The finite-dimensional decomposition property for non-Archimedean Banach spaces

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Eight International Conference on p-adic mathematical physics and its applications, 2021

Outline

- Overview
- The orthogonal finite-dimensional decomposition property
- The metric approximation property vs OFDDP (C. Perez-Garcia - W.Schikhof)
- Hreditary aspects of the OFDDP (C. Perez-Garcia - A.Kubzdela)

Overview

(Schauder 1927) Let X be an infinite-dimensional normed linear space. A sequence $\{e_n\}_n$ in X is called a Schauder basis of X if for every $x \in X$ there is a unique sequence of scalars (a_n), the coordinates of x, such that $x = \sum_n a_n e_n$.

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A sequence (D_n) of non-trivial linear subspaces of a Banach space X is a Schauder decomposition of X if for each $x \in X$ there is a unique sequence $(x_n)_n$, $x_n \in D_n$ for every $n \in \mathbb{N}$ such that $x = \lim_m \sum_{n=1}^m x_n$ and associated orthogonal projections $\{P_n\}$, $P_n(X) = D_n$, $n \in \mathbb{N}$ and $P_k \circ P_j = 0$ when $k \neq j$ defined by $P_n(x) = x_n$ are continuous.

A Schauderbasis decompose a Banach space into a direct sum of one-dimensional subspaces.
$\Longrightarrow$ Hence, every Banach space with a Schauderbasis has a Schauder decomposition

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A finite-dimensional decomposition

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A sequence (D_n) of finite-dimensional subspaces of a Banach space X is called a finite dimensional decomposition (FDD) for X, we write $X = \sum_n D_n$ there is a unique sequence $(x_n)_n, x_n \in D_n$ for every $n \in \mathbb{N}$ such that $x = \lim_m \sum_{n=1}^{m} x_n$.

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Non-Archimedean notations and concepts (1)

K - denotes a non-trivial non-archimedean valued field, complete with the metric induced by a non-Archimedean valuation:
a map $|.| : K \to [0, \infty)$ such that:

$$|\lambda| = 0 \text{ if and only if } \lambda = 0,$$
$$|\lambda\mu| = |\lambda| \cdot |\mu|,$$
$$|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$$

K is called spherically complete if every nested sequence of balls $(B_{K,r_n}(\lambda_n))_n$ in K has a nonempty intersection, otherwise, we will say that K is non-spherically complete

Examples:

- spherically complete: $\mathbb{Q}_p$
- non-spherically complete: $\mathbb{C}_p$

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Non-Archimedean notations and concepts (2)

Let E be a non–Archimedean normed space over K.

The norm, defined on E is called non-archimedean if it satisfies the strong triangle inequality:

$$||x + y|| \leq \max\{||x||, ||y||\}$$

for all $x, y \in E$

If $(D_i)_{i\in I}$ is a family of subspaces of E, then the linear hull of $\bigcup_{i\in I} D_i$ is denoted by $\sum_{i\in I} D_i$.

Two subspaces D_1, D_2 of E are called orthogonal ($D_1 \perp D_2$) if $||d_1 + d_2|| = \max\{||d_1||, ||d_2||\}$ for all $d_1 \in D_1$, $d_2 \in D_2$.

The set $(x_i)_{i\in I} \subset E$, $x_i \neq 0$, is called an orthogonal base of E if

$$||\sum_{i\in J} \lambda_i x_i|| = \max_{i\in J}(||\lambda_i x_i||)$$

for every finite subset $J \subset I$ and all $\lambda_i \in K$ ($i \in J$) and every $x \in E$ has an unequivocal expansion

$$x = \sum_{i\in I} \lambda_i x_i \quad (\lambda_i \in K,\ i \in I).$$

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The orthogonal finite-dimensional decomposition property

Let $(D_i)_{i\in I}$ be a system of non-Archimedean Banach spaces.
An orthogonal direct sum $\bigoplus_{i\in I} D_i$ is the space of all $(x_i)_{i\in I} \in \prod_{i\in I} D_i$ for which $\lim_i ||x_i|| = 0$, normed by $(x_i)_{i\in I} \longmapsto \max_{i\in I} ||x_i||$.

