On metrizable subspaces and quotients of non-Archimedean spaces $C_p(X, K)$

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1. $\mathbb{K}$ – non-trivially valued **non-Archimedean** complete field.
2. For all $\alpha, \beta \in \mathbb{K}$ we have $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$; if additionally $|\alpha| \neq |\beta|$, then $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$.
3. E – linear space over $\mathbb{K}$. A **seminorm** on E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$.
4. X – infinite **ultraregular** space i.e. an infinite Hausdorff topological space such that the clopen subsets of X form a basis for the topology of X. $C_p(X, \mathbb{K})$ is isomorphic to some dense subspace of $\mathbb{K}^X$ with the product topology. Thus $C_p(X, \mathbb{K})$ is metrizable if and only if X is countable.

Problem 1

Let X be an infinite ultraregular space. Does $C_p(X, \mathbb{K})$ admit

1. *an infinite-dimensional [closed]* **metrizable** *subspace?*
2. *an infinite-dimensional complemented* **metrizable** *subspace?*
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We say that a locally convex space E contains a complemented copy of a locally convex space F if there exist a closed vector subspace $G \subset E$ such that G is isomorphic to F and a closed vector subspace L of E such that $E = G \oplus L$.

Let $c_0(\mathbb{N}, \mathbb{K})$ be the space of all sequences in $\mathbb{K}$ that are convergent to 0 with the topology of pointwise convergence.

Theorem 2

Let X be an infinite ultraregular space [with an infinite compact subset]. Then $C_p(X, \mathbb{K})$ has an infinite-dimensional [closed] metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

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1. In particular, for any infinite ultraregular compact space X the space $C_p(X, \mathbb{K})$ has an infinite-dimensional closed metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.
2. If X is discrete, $C_p(X, \mathbb{K}) = \mathbb{K}^X$, so any closed subspace of $C_p(X, \mathbb{K})$ is isomorphic to $\mathbb{K}^A$ for some $A \subset X$. Thus any infinite-dimensional closed metrizable subspace of $C_p(X, \mathbb{K})$ is isomorphic to $\mathbb{K}^{\mathbb{N}}$; in particular, $C_p(X, \mathbb{K})$ has no closed subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

1. Let X be an ultraregular space. For a point $x \in X$ let $\delta_x : C_p(X, \mathbb{K}) \to \mathbb{K}, \delta_x(f) = f(x)$, be the Dirac measure concentrated at x. The linear hull $L_p(X, \mathbb{K})$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{K}^{C_p(X,\mathbb{K})}$ can be identified with the topological dual of $C_p(X, \mathbb{K})$.

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2. Each $\mu \in L_p(X, \mathbb{K})$ can be uniquely written as a linear combination of Dirac measures $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite subset F of X and some non-zero scalars α_x for $x \in F$; the set F is called the **support** of μ and is denoted by $\mathrm{supp}(\mu)$. The real number $\max_{x \in F} |\alpha_x|$ will be denoted by $\|\mu\|$; for $\mu = 0$ we have $\mathrm{supp}(\mu) = \emptyset$ and $\|\mu\| = 0$.

Josefson-Nissenzweig theorem: For each infinite-dimensional Banach space F over $\mathbb{R}$ or $\mathbb{C}$ there exists a sequence (μ_n) in the topological dual F^* of F such that $\|\mu_n\| = 1, n \in \mathbb{N}$, with $\mu_n(f) \to 0 \ \forall \ f \in F$.

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Definition 3

Let X be an infinite ultraregular space. We say that the space $C_p(X, \mathbb{K})$ has the *Josefson-Nissenzweig property* (JNP in short) if there exists a sequence $(\mu_n) \subset L_p(X, \mathbb{K})$ such that $\|\mu_n\| = 1, n \in \mathbb{N}$, and $\mu_n(f) \to_n 0$ for every $f \in C_p(X, \mathbb{K})$.

Theorem 4

For inf. ultraregular X the following are equivalent:

1. $C_p(X, \mathbb{K})$ *contains a complemented subspace isomorphic to* $c_0(\mathbb{N}, \mathbb{K})$*;*
2. $C_p(X, \mathbb{K})$ *has a quotient isomorphic to* $c_0(\mathbb{N}, \mathbb{K})$*;*
3. $C_p(X, \mathbb{K})$ *admits a linear continuous map onto* $c_0(\mathbb{N}, \mathbb{K})$;
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1. For each infinite ultraregular compact space X the Banach space $(C(X, \mathbb{K}), \|\cdot\|_\infty)$ contains a complemented subspace isomorphic to $(c_0(\mathbb{N}, \mathbb{K}), \|\cdot\|_\infty)$.

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1. For each infinite ultraregular compact space X the Banach space $(C(X, \mathbb{K}), \|\cdot\|_\infty)$ contains a complemented subspace isomorphic to $(c_0(\mathbb{N}, \mathbb{K}), \|\cdot\|_\infty)$.
2. Hence, there exists a sequence (μ_n) converging to zero in the weak*-topology with $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$.

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3. Assume that an ultraregular space X contains an non-trivial convergent sequence. Then $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.
4. This motivates a natural question if $C_p(X, \mathbb{K})$ contains a complemented copy of $c_0(\mathbb{N}, \mathbb{K})$ (with the pointwise topology of the space $\mathbb{K}^{\mathbb{N}}$) for each infinite ultraregular compact space.

