Causality and time: An ultrametric view

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Contents

Systems, automata and word maps

Notion: *system*

A (discrete) system (or, a system with a discrete time $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$) is a 5-tuple $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O \rangle$ where

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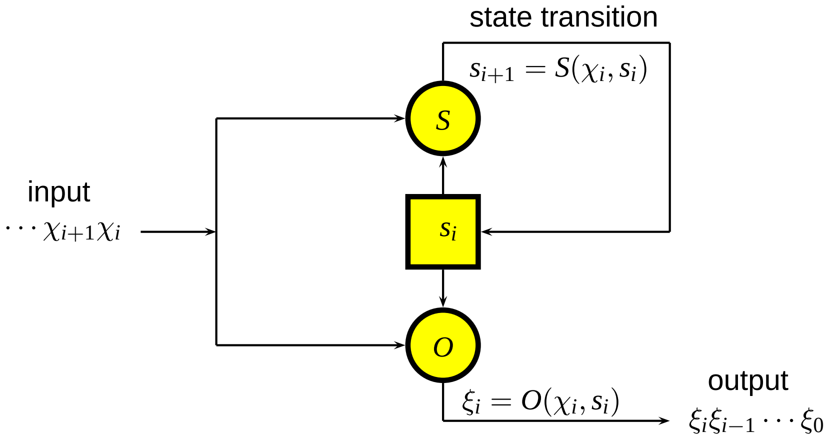
The system is autonomous if neither S nor O depend on input (that is $S \colon \mathcal{S} \to \mathcal{S}$, $O \colon \mathcal{S} \to \mathcal{O}$); otherwise the system is non-autonomous.

Systems, automata and word maps

An (initial) automaton $\mathfrak{A}(s_0)$ is a system where one of the states, $s_0 \in \mathcal{S}$, is fixed; it is called the initial state.



Automaton $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O, s_0 \rangle$: $\mathcal{I}$ – *input alphabet;* $\mathcal{O}$ – *output alphabet;* $\mathcal{S}$ – *state set;* $S \colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ – *transition function;* $O \colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}$ – *output function;* $s_0 \in \mathcal{S}$ – *initial state*

$\chi_0 \in \mathcal{I}$ — 0-*th input symbol*

$$\chi_0 \longrightarrow \boxed{\begin{array}{l}\textit{Time } t = 0 : \\ \textit{state } s_0\end{array}} \longrightarrow \xi_0 = O(\chi_0, s_0)$$

$\xi_0 \in \mathcal{O}$ — 0-*th output symbol*

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$\chi_1 \in \mathcal{I}$ — 1-*st input symbol*

$\chi_1 \longrightarrow$ **[** *Time* $t = 1:$ $s_1 = S(\chi_0, s_0)$ **]** $\longrightarrow$ $\xi_1 = O(\chi_1, s_1)$

$\xi_1 \in \mathcal{O}$ — 1-*st output symbol*

Systems, automata and word maps

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$\chi_2 \in \mathcal{I}$ — 2-*nd input symbol*

$\chi_2 \longrightarrow$ $\boxed{\text{Time } t = 2: \; s_2 = S(\chi_1, s_1)}$ $\longrightarrow$ $\xi_2 = O(\chi_2, s_2)$

$\xi_2 \in \mathcal{O}$ — 2-*nd output symbol*

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$\chi_i \in \mathcal{I}$ — *i-th input symbol*

$$\chi_i \longrightarrow \boxed{\begin{array}{l}\textit{Time } t = i:\\ s_i = S(\chi_{i-1}, s_{i-1})\end{array}} \longrightarrow \xi_i = O(\chi_i, s_i)$$

$\xi_i \in \mathcal{O}$ — *i-th output symbol*

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The automaton $\mathfrak{A}$ determines the <u>automaton function</u> $f_{\mathfrak{A}}$ that maps words over the alphabet $\mathcal{I}$ to words over the alphabet $\mathcal{O}$: $f_{\mathfrak{A}}\colon \ldots \chi_2\chi_1\chi_0 \mapsto \ldots \xi_2\xi_1\xi_0$

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$\xi_i \in \mathcal{O}$ — *i-th output symbol*

Every output symbol ξ_i *depends only on symbols* $\chi_0, \ldots, \chi_i \in \mathcal{I}$ *which have been already feeded to the automaton*: $\xi_i = \varphi_i(\chi_0, \ldots, \chi_i) \in \mathcal{O}$

Therefore an automaton can be considered as mathematical formalism for the *causality law:* (input symbols=causes; output symbols=effects)

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$\chi_i \in \mathcal{I}$ — *i-th input symbol*

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The automaton function $f_{\mathfrak{A}}\colon \ \ldots\chi_2\chi_1\chi_0 \mapsto \ldots\xi_2\xi_1\xi_0$ is completely determined by the sequence of maps $\varphi_i\colon \mathcal{I}^{i+1} \to \mathcal{O}$, $i \in \mathbb{N}_0$; and vice versa, every such sequence of maps determines an automaton function.

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$\chi_i \in \mathcal{I}$ — *i-th input symbol*

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The automaton function maps infinite words (=left-infinite sequences) over $\mathcal{I}$ to infinite words over $\mathcal{O}$.

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Take $\mathcal{I} = \mathcal{O} = \{0, 1, \ldots, p-1\}$ for a prime p; associate infinite words to canonical representations of p-adic integers; then the automaton function is a map $\mathbb{Z}_p \to \mathbb{Z}_p$. The map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is an automaton function *if and only if* it is 1-Lipschitz w.r.t. p-adic metric.

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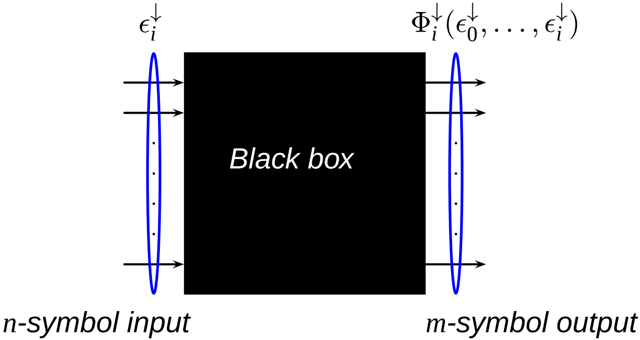
The map $f \colon \mathbb{Z}_p \to \mathbb{Z}_p$ is an automaton function *if and only if* it is 1-Lipschitz w.r.t. p-adic metric. Therefore (univariate) 1-Lipschitz functions can be judged as (univariate) *causal functions over discrete time*; i.e., as function which describes temporal evolution of a system.

Automata maps = causal functions over discrete time

More general model of a (non-autonomous) system which evolves in time is the case $\#\mathcal{I} = p^n$, $\#\mathcal{O} = p^m$ where $m, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. This case corresponds to an automaton having n input words and m output words over p-symbol alphabet; or, equivalently, to a 1-Lipschitz map $\mathbb{Z}_p^n \to \mathbb{Z}_p^m$. In other words, we consider a system as a black box and observe reactions of the system exposed to (long) series of impacts.



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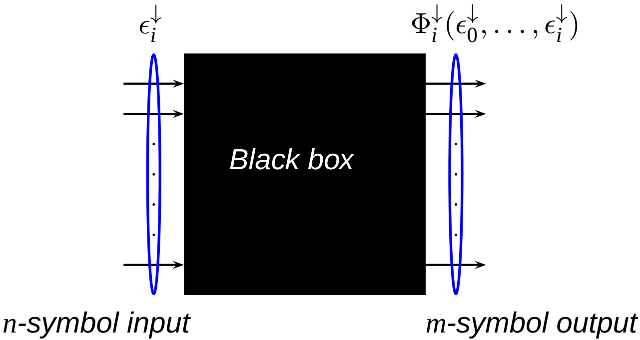
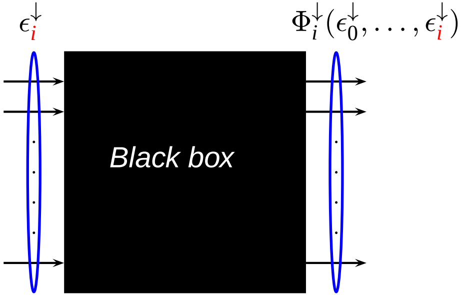
More general model of a (non-autonomous) system which evolves in time is 1-Lipschitz map $\mathbb{Z}_p^n \to \mathbb{Z}_p^m$. In other words, we consider a system as a black box and observe reactions of the system exposed to (long) series of impacts.



Automata maps = causal functions over discrete time

Causality=1-Lipschitzness: Every i-th output vector $\Phi_i^{\downarrow}$ does NOT depend on 'future input vectors' $\epsilon_{i+1}^{\downarrow}, \epsilon_{i+2}^{\downarrow}, \ldots$, for all $i = 0, 1, 2, \ldots$.



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Given a 1-Lipschitz map $F\colon \mathbb{Z}_p^n \to \mathbb{Z}_p^m$, there are infinitely many different automata (i.e., the ones whose sets of epistemic states are different, whose state transition functions are different, whose output functions are different, but) whose automaton function is F.

Therefore external observer can only make guesses about 'internal structure' of the system by observing pairs of 'causes and effects', i.e., pairs $(x, F(x))$.

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One of the most important features of the 'internal structure' of the system which imposes sharp restrictions on the automaton function F is finiteness of the set of (epistemic) states of the automaton.

Definition (Finite automata function)

A 1-Lipschitz map $F\colon \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ is called a *finite automaton function* if there exists an automaton having p^n-symbol input alphabet $\mathcal{I}$, a p^m-symbol output alphabet $\mathcal{O}$, a *finite* set of states $\mathcal{S}$, and whose automaton function is F.

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Basically a 'real-world' physical system cannot have infinite number of states; however, the number of states can be too large in comparison the 'time elapsed', i.e., with the length of a finite words the automaton accepts and produces during observation. Therefore 1-Lipschitz maps which are not finite automata functions must also be included into considerations.

