On non-Archimedean valued fields: a survey of algebraic, topological and metric structures, analysis and applications

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Outline for Section 2

Valued Fields

Valuation: Let K be a field. A valuation on K is a map $|\cdot| : K \to [0, \infty)$ that satisfies the following properties

1. $|a| = 0$ if and only if $a = 0$;
2. $|ab| = |a|\,|b|$ for all $a, b \in K$;
3. $|a + b| \leq |a| + |b|$ for all $a, b \in K$ (triangle inequality).

The pair $(K, |\cdot|)$ is called a valued field which, for simplicity, will be denoted by K.

The Value Group: The set $|K^*| = \{|a| : a \in K^*\}$, where $K^* = K \setminus \{0\}$, is a subgroup of the multiplicative group of positive real numbers; it is called the value group of K.

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Non-Archimedean Valued Fields

Let $K \equiv (K, |\cdot|)$ be a valued field.

Definition: We say that K is non-Archimedean if the set $\{n.1 : n \in \mathbb{N}\} := \{1, 1+1, 1+1+1, \ldots\}$ is bounded in K, i.e.

$$\sup_{n\in\mathbb{N}}\{|n.1| : n \in \mathbb{N}\} < \infty.$$

Otherwise, we say that K is Archimedean.

Theorem: The following are equivalent

1. K is non-Archimedean;
2. $|a+b| \le \max\{|a|, |b|\}$ for all $a, b \in K$ (the strong triangle inequality);
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Outline for Section 3

Metric Spaces

A metric on a set X is a map $\Delta : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$

(i) $\Delta(x, y) = 0$ if and only if $x = y$;

(ii) $\Delta(x, y) = \Delta(y, x)$;

(iii) $\Delta(x, z) \leq \Delta(x, y) + \Delta(y, z)$ (triangle inequality).

The pair $(X, \Delta) \equiv X$ is called a metric space.

For $a \in X$, and $r > 0$ in $\mathbb{R}$ we set

$$\begin{aligned} B(a, r) &:= \{x \in X : \Delta(x, a) \leq r\} \text{ and} \\ B(a, r^-) &:= \{x \in X : \Delta(x, a) < r\}. \end{aligned}$$

A subset $U \subset X$ is called open if for each $a \in U$ there exists an $r > 0$ in $\mathbb{R}$ such that $B(a, r^-) \subset U$.

The collection of open sets forms a topology on X which is called the topology induced by Δ.

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Ultrametric Spaces

Definition: The metric Δ is said to be an ultrametric [and (X, Δ) an ultrametric space] if it satisfies the strong triangle inequality

$$\Delta(x,z) \leq \max\{\Delta(x,y), \Delta(y,z)\} \ \forall x,y,z \in X.$$

Theorem: Let (X, Δ) be a metric space. Then Δ is an ultrametric if and only if it satisfies the **Isosceles Triangle Principle**:

For all $x, y, z \in X$

$$\Delta(x,y) \neq \Delta(y,z) \Rightarrow \Delta(x,z) = \max\{\Delta(x,y), \Delta(y,z)\}.$$

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Non-Archimedean Valued Fields $\leftrightarrow$ Ultrametric Spaces

Let $K \equiv (K, |\cdot|)$ be a valued field.

- The map $\Delta : K \times K \to [0, \infty),\ (a, b) \mapsto |a - b|$, is a metric on K that induces a topology on K and makes K a topological field. We say that $(K, |\cdot|)$ is complete if it is complete with respect to the metric Δ.
- If K is Archimedean and complete then K is topologically isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Thus, almost all complete valued fields are non-Archimedean.
- Assume $(K, |\cdot|)$ is a non-Archimedean valued field. Then
  - $(K, |\cdot|)$ is an ultrametric space; i.e. the metric induced by $|\cdot|$ satisfies the strong triangle inequality.
  - We have this way all examples of ultrametric spaces since each ultrametric space can isometrically be embedded into a non-Archimedean valued field.

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- Assume $(K, |\cdot|)$ is a non-Archimedean valued field. Then
    - $(K, |\cdot|)$ is an ultrametric space; i.e. the metric induced by $|\cdot|$ satisfies the strong triangle inequality.
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Non-Archimedean Valued Fields $\leftrightarrow$ Ultrametric Spaces

Let $K \equiv (K, |\cdot|)$ be a valued field.

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Some of the Unusual Properties of Ultrametric Spaces

Let (X, Δ) be an ultrametric space.

- Each point of a ball is a center.
- Each ball in X is both open and closed ('clopen') and has an empty boundary.
- Two balls are either disjoint, or one is contained in the other.
  - If two balls B_1 and B_2 are disjoint, then

$$\text{dist}(B_1, B_2) = \Delta(x, y) \text{ for each } x \in B_1, y \in B_2$$

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(X, Δ) is totally disconnected.

