mathbb{R}$.

Proof of Theorem; 2/3

Then, by the spectral theorem there exists a measurable space (X, Σ) with σ-finite measure μ, and a measurable finite a.e. function $\omega : X \to \mathbb{R}$ such that there is a unitary transformation

$$W : \mathcal{H} \to L^2(X, \mu)$$

such that $U_t = W^* V_t W$, where $V_t = e^{-itM_\omega}$, where M_ω is the operator of multiplication by the function ω.

On the corresponding domain one has

$$H = W^* M_\omega W.$$

Proof of Theorem; 3/3

The initial Schrödinger equation in $\mathcal{H}$ goes over under the unitary transformation of W into the Schrödinger equation in $L^2(X, \mu)$ of the form

$$i\frac{\partial}{\partial t}\varphi_x(t) = \omega_x \varphi_x(t),\ x \in X$$

Passing to the real and imaginary parts of the function

$$\varphi_x(t) = \frac{1}{\sqrt{2}}(q_x(t) + ip_x(t)),$$

this Schrodinger equation is rewritten in the form of equations of the family of classical harmonic oscillators

$$\dot{q}_x = \omega_x p_x,\ \dot{p}_x = -\omega_x q_x,\ x \in X.$$

Integrability of Quantum Dynamical Systems on $\mathbb{Q}_p$ and Locally Compact Commutative Groups

- $(\mathcal{H},\ U_t)$, $t \in \mathbb{Q}_p$ or G.

Categorical quantum dynamical system

- The categorical quantum dynamical system is the triple $(\mathcal{T}, F, \mathcal{M})$,

  where $\mathcal{M}$ is a monoidal additive involutive category,

  and F is a functor from the category $\mathcal{T}$ to the category $\mathcal{M}$, $\quad F : \mathcal{T} \to \mathcal{M}$,

  satisfying the following condition:
  - if the functor F maps the object T to some object A, then $F : Hom(T, T) \to UHom(A, A)$.
  - Moreover, $F(t)$ is a unitary morphism, $F(t) \in UHom(A, A)$ and $F(t + s) = F(t) \circ F(s)$.

  Thus, the functor F defines a unitary representation of the abelian group T in monoidal additive categories with involution $\mathcal{M}$.

Integrals of motion

- **Integrals of motion in a monoidal additive involutive category.**

  - Let $\eta = (\mathcal{T}, F, \mathcal{M})$ be a categorical quantum dynamical system, F a functor from $\mathcal{T}$ to $\mathcal{M}$ such that $F(T) = A \in \mathcal{M}$ and $F : Hom(T,T) \to UHom(A,A)$.

    Further, let G be a functor from $\mathcal{T}$ to $\mathcal{M}$ such that $G(T) = A \in \mathcal{M}$ and $G : Hom(T,T) \to UHom(A,A)$.

  - If the unitary morphisms $F(t)$ and $G(s)$ commute for all $t, s \in T$, i.e.

$$F(t) \circ G(s) = G(s) \circ F(t),$$

    then the functor G will be called **integral of motion** for the quantum dynamical system η.

Various Formulations of p-adic Quantum Mechanics

Vladimirov, I.V. *"p-adic quantum mechanics?*, Comm. Math. Phys., 123 (1989), 659?676;
Vladimirov, I. V., Zelenov, *p-adic analysis and mathematical physics*,"Nauka", Moscow, 1994
Khrennikov, *Non-Archimedean Analysis,...*, 2013

- 4 numbers fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$
- Unitary representations, Vladimirov operator, ...

Quantum Gravity: real case vs. p-adic

I follow I. Aref'eva, B. Dragovich, P. Frampton, I.V. *"Wave function of the universe and p-adic gravity,"* Int. J. Mod. Phys. A **6**, 4341-4358 (1991)

- Partition function

$$Z = \sum_{\text{manifolds}} \int e^{i\{S_{gr}(g)+S_{matter}(g,\phi)\}} \mathcal{D}g\mathcal{D}\phi$$

- Black hole thermodynamics, $dM = T_H dS$,
  $S = A/4$, Bekenstein-Hawking entropy
- p-adic partition function

$$Z_p = \sum_{\text{alg.manif.}} \int \chi\{S_{gr}(g) + S_{matter}(g,\phi)\}\,\mathcal{D}g\mathcal{D}\phi$$

Motives

I.V. *"D-branes, black holes and $SU(\infty)$ gauge theory,"*
2nd International Sakharov Conference on Physics,
20-23 May 1996, Proc. pp 618-621, hep-th/9608137.

- Grothendieck's motives.
  - Motives are defined by algebraic correspondences modulo homological equivalence.
  - Motivic cohomology is a kind of universal cohomology theory for algebraic varieties.
  - Realizations of a motive M over the field of rational numbers Q are linear spaces over Q and over the field of l-adic numbers Q_l.
  - If X is a smooth projective algebraic variety over Q then the realization of its motive is given by the Betti $H_B(X)$, De Rham $H_{DR}(X)$ and l-adic cohomology $H_l(X)$ of X.
  - Category of motives is isomorphic to category of finite dimensional representations of proalgebraic motivic Galois-Serre group G.

Motives

- L-functions
  - The group G plays the role of conformal group in the ordinary string theory.
  - Motivic-theory deals with algebraic varieties over the field of rational numbers Q, so the theory is background independent and it is not based on a spacetime continuum.
  - The partition function in motivic-theory is given by L-function of a motive.
  - L-function of a motive M is defined by the Euler product:

$$L(M,s) = \prod_p det(1 - p^{-s} Frob_p | H_l(M)^{I_p})^{-1},$$

where $Frob_p$ is a Frobenius element in G_{Q_p} and I_p is the inertia group in G_{Q_p}.

Motives

- Motivic-string
  - Motivic-string partition function can be expressed as inverse to the Mellin transform of L-function of a motive.
  - Deligne has used L-function in the proof of the Ramanujan conjecture:

    $$|\tau(p)| \leq 2p^{11/2}.$$

    The Ramanujan function $\tau(n)$ is defined by the relation

    $$\Delta(q) = q\prod_n (1-q^n)^{24} = \sum_n \tau(n)q^n$$

    We have: $\Delta(q)^{-1} = \mathrm{tr} q^H$, where $H = L_0 - \frac{1}{24}$ and L_0 is the Virasoro operator.
  - L-function of the motive M is the Dirichlet series:

    $$L(M,s) = \sum_n \tau(n)n^{-s} = \prod_p (1-\tau(p)p^{-s} - p^{11-2s})^{-1},$$

  - It is amusing that the motive M is eleven-dimensional.
- We suggest a relation between the cosmological constant problem and the Beilinson conjectures on values of L-functions.

Conclusion

- Principle of invariance of fundamental physical laws:
  fundamental physical laws should be invariant with respect to change of number field
- p-adic black hole solutions of p-adic Einstein equation
- All quantum dynamical system $(\mathcal{H},\ U_t)$ (real and p-adic) are **completely integrable**
- a relation between L-functions of motives and partition function of bosonic string is proposed