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Examples

- A constant map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is a finite automaton function if and only if $f(x) = const \in \mathbb{Z}_p \cap \mathbb{Q}$ for all $x \in \mathbb{Z}_p$.

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- An affine map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $f(x) = ax + b$, $(x \in \mathbb{Z}_p)$ is a finite automaton function if and only if $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$.

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- A polynomial map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $f(x) \in \mathbb{Z}_p[x]$, is an automaton function, but it is never a finite automaton function if $\deg f \geq 2$.

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- Note that the maps defined by polynomials all whose coefficients are rational p-adic integers (i.e., lie in $\mathbb{Z}_p \cap \mathbb{Q}$) are examples of the functions of our interest: The maps are automata functions (since they are p-adic 1-Lipschitz); they are continuous real-valued functions of real variable; and both that functions (real and p-adic) agree on $\mathbb{Z}_p \cap \mathbb{Q}$; all their derivatives agree as well.

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Are there other functions of that highlighted sort? Yes!

On p-adic 1-Lipschitz continuous real functions

Given a prime p, denote via $\mathcal{C}_p(\mathbb{R})$ the class of all functions continuous (w.r.t. to metric on $\mathbb{R}$) real functions $\breve{f}\colon \mathbb{R} \to \mathbb{R}$ which satisfy the following properties:

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2. There exists a p-adic 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that
   - $f(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$
   - $f(z) = \breve{f}(z)$ for every $z \in \mathbb{Z}_p \cap \mathbb{Q}$

The class $\mathcal{C}_p(\mathbb{R})$ is the main class we are focused at; further in the talk we do not differ $\breve{f}$ from f when it is clear from the context what domain is considered.

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1. $\breve{f}(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$;
2. There exists a p-adic 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ such that
   - $f(\mathbb{Z}_p \cap \mathbb{Q}) \subset \mathbb{Z}_p \cap \mathbb{Q}$
   - $f(z) = \breve{f}(z)$ for every $z \in \mathbb{Z}_p \cap \mathbb{Q}$

The class $\mathcal{C}_p(\mathbb{R})$ is the main class we are focused at; further in the talk we do not differ $\breve{f}$ from f when it is clear from the context what domain is considered.

Example

Given polynomials $u, v \in \mathbb{Z}[x]$ s.t. $v(z) \not\equiv 0 \pmod p$ for all $z \in \mathbb{Z}_p$ and $v(z) \neq 0$ for all $z \in \mathbb{R}$, the rational function $f(x) = \frac{u(x)}{v(x)}$ is in $\mathcal{C}_p(\mathbb{R})$.

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► The rational functions f from the example above *are differentiable both w.r.t. p-adic metric and real metric; moreover, $\breve{f}\,' = f\,'$ everywhere on $\mathbb{Z}_p \cap \mathbb{Q}$ and $f\,' \in \mathcal{C}_p(\mathbb{R})$*.

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We denote via $\mathcal{C}_p^k(\mathbb{R})$ (resp., via $\mathcal{C}_p^\infty(\mathbb{R})$ the sub-class of all k-times (resp., infinitely) differentiable (w.r.t. both p-adic and real metric) functions whose derivatives are also in $\mathcal{C}_p(\mathbb{R})$. The functions from the above example are all in $\mathcal{C}_p^\infty(\mathbb{R})$.

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Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

Pre-history: Once K. Weierstrass asked D. Hilbert whether there exists a smooth real function which is not a rational function but which maps rationals to rationals. The answer was positive, but neither records of the conversation nor Hilbert's example are known.

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Relying on ideas from these works, it is possible to construct functions from $\mathcal{C}_p^k(\mathbb{R})$ which are not rational functions.

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

It is possible to construct functions from $\mathcal{C}_p^k(\mathbb{R})$ which are not rational functions.

- The said functions are complex entire functions of the form

$$t_k(x) = \sum_{i=1}^{\infty} C_i((q_1 - x)(q_2 - x)\cdots(q_i - x))^{k+1}$$

where

1. $q_1, q_2, q_3, \ldots$ is enumeration of elements of $\mathbb{Z}_p \cap \mathbb{Q}$ by positive rational integers $1, 2, 3, \ldots$;
2. C_i are specially constructed rational p-adic integers to ensure the function $t_k(x)$ is entire complex function and are such that the sequence $(C_i)_{i=1}^{\infty}$ converges to 0 both in p-adic metric and in real metric: All $C_i \in \mathbb{Z}_p \cap \mathbb{Q}$ and $C_i \xrightarrow[\mathbb{R}]{\mathbb{Z}_p} 0$ as $i \to \infty$.

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That is a view from the 'real side'; now we will look from the 'p-adic side'.

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

Firstly recall some facts about p-adic 1-Lipschitz functions.

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

Firstly recall some facts about p-adic 1-Lipschitz functions.

► A mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ belongs to the class of all 1-Lipschitz mappings $\mathbb{Z}_p \to \mathbb{Z}_p$ (denoted via $\mathcal{L}_p$) if and only if f can be represented via Mahler series of the form

$$f(x) = \sum_{i=0}^{\infty} a_i p^{\lfloor \log_p i \rfloor} \binom{x}{i} \quad (\, a_i \in \mathbb{Z}_p;\; i = 0, 1, 2, \ldots)$$

where

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

for $i = 1, 2, \ldots$;

$$\binom{x}{0} = 1, \lfloor \log_p 0 \rfloor = 0$$

by the definition.

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

- A mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ belongs to the class $\mathcal{L}_p$ (=is 1-Lipschitz) iff

$$f(x) = \sum_{i=0}^{\infty} a_i p^{\lfloor \log_p i \rfloor} \binom{x}{i} \quad (\text{here } a_i \in \mathbb{Z}_p;\ i = 0, 1, 2, \ldots)$$

- Functions defined by power series over $\mathbb{Z}_p$

$$s(x) = \sum_{i=0}^{\infty} c_i x^i \quad (\text{where } c_i \in \mathbb{Z}_p;\ i = 0, 1, 2 \ldots;),$$

that converge everywhere on $\mathbb{Z}_p$ (i.e., s.t. $\overset{p}{\lim\limits_{i\to\infty}} c_i = 0$) constitute a subclass (denoted via $\mathcal{C}$) of $\mathcal{L}_p$.

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- Following functions constitute a subclass $\mathcal{B} \subset \mathcal{L}_p$ s.t. $\mathcal{B} \supset \mathcal{C}$:

$$g(x) = \sum_{i=0}^{\infty} b_i \binom{x}{i} \quad (\text{here } b_i \in \mathbb{Z}_p \text{ are s.t. } \frac{b_i}{i!} \in \mathbb{Z}_p;\ i = 0, 1, 2, \ldots)$$

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

$$\mathcal{C} = \left\{ s(x) = \sum_{i=0}^{\infty} c_i x^i : c_i \in \mathbb{Z}_p;\ i = 0, 1, 2 \ldots;\ \lim_{i\to\infty}^{p} c_i = 0 \right\},$$

$$\mathcal{B} = \left\{ g(x) = \sum_{i=0}^{\infty} b_i \binom{x}{i} : \frac{b_i}{i!} \in \mathbb{Z}_p;\ i = 0, 1, 2, \ldots \right\},$$

The class $\mathcal{B}$ is endowed with the non-Archimedean metric $D_p(u, v) = \max\{|u(z) - v(z)|_p : z \in \mathbb{Z}_p\}$.

Theorem (V. A., 2002)

The class $\mathcal{B}$ is a complete (w.r.t. D_p) *metric space; it consists of $C^\infty(\mathbb{Z}_p)$-functions and is closed w.r.t. additions, multiplications, derivations, and compositions of functions; $\mathbb{Z}[x]$ is dense in $\mathcal{B}$.*

The notion of C^k-functions is used in standard meaning; we write $C^k(\mathbb{Z}_p)$ (resp., $C^k(\mathbb{R})$) to emphasize what derivatives are meant.

$$\mathcal{B} = \left\{ g(x) = \sum_{i=0}^{\infty} b_i \binom{x}{i} : \frac{b_i}{i!} \in \mathbb{Z}_p;\ i = 0, 1, 2, \ldots \right\},$$

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$\mathcal{B}$-functions are p-adic locally analytic of order 1:

$$f(z + p^k h) = f(z) + f'(z) \cdot p^k h + \frac{f''(z)}{2!} \cdot p^{2k} h^2 + \frac{f'''(z)}{3!} \cdot p^{3k} h^3 + \cdots$$

for all $z, h \in \mathbb{Z}_p$, $k = 1, 2, \ldots$. Moreover, $\frac{f^{(j)}(z)}{j!} \in \mathbb{Z}_p$, for all $j = 0, 1, 2, \ldots$.

Non-rational functions from $\mathcal{C}_p^k(\mathbb{R})$

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Examples ($\mathcal{B}$-functions)

Given $u(x), v(x) \in \mathcal{B}$,

- the function $\frac{u(x)}{v(x)}$ is in $\mathcal{B}$ if and only if $v(z) \not\equiv 0 \pmod p$ for all $z \in \mathbb{Z}_p$;

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- $$\sin_p u(x)=\sum_{j=0}^{\infty}\frac{(-1)^ju(x)^{2j+1}}{(2j+1)!},\quad \cos_p v(x)=\sum_{j=0}^{\infty}\frac{(-1)^jv(x)^{2j}}{2j!}\in\mathcal{B}$$

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- the function $\ln_p(1 + pu(x)) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{p^i u(x)^i}{i}$ is in $\mathcal{B}$;

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The non-rational functions $t_k(x) \in \mathcal{C}_p^k(\mathbb{R})$ which are constructed above are also in $\mathcal{B}$: Namely, taking a restriction of the function $t_k(x)$ on $\mathbb{Z}_p \cap \mathbb{Q}$ and expanding the obtained function $\mathbb{Z}_p \cap \mathbb{Q} \to \mathbb{Z}_p \cap \mathbb{Q}$ to the whole $\mathbb{Z}_p$ we obtain a $\mathcal{B}$-function.