(C. Perez-Garcia, W. Schikhof, 2014) We say that a NA Banach space E has the orthogonal finite-dimensional decomposition property (OFDDP) if E can be expressed as the orthogonal direct sum of a system of finite-dimensional linear subspaces $(D_i)_{i\in I}$:
Then,

- $D_i \perp \sum_{j\neq i} D_j$ for all $i \in I$
- every $x \in \bigoplus_{i\in I} D_i$ can be written as $x = \sum_{i\in I} d_i$, where $d_i \in D_i$ for all $i \in I$.

Remark

An orthogonal base decomposes a Banach space into a direct sum of one-dimensional subspaces $\Longrightarrow$ Every Banach space with an orthogonal base has the orthogonal finite-dimensional decomposition property

The orthogonal finite-dimensional decomposition property - properties

Remark

- If K is spherically complete then every finite-dimensional space has an orthogonal base. $\Longrightarrow$ E has the orthogonal finite-dimensional decomposition property if and only if E has an orthogonal base.
- If K is spherically complete then the class of Banach spaces with the orthogonal finite-dimensional decomposition property = the class of Banach spaces with an orthogonal base

- If K is not spherically complete $\Longrightarrow$there exist various kinds of finite-dimensional spaces without orthogonal base,
- for these K the class of NA Banach spaces with the OFDDP can be viewed as a natural proper generalization of the class of NA Banach spaces with an orthogonal base.

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Example of a Banach space of countable type with OFDDP, having no orthogonal base.

Let K be non-spherically complete

- let $\left(B_{K,r_n}(c_n)\right)_n$ be a nested sequence of closed balls in K with an empty intersection.
- The formula

$$\left\|(x_1, x_2)\right\|_v := \lim_{n\to\infty} |x_1 - x_2 c_n|, \quad (x_1, x_2) \in K^2,$$

  defines a non-archimedean norm on the linear space K^2.
- The normed space $K_v^2 = \left(K^2, \|.\|_v\right)$ has no orthogonal base

$\Longrightarrow$ The space $K_v^2 \bigoplus c_0$ has the OFDDP, but it has no orthogonal base.

Remark

(W. Śliwa, 2000): A non-archimedean Frechet space of countable type without a Schauderbasis but having a finite-dimensional Schauder decomposition

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(W. Śliwa, 2000): A non-archimedean Frechet space of countable type without a Schauderbasis but having a finite-dimensional Schauder decomposition

Example of a Banach space of countable type with OFDDP, having no orthogonal base.

Let K be non-spherically complete

- let $(B_{K,r_n}(c_n))_n$ be a nested sequence of closed balls in K with an empty intersection.
- The formula
$$||(x_1, x_2)||_v := \lim_{n\to\infty} |x_1 - x_2 c_n|, \ \ (x_1, x_2) \in K^2,$$
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A non-archimedean normed space E has the metric approximation property (MAP) if the identity on E can be approximated pointwise by finite rank operators of norm ≤ 1.

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A non-Arichimedean Banach space with the OFDDP has the MAP

(sketch of the proof: $\bigoplus_{i \in I} D_i$ has the MAP if and only if each D_i has the MAP, finite-dimensional spaces trivially have the MAP.)

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The converse is not true.

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Problem: Is the OFDDP hereditary to linear subspace?

Let E be a NA Banach space with the OFDDP and D be its linear subspace.

Theorem (C. Perez-Garcia, W. Schikhof, 2014)

Then D has the OFDDP if one of the following conditions satisfied:

- *D is finite-dimensional*
- *D is orthocomplemented in E*
- *D is dense in E*

Theorem (C. Perez-Garcia,A.Kubzdela, 2014)

Assume $E = F_E \bigoplus G_E$, where F_E and G_E are closed subspaces of E and G_E has an orthogonal base. Let D be an $n-$codimensional subspace of E $(n \in \mathbb{N})$. Then, there exist $u_1, \ldots, u_n \in E$ and closed subspaces $F_D, G_D \subset E$ such that $F_D \subset F_E + [u_1, \ldots, u_n]$, G_D has an orthogonal base and $D = F_D \bigoplus G_D$.