- There are no new values of an ultrametric after completion.
- A sequence $(x_n)_n$ in X is Cauchy if and only if $\lim\limits_{n\to\infty} \Delta(x_n, x_{n+1}) = 0$.

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- **A student's dream come true**:

Given $a_1, a_2, \ldots$ in a complete ultrametric space (non-Archimedean valued field) K, then

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Spherical completeness

Definition: An ultrametric space is called spherically complete if each nested sequence of balls has a non-empty intersection.

Remark: The concept of spherical completeness plays a key role as a necessary and sufficient condition for the validity of the Hahn-Banach theorem in the non-Archimedean context.

A spherically complete ultrametric space is Cauchy complete, but the converse is not always true. Nevertheless, the following lemma is a partial converse.

Lemma: Suppose that (X, Δ) is a Cauchy complete ultrametric space. If 0 is the only accumulation point of the set $\Delta(X \times X)$ then (X, Δ) is spherically complete.

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Two Important Attributes of Spherically Complete Ultrametric Spaces

1. A stronger version of the fixed point theorem: every shrinking map of a spherically complete ultrametric space has a unique fixed point.

2. Existence of best approximations: Let $Y \neq \emptyset$ be a spherically complete ultrametric space embedded in an ultrametric space X. Then each $x \in X$ has a best approximation in Y, i.e. $\min\{\Delta(y, x) : y \in Y\}$ exists.

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Outline for Section 4

The p-adic Fields

Let p be a prime number.

- The p-adic valuation on $\mathbb{Q}$ is determined by

$$|a|_p = p^{-r} \text{ if } a = \frac{m}{n} p^r \text{ and } m, n \text{ not divisible by } p.$$

- The completion of $(\mathbb{Q}, |\cdot|_p)$ is called $(\mathbb{Q}_p, |\cdot|_p)$, the field of the p-adic numbers. Its value group is $\{p^n : n \in \mathbb{Z}\}$.
- $\mathbb{Q}_p$ is locally compact and hence spherically complete; also separable.

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- $\mathbb{Q}_p$ is not algebraically closed. $|\cdot|_p$ can be extended uniquely to the algebraic closure $\mathbb{Q}_p^a$; and the completion of $\left(\mathbb{Q}_p^a, |\cdot|_p\right)$ is called $\mathbb{C}_p$, the field of the p-adic complex numbers.
- $\mathbb{C}_p$ is no longer locally compact, but separable and algebraically closed. Its value group is

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so the valuation is dense.
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Ordered Fields

Let K be an ordered field.

- For $x, y \in K^*$, we say that x and y are *comparable* and we write $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $|x|_0 < n|y|_0$ and $|y|_0 < m|x|_0$, where

$$|a|_0 := \max\{a, -a\} = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0\,. \end{cases}$$

- $\sim$ is an equivalence relation on K^*. The equivalence class of $x \in K^*$ is denoted by $[x]$; and the set of all the equivalence classes (aka Archimedean classes) is denoted by G_K.

- G_K is an ordered abelian group under the order $\prec$ and addition $+$ defined as follows: for every $x, y \in K^*$,
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- **Definition**: Let E/K be an extension of ordered fields. The field E is an *Archimedean extension of* K if every $x \in E$ is comparable to some $y \in K$. In that case, G_E and G_K are isomorphic ordered groups. An ordered field K is called *Archimedean complete* if it has no proper Archimedean extension fields.
- **Definition**: Let K be an ordered field. If G is an ordered abelian group isomorphic to G_K, then we say that K *is of type* G and G is called an *Archimedean group of* K.
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General Hahn Fields

- **Theorem**: Let K be a field (not necessarily ordered) and G an ordered abelian group. The set

$$K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\},$$

where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows:

1. $(f+g)(x) := f(x) + g(x)$,
2. $fg(x) := \sum_{a+b=x} f(a)g(b)$.

- Fields of the form $K((G))$ are called *general Hahn fields*.
- When K is an ordered field we can define an order on $K((G))$.

Definition: Let K be an ordered field and consider $\lambda : K((G))^* \to G$, $\lambda(f) = \min\{supp(f)\}$. For $f, g \in K((G))$ we define:

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Then $(K((G)), \leq)$ is an ordered field.

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Definition: Let K be an ordered field and consider $\lambda : K((G))^* \to G$, $\lambda(f) = \min\{supp(f)\}$. For $f, g \in K((G))$ we define:

$$f < g \Leftrightarrow f \neq g \text{ and } (f - g)(\lambda(f - g)) < 0.$$

Then $(K((G)), \leq)$ is an ordered field.

General Hahn Fields

- **Theorem**: Let K be a field (not necessarily ordered) and G an ordered abelian group. The set

$$K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\},$$

where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows:

1. $(f+g)(x) := f(x) + g(x)$,
2. $fg(x) := \sum_{a+b=x} f(a)g(b)$.