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However, currently it is not known whether this fact implies that $t_k(x) \in \mathcal{C}_p^\infty(\mathbb{R})$ although $\mathcal{B}$-functions are p-adic C^∞-functions.

Open question

What are functions from $\mathcal{C}_p^\infty(\mathbb{R})$? Are there in that class other functions than rational functions over $\mathbb{Z}$? What $\mathcal{B}$-functions are in $\mathcal{C}_p(\mathbb{R})$?

Some properties of $\mathcal{C}_p(\mathbb{R})$-functions

As finite automata are of high importance, it would be reasonable to study finite automata functions from $\mathcal{C}_p(\mathbb{R})$.

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Theorem (Finite automata C^1-functions = affine functions over $\mathbb{Z}_p \cap \mathbb{Q}$)

Let a finite automaton function $f \in \mathcal{C}_p^1(\mathbb{R})$; i.e., let f be differentiable both over $\mathbb{R}$ and over $\mathbb{Z}_p$; let $f' \in \mathcal{C}_p(\mathbb{R})$ (that is, let the derivatives w.r.t. both real and p-adic metric exist, coincide on $\mathbb{Z}_p \cap \mathbb{Q}$, and be continuous w.r.t. respective metrics). *Then f is an affine function over $\mathbb{Z}_p \cap \mathbb{Q}$; i.e., $f(x) = ax + b$ for suitable $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$. Vice versa, all these affine functions are finite automata functions from $\mathcal{C}_p^\infty(\mathbb{R})$.*

Note

The theorem is true in multidimensional case as well.

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- This result may serve as an explanation why mathematical formalism of QM is the theory of linear operators: As all ‘real-world’ systems have finite number of epistemic states, then when, e.g., duration of temporal interval measured in Planck units becomes comparable to that number, the finiteness reveals itself.

Some properties of $\mathcal{C}_p(\mathbb{R})$-functions

It turns out that $\mathcal{C}_p(\mathbb{R})$-functions are 'hologram-like': If they coincide on arbitrarily small real interval from $\mathbb{R}$, they coincide on $\mathbb{R}$.

Theorem (Hologram-like property of $\mathcal{C}_p(\mathbb{R})$-functions)

Let $f, g \in \mathcal{C}_p(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$; then $f = g$ if and only if $f(x) = g(x)$ for all $x \in (\alpha, \beta) \cap \mathbb{Z}_p \cap \mathbb{Q}$.

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The hologram-like property resembles the property of a plate with a hologram on it: Even a small piece of the plate is enough to restore the image stored by the whole plate wit the hologram.

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- This property can be used in order to construct various classes of automata functions which approximate real functions. For instance, one can take polynomials with rational integer coefficients (i.e., the polynomials from $\mathbb{Z}[x]$) as such a class.

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- This property can be used in order to construct various classes of automata functions which approximate real functions. For instance, one can take polynomials with rational integer coefficients (i.e., the polynomials from $\mathbb{Z}[x]$) as such a class. Note that polynomials over $\mathbb{Z}$ are in $\mathcal{C}_p(\mathbb{R})$ and in $\mathcal{B}$. Note also that the theory of approximations by polynomials over $\mathbb{Z}$ is well-developed, see, e.g., L. B. O. Ferguson, 1980. We remind some classical results of the theory.

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- This property can be used in order to construct various classes of automata functions which approximate real functions. For instance, one can take polynomials with rational integer coefficients (i.e., the polynomials from $\mathbb{Z}[x]$) as such a class. We remind some classical results of the theory.
  - (J. Pál,1914) A continuous real-valued function g defined on the interval $[-\alpha, \alpha] \subset \mathbb{R}$, $0 < \alpha < 1$, can be uniformly approximated by polynomials from $\mathbb{Z}[x]$ if and only if $g(0) \in \mathbb{Z}$. In particular:
  - (M. I.Chlodovsky, 1925) A continuous real-valued function on a real interval which does not contain integer can be uniformly approximated by polynomials from $\mathbb{Z}[x]$.

Some properties of $\mathcal{C}_p(\mathbb{R})$-functions

Theorem (Hologram-like property of $\mathcal{C}_p(\mathbb{R})$-functions)

Let $f, g \in \mathcal{C}_p(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$; then $f = g$ if and only if $f(x) = g(x)$ for all $x \in (\alpha, \beta) \cap \mathbb{Z}_p \cap \mathbb{Q}$.

- This property can be used in order to construct various classes of automata functions which approximate real functions. For instance, one can take polynomials with rational integer coefficients (i.e., the polynomials from $\mathbb{Z}[x]$) as such a class. We remind some classical results of the theory.
  - (M. I.Chlodovsky, 1925) A continuous real-valued function on a real interval which does not contain integer can be uniformly approximated by polynomials from $\mathbb{Z}[x]$.
- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.

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- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.
- The *time reversibility of an automaton means that the automaton mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is bijective*; but a 1-Lipschitz map $\mathbb{Z}_p \to \mathbb{Z}_p$ is bijective if and only if it is an isometry w.r.t. p-adic metric, or, equivalently, *if and only if f is measure-preserving w.r.t. normalized Haar measure on $\mathbb{Z}_p$*.

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The theory of 1-Lipschitz measure-preserving functions $\mathbb{Z}_p \to \mathbb{Z}_p$ is well developed, there are known various criteria and sufficient conditions for measure-preservation; we can use them to construct various approximations by time reversible automata.

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- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.
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- Given a continuous real-valued function $w(x)$ on real interval $[\alpha, \beta] \subset [0, 1]$ where $\alpha > 0$, consider the function $\tilde{w}(x) = \frac{w(x)-x}{p}$. Since $\tilde{w}(x)$ is continuous on $[\alpha, \beta]$, it can be uniformly approximated by polynomials $u_i(x) \in \mathbb{Z}_p[x]$, in view of Chlodovsky theorem.

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- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.
- The *time reversibility of an automaton means that the automaton mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving* .
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- Hence $w(x)$ can be uniformly approximated by polynomials $v_i(x) = x + p \cdot u_i(x) \in \mathbb{Z}[x]$; but all $v_i(x)$ are measure-preserving, thus, they are automata functions of time-reversible automata. $\square$

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- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.
- Can this be treated as time reversibility at Planck time scale but generally time non-reversibility at macro-scale? Ask physicists!

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- By using approximations via polynomials from $\mathbb{Z}[x]$ it is possible to show that any continuous real-valued function defined on real interval $[\alpha, \beta]$ can be uniformly approximated by *'time-reversible' automata*.
- In a similar way it can be shown that a real-valued continuous function $v(x)$ on $[\alpha, \beta]$ can be uniformly approximated by *ergodic* automata functions.

Some properties of $\mathcal{C}_p(\mathbb{R})$-functions

- It can be shown that a real-valued continuous function $v(x)$ on $[\alpha, \beta]$ can be uniformly approximated by *ergodic* automata functions.
- It suffices to approximate the function $v(x)$ by polynomials $w_j(x)$, then approximate by polynomials $u_{ji}(x) \in \mathbb{Z}[x]$ every function $\tilde{w}_j(x)$ which satisfy the equation $\frac{w_j(x)-x-1}{p} = \Delta\tilde{w}_j(x)$, where Δ is difference operator, $\Delta d(x) = d(x+1) - d(x)$. Then all $w_j(x)$ can be uniformly approximated by polynomials $1 + x + p \cdot \Delta u_{ij}(x)$ over $\mathbb{Z}$; thus $v(x)$ can be uniformly approximated by polynomials of the form $1 + x + p \cdot \Delta u(x) \in \mathbb{Z}[x]$ (where $u(x)\mathbb{Z}[x]$) which are all ergodic since every function $\mathbb{Z}_p \to \mathbb{Z}_p$ of the form $1 + x + p \cdot \Delta d(x)$ is 1-Lipschtz and ergodic, for every 1-Lipschtz $d\colon \mathbb{Z}_p \to \mathbb{Z}_p$. □

Some properties of $\mathcal{C}_p(\mathbb{R})$-functions

- It can be shown that a real-valued continuous function $v(x)$ on $[\alpha, \beta]$ can be uniformly approximated by *ergodic* automata functions.
- One may say that ergodicity of an automaton function $d\colon \mathbb{Z}_p \to \mathbb{Z}_p$ means that the automaton 'behaves like a clock'since for p-adic 1-Lipschitz functions ergodicity is equivalent to the property that for every $n = 1, 2, 3, \ldots$, the sequence of iterates $c_0, c_1 = d(c_0), c_2 = d(c_1) = d^2(c_0), \ldots$ taken modulo p^n, is a purely periodic sequence over residue ring $\mathbb{Z}/p^n\mathbb{Z}$ of rational integers modulo p^n, the length of the period is p^n, and every residue from $\mathbb{Z}/p^n\mathbb{Z} = \{0, 1, \ldots, p^n - 1\}$ occurs at the period exactly once.

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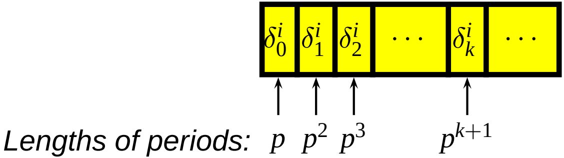
Thus, taking p-adic canonical representation $c_i = \delta_0^i + \delta_1^i \cdot p + \delta_2^i \cdot p^2 + \cdots$ (where $\delta_i^j \in \{0, 1, \ldots, p - 1\}$) one may think of the sequence of iterates as of consecutive readouts of a timer



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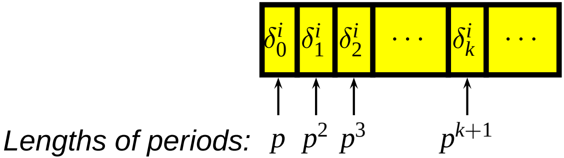

- Only higher order digits δ_i^j where i is many orders larger than Planck's time can be measured; hence *lower order digits play a role of 'hidden variables' whereas the higher order digits exhibit 'chaotic behaviour'*. Postpone rigourous definition of what does 'chaotic behavior' mean but note that a well-known example of chaotic maps, the logistic map $L(x) = rx(x-1)$, is not chaotic 'at Planck distances'.