If $E = F_E \bigoplus G_E$ has the OFDDP, where F_E and G_E are closed subspaces of E and G_E has an orthogonal base then every finitely$-$codimensional subspace of E has the OFDDP.

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Example of a Banach space of countable type with the OFDDP having a one-codimensional subspace without the OFDDP (1).

Let K be non-spherically complete NA valued field, $\widehat{K}$ denotes the spherical completion of K.

- $\widehat{K}$ will be considered as a Banach space over K with the norm given by its valuation,
- there is no non-zero element of $\widehat{K}$ that is orthogonal to K
- Choose $\lambda_1, \lambda_2, \ldots \in \widehat{K} \backslash K$ with $||\lambda_k|| = 1$ for all $k \in \mathbb{N}$ and such that $r := dist\,(\lambda_1, K) = dist\,(\lambda_k, K)$ for all $k \geq 2$.
- Recall that, for each k, $dist\,(\lambda_k, K)$ is not attained.
- $\Lambda := \{\lambda_1, \lambda_2, \ldots\}$.
- $l^\infty(\widehat{K})$ - the space of all bounded sequences of elements of $\widehat{K}$
- $E_\Lambda := \overline{[e_1, \lambda_1 e_1, e_2, \lambda_2 e_2, \ldots]} \subset l^\infty(\widehat{K})$, with the restricted sup-norm ($e_1, e_2, \ldots$ are the usual unit vectors of $K^{\mathbb{N}}$).

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- *D_Λ has the OFDDP if and only if Λ has finitely many equivalence classes with respect to $\sim$.*
- *Let $K := \mathbb{C}_p$ - the completion of the algebraic closure of the field $\mathbb{Q}_p$. There exist infinite sets $\Lambda_1, \Lambda_2 \subset \widehat{K} \backslash K$ such that $D_{\Lambda_1} \subset E_{\Lambda_1}$ has the OFDDP, and $D_{\Lambda_2} \subset E_{\Lambda_2}$ has not the OFDDP.*

Example of a Banach space of countable type with the OFDDP having a one-codimensional subspace without the OFDDP(2).

K is non-spherically complete NA valued field, $\widehat{K}$ is the spherical completion of K.

- $E_\Lambda := \overline{[e_1, \lambda_1 e_1, e_2, \lambda_2 e_2, \ldots]} \subset l^\infty(\widehat{K})$,
- E_Λ is a K-Banach space of countable type with the OFDDP, as $E_\Lambda = \bigoplus_k D_k$, where $D_k := [e_k, \lambda_k e_k]$, $k \in \mathbb{N}$.
- Define orthogonal sets $X_1 := \{e_1, e_2, \ldots\}$, $X_2 := \{\lambda_1 e_1 + \lambda_2 e_2, \lambda_1 e_1 + \lambda_3 e_3, \ldots\}$ and $D_\Lambda := \overline{[X_1 \cup X_2]}$
- Then D_Λ is a one-codimensional (hence closed) subspace of E_Λ, since $E_\Lambda = D_\Lambda + [\lambda_1 e_1]$ and $\lambda_1 e_1 \notin D_\Lambda$.

For such $\lambda_1, \lambda_2, \ldots \in \widehat{K} \backslash K$ we define a relation $\sim$ on $\Lambda := \{\lambda_1, \lambda_2, \ldots\}$ by

$\lambda_i \sim \lambda_j$ if there exist $a, b \in K$ such that $a\lambda_i + b \in \widehat{B}(\lambda_j, r)$, $\widehat{B}(\lambda_j, r)$ is a ball in $\widehat{K}$.

Theorem (C. Perez-Garcia, A.Kubzdela, 2014)

- *D_Λ has the OFDDP if and only if Λ has finitely many equivalence classes with respect to $\sim$.*
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Final Reflection

1. Other possible applications of Orthogonal Finite Dimensional Decompositions of non-Archimean Banach spaces
2. Further partial affirmative solutions of the problem: " Is the OFDDP stable for linear subspaces?"

... thanks for the attention