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The next two results are the main features of the general Hahn fields as ordered fields and mimic the relation between $\mathbb{R}$ and other ordered Archimedean fields.

1. **Hahn's Embedding Theorem**: If K is an ordered field, then for every Archimedean group G of K, there exists an order-preserving field monomorphism σ from K into $\mathbb{R}((G))$ such that $\mathbb{R}((G))$ is an Archimedean extension of $\sigma(K)$.
2. **Hahn's Completeness Theorem**: If G is an ordered abelian group then the field $\mathbb{R}((G))$ is (up to isomorphism) the only Archimedean complete, ordered field of type G.

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Hahn Fields

Definition: A *Hahn field* is a general Hahn field $K((G))$ for which G is a subgroup of $(\mathbb{R}, +)$ and K is any field.

Theorem: Let G be a subgroup of $(\mathbb{R}, +)$ and K any field. If the map $|\ \ |: K((G)) \to \mathbb{R}$ is defined by

$$|f| := \begin{cases} e^{-\lambda(f)} & \text{if } f \neq 0 \\ 0 & \text{if } f = 0, \end{cases}$$

then $(K((G)), |\ \ |)$ is a spherically complete non-Archimedean valued field with residue class field isomorphic to K and value group $|K((G))^*| = \{e^g \in \mathbb{R} : g \in G\}$.

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Levi-Civita Fields

Let K be any field and let G be a subgroup of $(\mathbb{R}, +)$. Then

- $$L[G,K] := \{f : G \to K \,|\, supp(f) \cap (-\infty, n] \text{ is finite for every } n \in \mathbb{Z}\}$$

  is a subfield of $K((G))$.
- When we restrict the valuation of $K((G))$ to $L[G,K]$, the latter becomes a Cauchy complete, non-Archimedean valued field with residue class field isomorphic to K and value group $|L[G,K]^*| = \{e^g : g \in G\}$.
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Lemma: Let K be a field and let $d : \mathbb{Q} \to K$ be the function defined by

$$d(x) := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases}$$

Then d is an element of the field $L[\mathbb{Q}, K]$; and for any $r \in \mathbb{Q}$, we have that

$$d^r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{if } x \neq r. \end{cases}$$

Every nonzero element $f \in L[\mathbb{Q}, K]$ is the sum of a convergent generalized power series with respect to the valuation on $L[\mathbb{Q}, K]$, specifically:

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Theorem: Let K be any field and G a subgroup of $(\mathbb{R}, +)$. Then

1. The following are equivalent:
   - $K((G))$ and $L[G, K]$ coincide.
   - G is discrete.
   - $L[G, K]$ is spherically complete.

2. If K is an ordered field, then $K((G))$ is an Archimedean extension of $L[G, K]$ with respect to the order. If, in addition, K is Archimedean then both $K((G))$ and $L[G, K]$ are of type G.

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Some general Hahn fields are real closed.

- If K is a field and G an ordered abelian group, then $K((G))$ is real closed if and only if K is real closed and G is divisible.
- $L[\mathbb{Q}, K]$ is real closed if and only if K is real closed.

$$\Downarrow$$

The Hahn field $\mathbb{R}((\mathbb{Q}))$ and the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}]$ are real closed.

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The Hahn field $\mathbb{R}((\mathbb{Q}))(i) = \mathbb{C}((\mathbb{Q}))$ and the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}](i) = L[\mathbb{Q}, \mathbb{C}]$ are algebraically closed.

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Outline for Section 5

Let $\mathscr{R} := L[\mathbb{Q}, \mathbb{R}]$ and $\mathscr{C} := L[\mathbb{Q}, \mathbb{C}]$.

- $\mathscr{R}$ is the smallest Cauchy complete and real closed non-Archimedean field extension of $\mathbb{R}$.
  - It is small enough so that the $\mathscr{R}$-numbers can be implemented on a computer, thus allowing for computational applications.
- $\mathscr{C}$ is the smallest Cauchy complete and algebraically closed non-Archimedean field extension of $\mathbb{C}$.

Uniqueness of the Levi-Civita Fields $\mathscr{R}$ and $\mathscr{C}$

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- Topological Structure (Valuation topology and a weaker topology)
- Power Series and Analytic Functions
- Calculus on $\mathscr{R}$
- Measure Theory and Integration on $\mathscr{R}$, $\mathscr{R}^2$ and $\mathscr{R}^3$
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Differentiation of Computer Functions

Problem: The need for differentiation tools arises in many fields of science. Usually, formula manipulators like Mathematica do not work everywhere. For example,

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Solution: Using the calculus on $\mathscr{R}$, we formulate a necessary and sufficient condition for the derivatives of functions representable on a computer to exist, and show how to find these derivatives whenever they exist.