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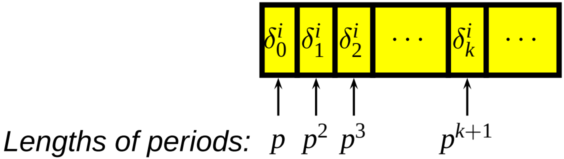

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- For no $r \in \mathbb{Z}_p \cap \mathbb{Q}$ that map is ergodic w.r.t. Haar measure on $\mathbb{Z}_p$, not speaking of mixing.

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No chaotic causal maps at Planck distances

It can be proved that there are no 1-Lipschitz maps $\mathbb{Z}_p \to \mathbb{Z}_p$ which are mixing w.r.t. normalized Haar measure on $\mathbb{Z}_p$; only the ergodic ones exist. Moreover, all ergodic 1-Lipschitz maps are conjugate in the group of all measure-preserving 1-Lipschitz maps $\mathbb{Z}_p \to \mathbb{Z}_p$ to the odometer map $x \mapsto x+1$ and thus all have zero entropy.

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- Although the logistic map $L(x)$ is a p-adic 1-Lipschitz map (thus, an automaton function), the map is a well-known chaotic map on $\mathbb{R}$, despite $L(x) \in \mathcal{C}_p(\mathbb{R})$ if $r \in \mathbb{Z}_p \cap \mathbb{Q}$. *Where does that chaos come from?*

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- In order to answer that question we will adjoin 'time shifts at Planck distances' to causal maps; or, to put it in other words, we will *expand automata functions to the whole field $\mathbb{Q}_p$ of p-adic numbers*.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle

- Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $s \in \mathbb{Q}_p$, let $[s]_p$ and $\{s\}_p = \zeta_{-k}p^{-k} + \cdots + \zeta_{-1}p^{-1}$ be integral and fractional parts of s respectively; put $f(s) = p^{-k}f(p^k s)$. This way we expand f to the whole field $\mathbb{Q}_p$.

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- Such an expansion exploits common physical notion of 'time shift' since $p^{-k}f(p^k s)$ can naturally be treated as 'effect' of causal system (=output of the automaton) to the 'cause' $p^k s \in \mathbb{Z}_p$ (=input of the automaton) for the system which has started to evolve k Planck time units before 'moment zero' when observer starts observations.

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Caution!

Generally, $p^{-n}f(p^n t) \neq f(t)$, for arbitrary 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, $n \in \mathbb{N}$, $z \in \mathbb{Z}_p$ since general automaton feeded by a zero input sequence may nonetheless update its states.

But $p^{-n}f(p^n z) = f(z)$ for all $n \in \mathbb{N}$ and $z \in \mathbb{Z}_p$ if the automaton remains in initial state until it accepts the first non-zero symbol, i.e., '*when in initial state, the system does not evolve until there is no input signal*'.

- Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $s \in \mathbb{Q}_p$, let $[s]_p$ and $\{s\}_p = \zeta_{-k}p^{-k} + \cdots + \zeta_{-1}p^{-1}$ be integral and fractional parts of s respectively; put $f(s) = p^{-k}f(p^k s)$. This way we expand f to the whole field $\mathbb{Q}_p$.
- Recall that any complex character of $\mathbb{Q}_p^+$ is of the form $\chi_r(s) = e^{2\pi i\{sr\}_p}$, where $r \in \mathbb{Q}_p$; χ_r is a continuous group epimorphism into the group of complex roots of unity (which is isomorphic to the group $\mathbb{Q}^+/\mathbb{Z}^+$). We take $r = 1$, denote $\chi_1 = \chi$ and, given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, consider a mapping $\check{f}$ defined as follows:

$$\check{f}\colon \chi(s) \mapsto \chi(f(s)); \text{ that is, } \quad \check{f}\colon e^{2\pi i\{s\}_p} \mapsto e^{2\pi i\{f(s)\}_p} \quad (s \in \mathbb{Q}_p).$$

- The pairs $(e^{2\pi i\{s\}_p}; e^{2\pi i\{f(s)\}_p})$ can be identified with the points on the surface of the torus $\mathbb{T}^2 \subset \mathbb{R}^3$ or, which is actually the same, with the points of the real unit square $\mathbb{I}^2 = \mathbb{I} \times \mathbb{I}$ where $\mathbb{I} = [0,1]$. Denote corresponding point set via $E^2(f)$. Denote $\mathcal{P}^2(f)$ the closure of $E^2(f)$ in $\mathbb{R}^2$; call $\mathcal{P}^2(f)$ a (2-dimensional) plot of the 1-Lipschitz map f.

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- In a similar way, given $\check{f} \in \mathcal{C}_p(\mathbb{R})$, define a mapping

$$\hat{f}\colon e^{2\pi i z} \mapsto e^{2\pi i f(z)} \quad (z \in \mathbb{Z}_p \cap \mathbb{Q}).$$

Here we use characters of $\mathbb{R}^+$ rather than characters of $\mathbb{Q}_p^+$.

- Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $s \in \mathbb{Q}_p$, let $[s]_p$ and $\{s\}_p = \zeta_{-k}p^{-k} + \cdots + \zeta_{-1}p^{-1}$ be integral and fractional parts of s respectively; put $f(s) = p^{-k}f(p^k s)$. This way we expand f to the whole field $\mathbb{Q}_p$.

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  Define $\hat{E}^2(f), \hat{\mathcal{P}}^2(f)$; note that $\hat{\mathcal{P}}^2(f) \subset \mathcal{P}^2(f)$.

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$\mathcal{P}^2$ 'keeps records' of evolution of the system.

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Define $\hat{E}^2(f), \hat{\mathcal{P}}^2(f)$; $\hat{\mathcal{P}}^2(f)$ shows where the system '*finally* occurs'.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle

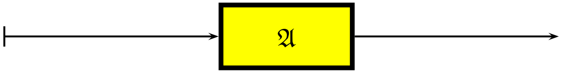
- Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ and $s \in \mathbb{Q}_p$, let $[s]_p$ and $\{s\}_p = \zeta_{-k}p^{-k} + \cdots + \zeta_{-1}p^{-1}$ be integral and fractional parts of s respectively; put $f(s) = p^{-k}f(p^k s)$. This way we expand f to the whole field $\mathbb{Q}_p$.
Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, consider a mapping $\check{f}$ defined as follows:
$$\check{f}\colon e^{2\pi i\{s\}_p} \mapsto e^{2\pi i\{f(s)\}_p} \quad (s \in \mathbb{Q}_p).$$
- The pairs $(e^{2\pi i\{s\}_p}; e^{2\pi i\{f(s)\}_p})$ can be identified with the points on the surface of the torus $\mathbb{T}^2 \subset \mathbb{R}^3$ (or with the points of the real unit square $\mathbb{I}^2$). Denote corresponding point set via $E^2(f)$. Denote $\mathcal{P}^2(f)$ the closure of $E^2(f)$ in $\mathbb{R}^2$; call $\mathcal{P}^2(f)$ a (2-dimensional) plot of the 1-Lipschitz map f. $\mathcal{P}^2$ 'keeps records' of evolution of the system.
- In a similar way, given $\check{f} \in \mathcal{C}_p(\mathbb{R})$, define a mapping
$$\hat{f}\colon e^{2\pi iz} \mapsto e^{2\pi i f(z)} \quad (z \in \mathbb{Z}_p \cap \mathbb{Q}).$$

Define $\hat{E}^2(f), \hat{\mathcal{P}}^2(f)$; call $\hat{\mathcal{P}}^2(f)$a *ultimate* plot of f. Discuss plots in detail.

Given a 1-Lipschitz map $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ (which also can be regarded as an automaton function $f = f_{\mathfrak{A}}$ of an automaton $\mathfrak{A}$) consider a set of all points $e_k^f(x)$ of the Euclidean unit square $\mathbb{I}^2 = [0,1]^2 \subset \mathbb{R}^2$,

$$e_k^f(x) = \left(\frac{x \bmod p^k}{p^k}; \frac{f(x) \bmod p^k}{p^k}\right), (x \in \mathbb{Z}_p, k \in \mathbb{N})$$

Note that $f(x) \bmod p^k$ is merely a k-letter output word that corresponds to the k-letter input word $x \bmod p^k$. Note also that $e_k^f = \left(e^{2\pi i \frac{x}{p^k}}, e^{2\pi i\, f\left(\frac{x}{p^k}\right)}\right) \in \mathbb{T}^2$ according to conventions we made above.



$$x \bmod p^k = \underbrace{\chi_{k-1}\cdots\cdots\chi_1\chi_0 \qquad \xi_{k-1}\cdots\cdots\xi_1\xi_0}_{(0.\chi_{k-1}\ldots\chi_1\chi_0;\, 0.\xi_{k-1}\ldots\xi_1\xi_0)} = f(x) \bmod p^k$$

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Denote via $\alpha_2(f)$ the Lebesgue measure of the closure $\mathcal{P}^2(f)$ (in the topology of $\mathbb{R}^2$) of the set of all points $e_k^f(x)$, where $x \in \mathbb{Z}_p$, $k = 1, 2, 3, \ldots$. The set $\mathcal{P}^2(f)$ is called a 2-dimensional plot of f.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Theorem (The automata 0-1 law; V. A., 2009)

Given a 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, the following alternative holds: either $\alpha_2(f) = 0$, or $\alpha_2(f) = 1$ otherwise

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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$$e^f_{k,n}(x) = \left(\frac{x \bmod p^k}{p^k}; \frac{f(x) \bmod p^k}{p^k}; \ldots; \frac{f^{n-1}(x) \bmod p^k}{p^k}\right), (x \in \mathbb{Z}_p, k \in \mathbb{N})$$

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Given a 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, the following alternative holds: either $\alpha_n(f) = 0$, or $\alpha_n(f) = 1$ otherwise $(n \geq 2)$

These alternatives correspond to the cases $\mathcal{P}_n(f)$ is nowhere dense in $\mathbb{I}^n$ and $\mathcal{P}_n(f) = \mathbb{I}^n$, respectively.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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We will say that a 1-Lipschitz map (resp., an automaton) $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is of measure 1 (in dimension n) if $\alpha_n(f) = 1$, and of measure 0 otherwise.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Theorem (Polynomials of measure 1 in all dimensions; E. Lerner, 2013)

If f is a polynomial over $\mathbb{Z}$ and $\deg f \ge 2$, then $\alpha_n(f) = 1$, for all $n = 2, 3, 4, \ldots$.