The answer is negative. In fact, we have the following fact which combined with Theorem 4 answers in the negative the above question.

Corollary 5

There exists a compact ultraregular space X such that for every sequence $(\mu_n) \subset L_p(X, \mathbb{K})$ with $\mu_n(f) \to_n 0$ for every $f \in C_p(X, \mathbb{K})$, we have $\|\mu_n\| \to 0$.

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1. We provide more concrete situations illustrating above general information.
2. We propose the following theorem.

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Let X be an extremally disconnected compact space. Then there exists no continuous linear surjection $T : C_p(X, \mathbb{K}) \to c_0(\mathbb{N}, \mathbb{K})$. In particular, $C_p(X, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ (and so, no complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$).

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Corollary 7

Let D be a discrete space. Then $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta\mathbb{N}, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

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2. Denote by $\ell_c(\mathbb{N}, \mathbb{K})$ the space of all relatively compact sequences in $\mathbb{K}$ and by $\ell_\infty(\mathbb{N}, \mathbb{K})$ the space of all bounded sequences in $\mathbb{K}$ with the topology of pointwise convergence.

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3. Clearly, $\ell_c(\mathbb{N}, \mathbb{K})$ and $\ell_\infty(\mathbb{N}, \mathbb{K})$ are linear subspace of the Fréchet spaces $\mathbb{K}^{\mathbb{N}}$ and $c_0(\mathbb{N}, \mathbb{K}) \subset \ell_c(\mathbb{N}, \mathbb{K}) \subset \ell_\infty(\mathbb{N}, \mathbb{K})$.

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4. Although Corollary 7 shows that for discrete space D the space $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$, our next theorem describes concrete infinite-dimensional metrizable quotients for spaces $C_p(\beta D, \mathbb{K})$ when D is an infinite discrete space.

Theorem 8

D – infinite discrete space. Then $E = C_p(\beta D, \mathbb{K})$ has an inf.inite-dim. metrizable quotient: For each sequence $F = (F_n)$ of non-empty, finite, pairwise disjoint $F_n \subset D$, $|F_n| \to \infty$ and

$$Y_F = \{f \in E : \sum_{x \in F_n} f(x) = 0 \text{ for every } n \in \mathbb{N}\},$$

E/Y_F is isomorphic to
$\ell_c^G(\mathbb{N}, \mathbb{K}) = \{(\sum_{s \in G_n} y_s)_{n=1}^{\infty} : (y_s) \in \ell_c(\mathbb{N}, \mathbb{K})\} \subset \mathbb{K}^{\mathbb{N}}$, where $G = (G_n)$ is a partition of $\mathbb{N}$, $|G_n| = |F_n|$. In particular E has a quotient isomorphic to

$$\ell_c^0(\mathbb{N}, \mathbb{K}) = \{(\sum_{2^{n-1} \leq s < 2^n} y_s)_{n=1}^{\infty} : (y_s) \in \ell_c(\mathbb{N}, \mathbb{K})\}.$$

1. If $\mathbb{K}$ is locally compact and Y is a compact subset of an ultraregular space X, every continuous function $g : Y \to \mathbb{K}$ has an extension to some bounded continuous function $f : X \to \mathbb{K}$. Hence $C_p(X, \mathbb{K}) \to C_p(Y, \mathbb{K}), f \to f|Y$ is a quotient map.

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4. D- infinite discrete space. If $\mathbb{K}$ is locally compact, $C_p(\beta D, \mathbb{K})$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta D, \mathbb{Q}_q)$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{Q}_q)$ for any prime number q.

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Let D be an infinite discrete space. Then for any prime number q the space $C_p(\beta D, \mathbb{C}_q)$ has a quotient isomorphic to the space $\ell_c(\mathbb{N}, \mathbb{C}_q)$.

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Recall that $\mathbb{C}_q$ is not locally compact.

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The only known examples of Efimov spaces have been found under additional set-theoretic assumptions (for example, diamond principle $\diamondsuit$- combinatorial principle (Jensen (1972)) that holds in the constructible universe (L) and that implies the continuum hypothesis) – **Fedorchuk, Dow, Efimov, Hart, De la Vega** etc. Applying above results one gets:

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Corollary 11

If there exists an ultraregular compact space X such that $C_p(X, \mathbb{K})$ does not admit an infinite-dimensional metrizable quotient, then X must be Efimov.

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Proof.

There exists ($\Diamond$) ultraregular compact hereditary separable X (so not containing $\beta\mathbb{N}$) such that:

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There exists ($\Diamond$) ultraregular compact hereditary separable X (so not containing $\beta\mathbb{N}$) such that:
X does not contain non-trivial convergent sequences. X has a base of clopen pairwise hemeomorphic sets. (De la Vega)

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Then we showed: X admits a sequence (K_n) of infinite compact subsets such that each K_n contains two disjoint subsets homeomorphic to K_{n+1}; which applies to show $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient. □

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3. *Does the space $C_p(\beta D, \mathbb{K})$ have a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$ for every field $\mathbb{K}$?*