Definition (Computer Functions): A function $f : \mathbb{R} \to \mathbb{R}$ is called a computer function if it can be obtained from intrinsic functions and the Heaviside function through a finite number of arithmetic operations and compositions.

Definition: Let f be a computer function, let $x_0 \in \mathbb{R}$ be in the domain of f, and let $s \in \mathscr{R}$. Then f is extendable to $x_0 + s$ means $x_0 + s$ belongs to the domain of $\bar{f}$, the continuation of f to $\mathscr{R}$

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Theorem (Standard Form of Computer Functions): Let f be a real computer function with domain of definition D, and let $x_0 \in D$ be such that f is extendable to $x_0 \pm d$. Then there exists a real number $\sigma > 0$ such that, for $0 < x < \sigma$,

$$f(x_0 \pm x) = A_0^{\pm}(x) + \sum_{i=1}^{i^{\pm}} x^{q_i^{\pm}} A_i^{\pm}(x),$$

where $A_i^{\pm}(x)$ is a power series with a radius of convergence no smaller than σ, $A_i^{\pm}(0) \neq 0$, and $q_i^{\pm} \in \mathbb{Q} \setminus (\mathbb{N} \cup \{0\})$, for $i = 1, \ldots, i^{\pm}$.

Realization of our Goal

Lemma: Let f be a computer function. Then f is defined at x_0 if and only if $f(x_0)$ can be evaluated on a computer.

Lemma: Let f be a computer function that is defined at the real point x_0. Then f is extendable to $x_0 \pm d$ if and only if $\bar{f}(x_0 \pm d)$ can be evaluated on a computer.

Theorem: Let f be a computer function, and let x_0 be such that $\bar{f}(x_0 - d)$, $f(x_0)$, and $\bar{f}(x_0 + d)$ are all defined. Then f is continuous at x_0 if and only if

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Realization of our Goal

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Lemma: Let f be a computer function that is defined at the real point x_0. Then f is extendable to $x_0 \pm d$ if and only if $\bar{f}(x_0 \pm d)$ can be evaluated on a computer.

Theorem: Let f be a computer function, and let x_0 be such that $\bar{f}(x_0 - d)$, $f(x_0)$, and $\bar{f}(x_0 + d)$ are all defined. Then f is continuous at x_0 if and only if

$$\bar{f}(x_0 - d) =_0 f(x_0) =_0 \bar{f}(x_0 + d).$$

Theorem: Let f be a computer function that is continuous at x_0. Then f is m times differentiable at x_0 if and only if

$$\bar{f}(x_0 - d) =_m f(x_0) + \sum_{j=1}^{m} a_j^- d^j$$

and

$$\bar{f}(x_0 + d) =_m f(x_0) + \sum_{j=1}^{m} a_j^+ d^j,$$

with $a_j^+ = (-1)^j a_j^-$ for $j \in \{1, \ldots, m\}$.

Moreover, in this case

$$f^{(j)}(x_0) = j! a_j^+ = (-1)^j j! a_j^-.$$

for all $j \in \{1, \ldots, m\}$.

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Moreover, in this case

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for all $j \in \{1, \ldots, m\}$.

Example

$$g(x) =$$

$$\frac{\sin\left(x^3+2x+1\right)+\frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2+\sin\left(\sinh\left(\cos\left(\tan^{-1}\left(\ln\left(\exp(x)+x^2+3\right)\right)\right)\right)\right)}.$$

Table: $g^{(n)}(0), 0 \leq n \leq 10$, computed using $\mathscr{R}$ calculus

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007115	1.820 msec
1	0.4601438089634254	2.070 msec
2	−5.266097568233224	3.180 msec
3	−52.82163351991485	4.830 msec
4	−108.4682847837855	7.700 msec
5	16451.44286410806	11.640 msec
6	541334.9970224757	18.050 msec
7	7948641.189364974	26.590 msec
8	−144969388.2104904	37.860 msec
9	−15395959663.01733	52.470 msec
10	−618406836695.3634	72.330 msec

Table: $g^{(n)}(0), 0 \leq n \leq 6$, computed using $\mathscr{R}$ calculus

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007115	1.820 msec
1	0.4601438089634254	2.070 msec
2	−5.266097568233224	3.180 msec
3	−52.82163351991485	4.830 msec
4	−108.4682847837855	7.700 msec
5	16451.44286410806	11.640 msec

Table: $g^{(n)}(0), 0 \leq n \leq 6$, computed with Mathematica

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007116	110 msec
1	0.4601438089634254	170 msec
2	−5.266097568233221	470 msec
3	−52.82163351991483	2,570 msec
4	−108.4682847837854	14,740 msec
5	16451.44286410805	77,500 msec

References

http://www2.physics.umanitoba.ca/u/khodr/#Publications

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d^{-n} thanks

for some $n \in \mathbb{N}$!

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