- Recall that all polynomials over $\mathbb{Z}$ are $\mathcal{C}_p(\mathbb{R})$-functions!

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Actually a stronger result is true:

Theorem (Ultimate uniform distribution; E. Lerner, 2013)

If f is a polynomial over $\mathbb{Z}$ and $\deg f \geq 2$ then the distribution of points $e^f_{k,n}(x)$ in the unit hypercube $\mathbb{I}^n$ tends to uniform as $k \to \infty$, for every $n \in \{2, 3, 4, \ldots\}$.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Two sources of chaos

- **1-st**: Chaos emerges from infinite ‘chaotic sequences’ like random real numbers by iterating them via Bernoulli-shift-like mappings. That is, when it is assumed *a priory* that ‘chaos do exist immanently’.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Two sources of chaos

- **1-st**: Chaos emerges from infinite 'chaotic sequences' like random real numbers by iterating them via Bernoulli-shift-like mappings.
- **2-nd**: Chaos emerges from the 'lack of knowledge' of 'what causes had happened at the beginning' during short temporal interval in Planck's time units; i.e., since an observer can't determine by measurements what are low-order digits of the input $x \in \mathbb{Z}_p$ of the causal function f which therefore are 'hidden parameters'.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Two sources of chaos

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► Lerner's theorem implies that *iterations of a polynomial $f \in \mathbb{Z}[x]$ which is ergodic w.r.t. normalized Haar measure on $\mathbb{Z}_p$ and s.t. $\deg f \geq 2$ results in a 'mixing' in the unit square $\mathbb{I}^2$*: Points of each closed region of $\mathbb{I}^2$ will be eventually uniformly distributed over $\mathbb{I}^2$ by iterations of f.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Pronounced straight lines (=windings of a torus) may occur in the plots.

- We are going to understand what are the lines.

Firstly note that plots of finite autonomous automata (=constants from $\mathbb{Z}_p \cap \mathbb{Q}$) and plots of finite affine automata (=whose automata functions are $x \mapsto ax + b$, where $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) are *windings of a torus* (=linear flows on torus); i.e., images of straight lines in $\mathbb{R}^2$ under the mapping

$$\operatorname{mod}1\colon (x;y) \mapsto (x \bmod 1; y \bmod 1) \in \mathbb{T}^2.$$

The windings may also be treated as graphs on the torus $\mathbb{T}^2$ of functions $\mathbb{S} \to \mathbb{S}$ on the circle $\mathbb{S}$ of the form $\{(e^{2\pi it}; e^{2\pi i(at+b)})\colon t \in \mathbb{R}\}$.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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The windings may also be treated as graphs on the torus $\mathbb{T}^2$ of functions $\mathbb{S} \to \mathbb{S}$ on the circle $\mathbb{S}$ of the form $\{(e^{2\pi i t}; e^{2\pi i (at+b)})\colon t \in \mathbb{R}\}$.

- Plots of finite automata are nowhere dense subsets of measure 0.
- There exist 1-Lipschitz maps which correspond to no finite automata but whose plots are nowhere dense subsets of zero measure.

In the plots of automata of measure 0 there are no ‘figures’; but ‘lines’ may occur even in the plots of infinite automata of measure 0.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

Both plots of constants from $\mathbb{Z}_p \cap \mathbb{Q}$ and plots of affine automata functions $x \mapsto ax + b$ (where $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) are *windings of a torus* , images of straight lines in $\mathbb{R}^2$ under the mapping

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Theorem (V. A., 2015)

Let $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be an automaton function of a finite automaton; let g be a C^2 real function with domain $[a, b] \subset [0, 1)$ and range $[0, 1)$. Let the graph $\mathbf{G}(g) = \{(x; g(x)) : x \in [a, b]\}$ of g lie completely in $\mathcal{P}^2(f)$. Then there exist $a, b \in \mathbb{Q} \cap \mathbb{Z}_p$ such that $g(x) = (ax + b) \bmod 1$ for all $x \in [a, b]$; moreover, there is a winding of the torus $\mathbb{T}^2$ which lies completely in $\mathcal{P}^2(f)$ and which contains the graph $\mathbf{G}(g)$ of the function g.

There are no more than a finite number of pairwise distinct windings of the unit torus $\mathbb{T}^2$ in $\mathcal{P}^2(f)$; all of these are images of real affine functions $x \mapsto ax + b$ for $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ under the mapping $\bmod 1\colon \mathbb{R}^2 \to \mathbb{T}^2$.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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There are no more than a finite number of pairwise distinct windings of the unit torus $\mathbb{T}^2$ in $\mathcal{P}^2(f)$; all of these are images of real affine functions $x \mapsto ax+b$ for $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ under the mapping $\operatorname{mod} 1 \colon \mathbb{R}^2 \to \mathbb{T}^2$.

- The theorem is true for finite automata functions $f\colon \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ as well.

This also may serve as a manifestation of the fact that linearity of operators of the mathematical apparatus of quantum mechanics emerges from the finiteness of epistemic states of causal functions.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

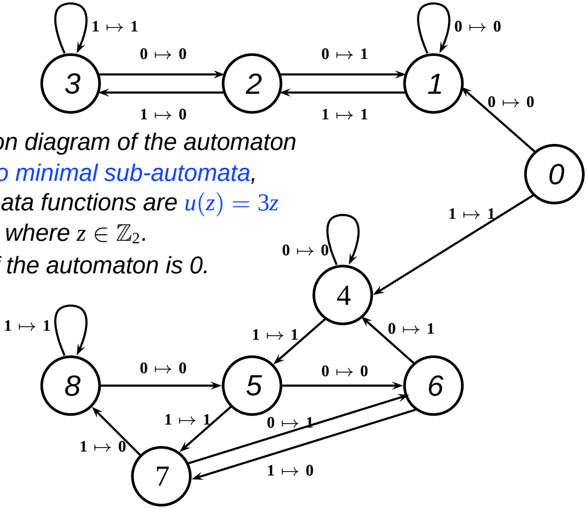
Note that in cylindrical coordinates the winding $t \mapsto \frac{\alpha}{\beta}t + \omega$ of a torus whose axis of rotation is Z can be represented by parametric equations

$$\begin{bmatrix} r_0 \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} R + r\cos\left(\frac{\alpha}{\beta}t + \omega\right) \\ t \\ r\sin\left(\frac{\alpha}{\beta}t + \omega\right) \end{bmatrix}, \ t \in \mathbb{R}.$$

The winding winds β times around Z-axis and $|\alpha|$ times around a circle in the interior of the torus (the sign of α determines whether the rotation is clockwise or counter-clockwise). Therefore 'physical meaning' of the coefficient $a = \frac{\alpha}{\beta}$ of the affine map $z \mapsto az+b$, which is a finite automaton function of affine automaton, is **frequency**. The choice of sign $+$ or $-$ depends only on what direction of rotation is assumed to be 'positive' or 'negative'. To the map it correspond functions $\mathbb{R} \to \mathbb{C}$

$$\psi_k(y) = e^{i(\frac{\alpha}{\beta}y - 2\pi p^k b)} = e^{2\pi i(\frac{\alpha}{2\pi\beta}y - p^k b)} \quad (k = 0, 1, 2, \ldots)$$

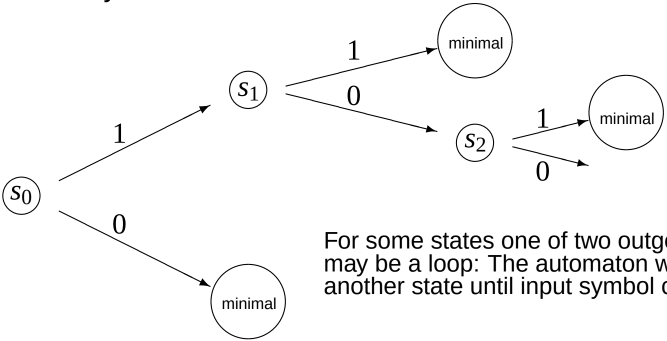
- Note that all the windings are automata functions of *minimal* automata; i.e., the automata whose state diagram is connected: Given two (epistemic) $s, t \in \mathcal{S}$, there is finite word w in the alphabet $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ such that when the automaton in state s accepts w, the automaton changes its state to t. If an automaton reaches a state which belongs to its minimal sub-automaton, the automaton will never get out of the sub-automaton.



State transition diagram of the automaton which has two minimal sub-automata, whose automata functions are $u(z) = 3z$ and $v(z) = 5z$ where $z \in \mathbb{Z}_2$.
Initial state of the automaton is 0.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

State transition diagram of finite automaton may be thought of as a tree each path in which ends with a minimal sub-automaton. There are no outgoing paths from any minimal sub-automaton. By feeding the automaton with random long words, to each minimal sub-automaton we assign a probability when the automaton occurs in states belonging to a minimal sub-automaton. If the sub-automaton is affine, we therefore assign a probability to its automaton function, thus to $e^{2\pi i(\xi x+b)}$.



For some states one of two outgoing arrows may be a loop: The automaton won't move to another state until input symbol changed

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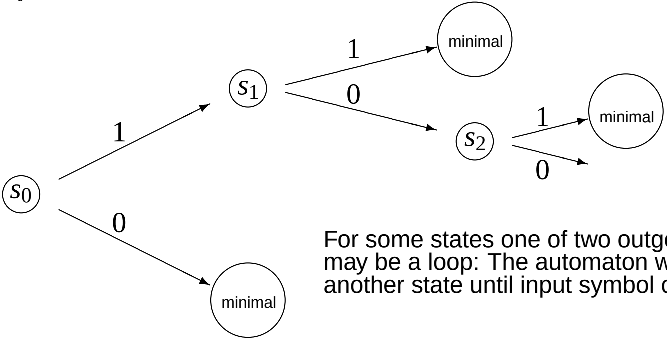

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- Any automaton function (=a 1-Lipschitz function $\mathbb{Z}_p \to \mathbb{Z}_p$) can be uniformly approximated on $\mathbb{Z}_p$ by finite automata functions.

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For $f \in \mathcal{C}_p(\mathbb{R})$, consider a sequence $(f_n)_{i=0}^{\infty}$ of *finite* automata functions such that $f(z) \equiv f_n(z) \pmod{p^n}$ for all $z \in \mathbb{Z}_p \cap \mathbb{Q}$ (i.e., for all z from the residue ring $\mathbb{Z}/p^n\mathbb{Z}$ since both f and f_n are 1-Lipschitz), $n = 1, 2, 3, \ldots$ and having assigned the sum $S_n = \sum_{(\xi)} P_{\xi,n} e^{-2\pi i\xi x}$ to every f_n, we by sending $n \to \infty$ assign to $f \in \mathcal{C}_p(\mathbb{R})$ the *distribution* (as $P(\xi)$ is a distribution)

$$\int_{\mathbb{R}} P(\xi) e^{-2\pi i x\xi} d\xi$$

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Whenever a finite automaton is represented by its state transition diagram, it is possible to find the integral explicitly by using Dirac's δ.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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Whenever a finite automaton is represented by its state transition diagram, it is possible to find the integral explicitly by using Dirac's δ. For instance, for the mentioned example automaton the integral is

$$\int_{\mathbb{R}} \left(\frac{1}{2}\delta\left(\xi - \frac{3}{2\pi}\right) + \frac{1}{2}\delta\left(\xi - \frac{5}{2\pi}\right)\right) e^{-2\pi i x \xi} d\xi = \frac{1}{2}\left(e^{-2\pi i x \frac{3}{2\pi}} + e^{-2\pi i x \frac{5}{2\pi}}\right) = \frac{1}{2}\left(e^{-3ix} + e^{-5ix}\right)$$

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

For $f \in \mathcal{C}_p(\mathbb{R})$, assign to $f \in \mathcal{C}_p(\mathbb{R})$ the *distribution* (as $P(\xi)$ is a distribution)

$$\int_{\mathbb{R}} P(\xi) e^{-2\pi i x \xi} d\xi$$

Whenever a finite automaton is represented by its state transition diagram, it is possible to find the integral explicitly by using Dirac's δ.

$$\int_{\mathbb{R}} P(\xi) e^{-2\pi i x \xi} d\xi = \sum \frac{\#\mathrm{Path}_{f_{\mathfrak{A}}}(\xi)}{\#\mathrm{Path}_{f_{\mathfrak{A}}}} e^{-2\pi \xi x}$$

where $\#\mathrm{Path}_{f_{\mathfrak{A}}}(\xi)$ is the number of paths leading to minimal sub-automata of the automaton $\mathfrak{A}$ whose automaton function is $z \mapsto \frac{\xi}{2\pi} z + b$ for some $b \in \mathbb{Z}_p \cap \mathbb{Q}$; and $\#\mathrm{Path}_{f_{\mathfrak{A}}}$ is total number of paths leading to minimal sub-automata.

- Note that to every such path $\mathrm{path}(\xi)$ of length N it corresponds input sequence $\chi_0, \chi_1, \ldots, \chi_{N-1}$ over a p-symbol alphabet; i.e., the integer $\mathrm{num}(\xi) = \chi_0 + \chi_1 + \cdots + \chi_{N-1} p^{N-1}$.

$\mathcal{C}_p(\mathbb{R})$-functions on a circle: Details

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- In general case, every probability $P(\xi)$ is Haar measure of all $z \in \mathbb{Z}_p$ s.t. $z \bmod p^N = \mathrm{num}(\xi)$ for some path of length N which to minimal sub-automaton whose automaton function is $z \mapsto 2\pi\xi z + b$ since given ξ, all that z constitute a union of balls in $\mathbb{Z}_p$.

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- Expanding the distribution for all $\xi \in \mathbb{R}$ rather than only for s.t. $2\pi\xi \in \mathbb{Z}_p \cap \mathbb{Q}$, we put into correspondence to the automaton the element $\int_{\mathbb{R}} P(\xi)e^{-2\pi i x\xi}d\xi$ of Hilbert space.

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Basically that's only an illustration of the approach; many details are omitted.

1. p-adic 1-Lipschitz maps=causal maps=automata maps
2. Expanding automata maps from $\mathbb{Z}_p \cap \mathbb{Q}$ to $\mathbb{R}$
3. Some properties of automata maps on $\mathbb{R}$
4. Automata maps on a circle
5. **Some physics**
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The uncertainty principle can be stated in terms of the function g from the Schwartz class $\mathcal{S}(\mathbb{R})$ and its Fourier transform $\hat{g}$.

Theorem (H. Weyl, 1931)

Let $g \in \mathcal{S}(\mathbb{R})$, *let* $||g||_2 = 1$, $\hat{g}(\xi) = \int_{\mathbb{R}} g(x)e^{-2\pi i x\xi}dx$. *Put*

$$\mu = \int_{\mathbb{R}} x|g(x)|^2 dx, \quad \sigma = \int_{\mathbb{R}} (x-\mu)^2 |g(x)|^2 dx$$
$$\hat{\mu} = \int_{\mathbb{R}} \xi|\hat{g}(\xi)|^2 d\xi, \quad \hat{\sigma} = \int_{\mathbb{R}} (\xi-\hat{\mu})^2 |\hat{g}(\xi)|^2 d\xi;$$

then

$$\sigma\hat{\sigma} \geq \frac{1}{4\pi}.$$

Some ‘physical’ properties of $\mathcal{C}_p(\mathbb{R})$-functions

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If $f \in \mathcal{C}_p(\mathbb{R})$ then $f \notin \mathcal{S}(\mathbb{R})$; we may try to embed f into $S(\mathbb{R})$ via, e.g, the mapping $S\colon f(x) \to f(x)e^{-x^2}$ since if, additionally, $f \in \mathbb{Z}[x]$ then $S(f) \in \mathcal{S}(\mathbb{R})$. Complete description of $\mathcal{C}_p(\mathbb{R})$-functions f s.t. $S(f) \in \mathcal{C}_p(\mathbb{R})$ is not known.

Some ‘physical’ properties of $\mathcal{C}_p(\mathbb{R})$-functions

‘Standard’ operations used in mathematical formalism of QM can be either interpreted in terms of automata or can be expanded to automata functions and to $\mathcal{C}_p(\mathbb{R})$-functions.

- Sequential composition of automata, when output sequence of the automaton $\mathfrak{A}$ is input sequence of automaton $\mathfrak{B}$ resultis an automaton whose automaton function is $f_{\mathfrak{B}}(f_{\mathfrak{A}})$.

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- Direct product of automata: An automaton whose automaton function is standard direct product of automata functions of components. Input/output alphabets are direct product of respective alphabets of components; action is componentwise.

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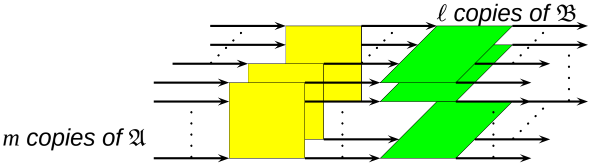
- Direct product of automata: An automaton whose automaton function is standard direct product of automata functions of components.
- Tensor product of automata. Given two automata whose automata functions are resp. $f_{\mathfrak{A}}\colon \mathbb{Z}_p^k \to \mathbb{Z}_p^\ell$, $f_{\mathfrak{B}}\colon \mathbb{Z}_p^m \to \mathbb{Z}_p^n$, the result is an automaton whose automaton function is $f_{\mathfrak{A}} \bigotimes f_{\mathfrak{B}}\colon \mathbb{Z}_p^{km} \to \mathbb{Z}_p^{\ell n}$



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► Note that for $\mathcal{C}_p(\mathbb{R})$-functions all above operations can be used to construct $\mathcal{C}_p(\mathbb{R}^r)$-functions. Note also that *skew shift is known in ergodic theory and is often used to construct a dynamical system which is controlled by another dynamical system: 1-Lipschtz ergodic functions* $\mathbb{Z}_p \to \mathbb{Z}_p$ *are all exactly of that nature*.

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► It can be proved that the plot of any automaton function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is *self-similar*: There exists a conformal bijection of any 'strip'

$$\{(x; f(x) \bmod 1) : x \in [\alpha, \beta] \subset [0, 1]\} \subset \mathbb{T}^2$$

onto the whole torus $\mathbb{T}^2$.

What p?

- At the moment, $p = 2$ is looking quite suitable since it is a distinguished case by the following reasons:
  - A 1-Lipschitz map $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is measure-preserving if and only if $f(x) = c + x + 2 \cdot g(x)$ where $g\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is a 1-Lipschitz map, $c \in \mathbb{Z}_2$.
  - A 1-Lipschitz map $f\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is ergodic if and only if $f(x) = 1 + x + 2 \cdot \Delta g(x)$ where $g\colon \mathbb{Z}_2 \to \mathbb{Z}_2$ is a 1-Lipschitz map.

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- However, since there are more than one Planck unit, it may happen that to make the model consistent, we must use several different prime numbers rather than only one.
- To that extend note that there exist automata which are letter-to-letter transducers for any set of primes, even for all primes: The latter are exactly the ones whose automata functions are as follows:

$$f(x) = a_0 + \sum_{i=1}^{\infty} a_i \cdot \operatorname{lcm}(1, 2, \dots, i) \cdot \binom{x}{i};$$

where all $a_i \in \mathbb{Z}$. B.t.w., these may lead to the automaton interpretation of QM over adeles rather than over $\mathbb{Z}_p$.

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where all $a_i \in \mathbb{Z}$. B.t.w., these may lead to the automaton interpretation of QM over adeles rather than over $\mathbb{Z}_p$.
- All that shows that the model we are discussing can be adjusted if necessary. The most 'physically meaningful' adjustment would be to include 'phase shifts' into the model aiming to deduce 'wave functions'.

Other automata interpretations of QM

The idea to apply automata to construct interpretations of QM has been already exploited by some other researches. But automata they use are *cellular*.

- G. ’t Hooft used *cellular automata* for that purpose, see his monograph ‘The Cellular Automaton Interpretation of Quantum Mechanics’, Springer, 2016.
- S. Wolfram with co-workers apply *cellular automata* to construct what he called a ‘new physics’; the approach which emerges from his book ‘A new Kind Of Science’, Wolfram Media, 2002.

► Cellular automata are much more powerful computers compared to automata (=sequential machines) we are discussing. We briefly recall the hierarchy of automata by their ‘computational power’.

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1. The automata considered in the talk are *synchronous* automata (=letter-to-letter transducers, sequential machines). The automata are <u>the weakest</u> computers of all automata; the *functions they perform constitute the class of <u>all 1-Lipschitz maps</u>* $\mathbb{Z}_p \to \mathbb{Z}_p$.

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2. More powerful are *asynchronous* automata (letter-to-word transducers). *Functions they perform constitute the class of all continuous maps* $\mathbb{Z}_p \to \mathbb{Z}_p$.
3. Cellular automata are equivalent to Turing machines. *They can perform any algorithm; functions* $\mathbb{Z}_p \to \mathbb{Z}_p$ *they perform are not even defined everywhere on* $\mathbb{Z}_p$.

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Last but not least

Felix Klein (1925)

The mathematics of our day seems to be like a great weapons factory in peace time. The show window is filled with parade pieces whose ingenious, skillful, eye-appealing execution attracts the connoisseur. The proper motivation for and purpose of these objects, to battle and conquer the enemy, has receded to the background of consciousness to the extent of having been forgotten.

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I dare to say that concerning what is discussed in the talk, this is not the case: Leaving aside whether it is of some importance for QM, the developed machinery is already being used in cryptography and may probably be used in 'digital economy' since *smart contracts are finite automata interacting in physical time*. The latter piece of work was supported by Russian Foundation for Basic Research grant No 18-20-03124.

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- As the set $\mathbb{Z}_p \cap \mathbb{Q}$ of rational p-adic integers is dense both in $\mathbb{Z}_p$ and $\mathbb{R}$, there exists a vast class $\mathcal{C}_p(\mathbb{R})$ of functions which are both 1-Lipschitz maps $\mathbb{Z}_p \to \mathbb{Z}_p$ and continuous functions $\mathbb{R} \to \mathbb{R}$; thus the class $\mathcal{C}_p(\mathbb{R})$ can be investigated by combining three tools simultaneously: Real/complex analysis, p-adic analysis, and automata theory.

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- The class $\mathcal{C}_p(\mathbb{R})$ has the following notable properties:
  - Functions from $\mathcal{C}_p(\mathbb{R})$ are completely defined by values they take on any (arbitrarily small) real interval (α, β): Given $g, f \in \mathcal{C}_p(\mathbb{R})$, if $g(x) = f(x)$ for all $x \in (\alpha, \beta)$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$ and $f(z) = g(z)$ for all $z \in \mathbb{Z}_p$.

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    - The class $\mathcal{C}_p(\mathbb{R})$ contains all polynomials over $\mathbb{Z} \cap \mathbb{Q}$. Therefore any continuous function $\mathbb{R} \to \mathbb{R}$ can be uniformly approximated on any real interval by polynomials from $(\mathbb{Z}_p \cap \mathbb{Q})[x]$ as well as any 1-Lipschitz (=automaton) function $\mathbb{Z}_p \to \mathbb{Z}_p$ can be uniformly approximated by polynomials from $(\mathbb{Z}_p \cap \mathbb{Q})[x]$.

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- Mathematical proofs are based on the only physical assumption that physical world at Planck distances is discrete and causal.

Publications

Some (far not all) results mentioned in the talk may be found in:

V. Anashin and A. Khrennikov. *Applied Algebraic Dynamics*, Walter de Gruyter GmbH & Co., Berlin—N.Y., 2009.

V. S. Anashin. *Quantization causes waves: Smooth finitely computable functions are affine*. p-Adic Numbers, Ultrametric Analysis Appl.., 7(3):169–227, 2015.

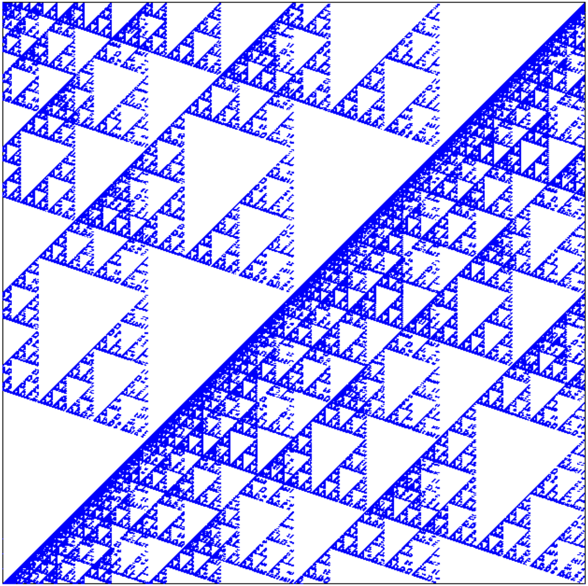
V. Anashin. *Discreteness causes waves*. Facta Universitatis, 14(6):143–196, 2016.

V. Anashin. *The non-Archimedean theory of discrete systems*. Math. Comp. Sci., 6(4):375–393, 2012.

V. Anashin. *The p-adic theory of automata functions* (preprint). https://www.researchgate.net/publication/345733138_THE_p-ADIC_THEORY_OF_AUTOMATA_FUNCTIONS

The rest of results are under preparation for publication.

Thank you!



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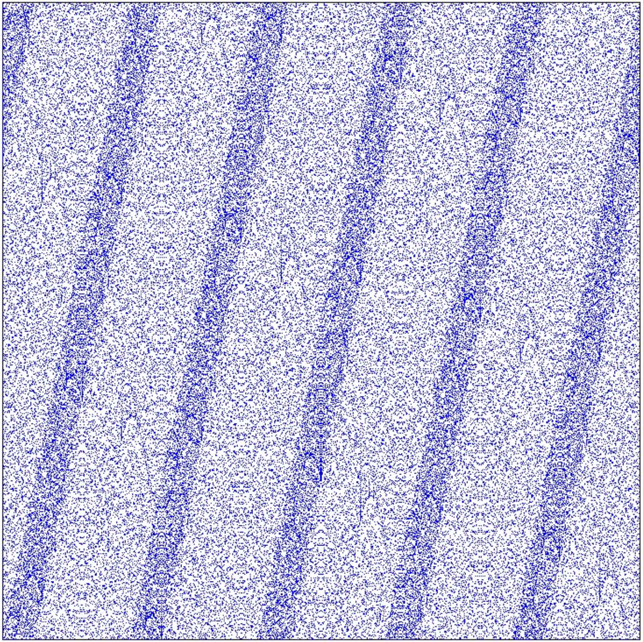

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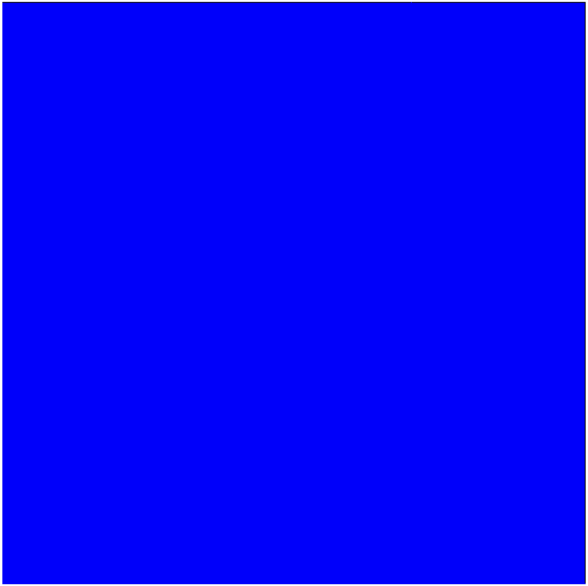


$p = 2$: $f(x) = 1 + x + 4x^2$; $\alpha(f) = 1$.

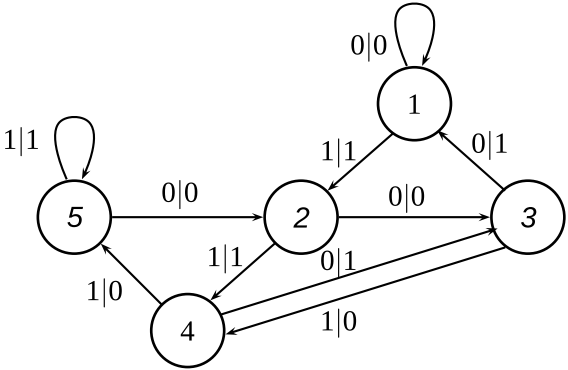


Figure: Example state diagram of a 5-state automaton with two-letter input/output alphabets $\{0, 1\}$. Initial state is 1. The automaton function is multiplication by 5 of numbers represented in base-2 expansion.

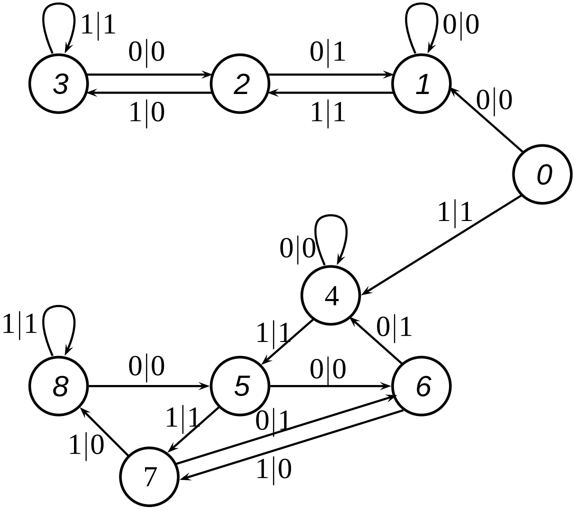
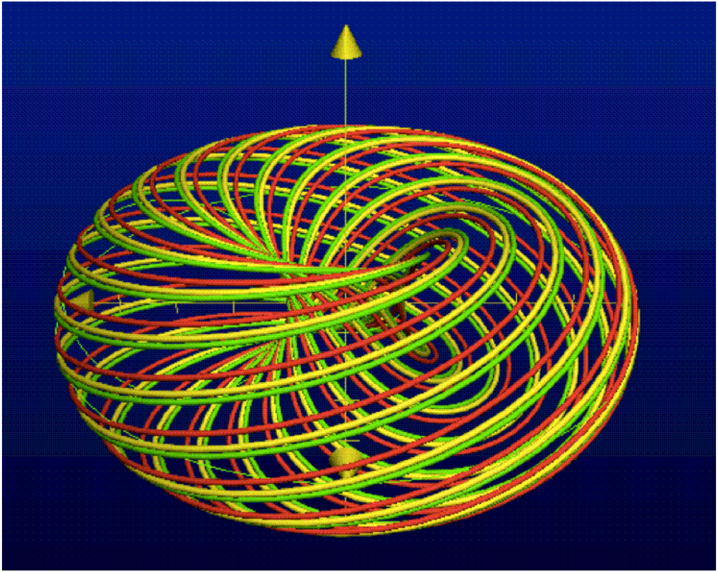
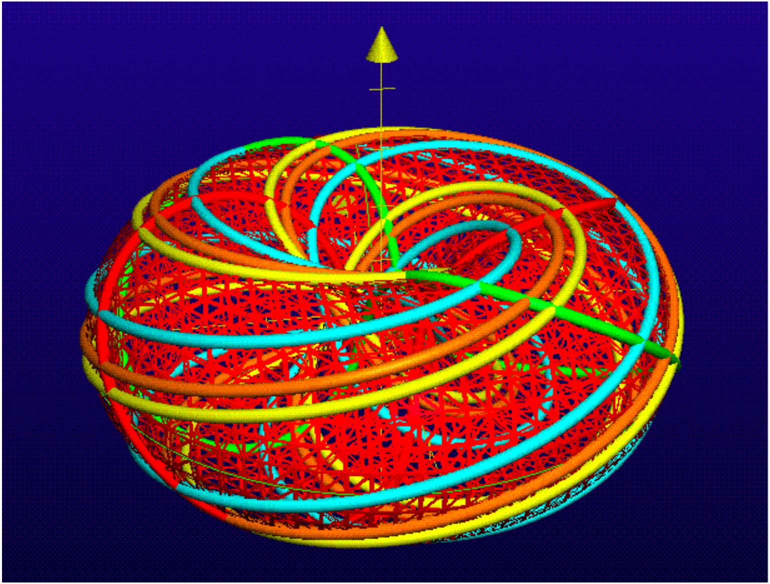


Figure: Example state diagram of an automaton with two minimal sub-automata. Initial state is 0.

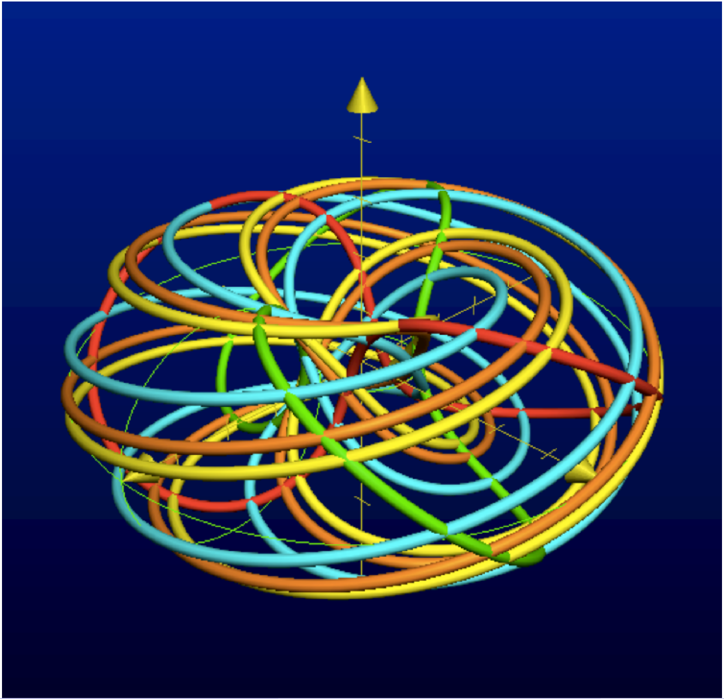


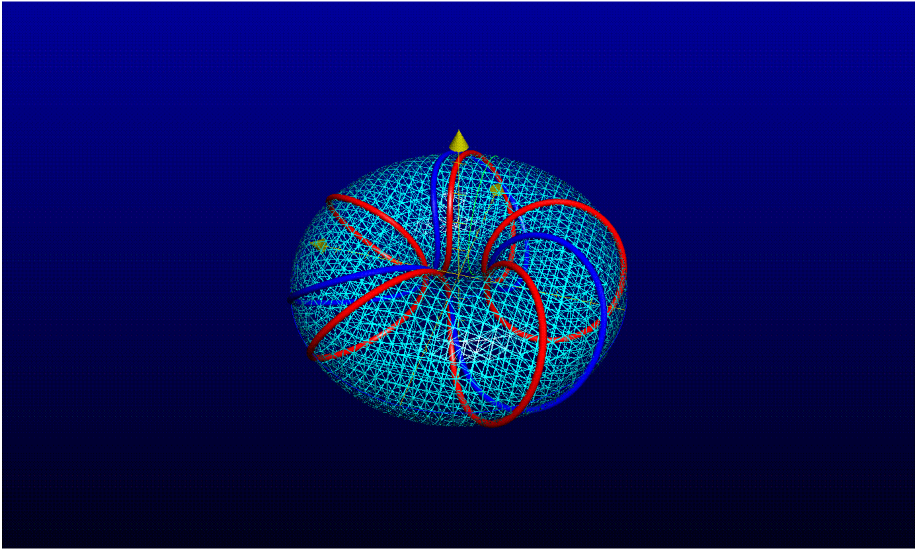
$f(z) = \frac{11}{15}z + \frac{1}{21}$, $p = 2$.

As $\gcd(15, 21) = 3$, the multiplicative order of 2 modulo $\frac{21}{\gcd(15,21)} = \frac{21}{3} = 7$ is 3; this is the number of windings in the torus link.

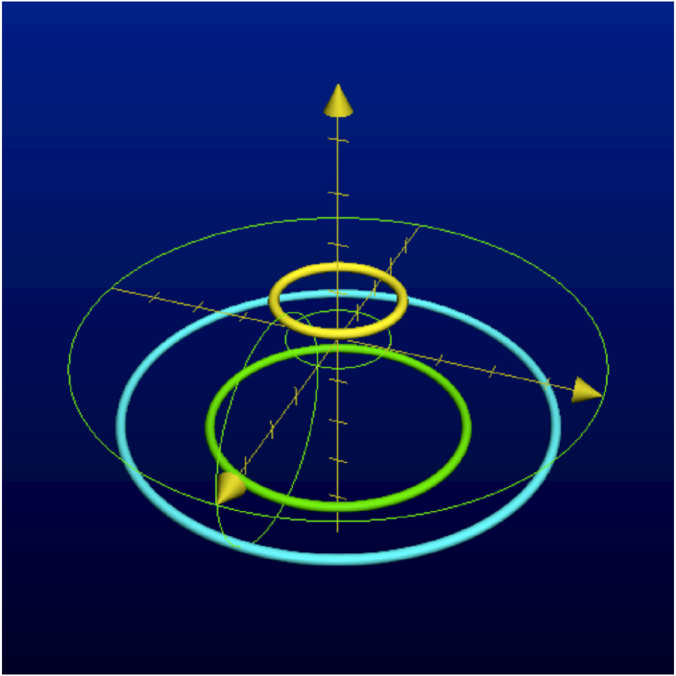


$f_1(z) = -2z + \frac{1}{3}$; $f_2(z) = \frac{3}{5}z + \frac{2}{7}$, $(p = 2)$.





$$u(z) = 3z, v(z) = 5z \quad (z \in \mathbb{Z}_2)$$



$f(z) = \frac{2}{7}, p = 2$