The p-adic Mehta Integral: Formulas, Functional Equations, and Combinatorics

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A statistical model of electrostatics on a line: Setup

1. Consider a system of N labeled point charges with random locations $x_1, \ldots, x_N \in \mathbb{R}$. Call each tuple $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ a **microstate**.

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4. Fix the Boltzmann constant $k > 0$ that makes $\frac{E(\boldsymbol{x})}{kT}$ dimensionless and define the **inverse temperature parameter** $\beta = \frac{1}{kT}$.

A statistical model of electrostatics on a line: Key idea

The energy E induces a probability distribution on the microstates:

$$d\mathbb{P}_\beta(\boldsymbol{x}) = \frac{1}{\mathcal{Z}_N(\beta)} e^{-\beta E(\boldsymbol{x})} d\boldsymbol{x} \quad \text{where} \quad \mathcal{Z}_N(\beta) = \int_{\mathbb{R}^N} e^{-\beta E(\boldsymbol{x})}\, d\boldsymbol{x}$$

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- **Practical use:** Taking expectations with $d\mathbb{P}_\beta$ for various β reveals the system's observable/macroscopic behavior.
- **Important task:** Determine the domain and explicit form of the **canonical partition function** $\mathcal{Z}_N$.

Example: log-Coulomb gas in a harmonic well

The **Mehta integral** is

$$\mathcal{Z}_N(\beta) = \int_{\mathbb{R}^N} e^{-\frac{1}{2}\|\boldsymbol{x}\|^2} \prod_{i<j} |x_i - x_j|^\beta \, d\boldsymbol{x}$$

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  - *harmonic potential* energies $\frac{1}{2\beta}x_i^2$ for $i = 1, 2, \ldots, N$ and
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Conjecture (Mehta and Dyson, early 1960's)

$$\mathcal{Z}_N(\beta) = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)} \quad \textit{if} \quad \mathrm{Re}(\beta) > -2/N$$

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Theorem (Bombieri, late 1970's)

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- Suppose the charges have random locations $x_1, \ldots, x_N \in \mathbb{Q}_p$ instead.

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- Now $\mathbb{Q}_p^N$ is the space of microstates $\boldsymbol{x} = (x_1, \ldots, x_N)$ with standard norm $\| \cdot \|_p$ and Haar measure $d\boldsymbol{x}$ defined by

$$\|\boldsymbol{x}\|_p = \max_{1 \leq i \leq N} |x_i|_p \qquad \text{and} \qquad \int_{\mathbb{Z}_p^N} d\boldsymbol{x} = 1$$

where $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is the ring of p-adic integers.

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  where $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$ is the ring of p-adic integers.
- Choose an analogue $V(\boldsymbol{x})$ of the total harmonic potential, so that $e^{-\beta V(\boldsymbol{x})} = \rho(\|\boldsymbol{x}\|_p)$ is "nice" (like $e^{-\frac{1}{2}\|\boldsymbol{x}\|^2}$ for $\boldsymbol{x} \in \mathbb{R}^N$) and define

$$E(\boldsymbol{x}) = V(\boldsymbol{x}) - \sum_{i<j} \log |x_i - x_j|_p$$

The p-adic Mehta integral

Main question:

$$\mathcal{Z}_N(\beta) = \int_{\mathbb{Q}_p^N} \rho(\|\boldsymbol{x}\|_p) \prod_{i<j} |x_i - x_j|_p^\beta \, d\boldsymbol{x} = ???$$

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- **Nice fact 1:** It suffices to compute $\int_{\mathbb{Z}_p^N} \prod_{i<j} |x_i - x_j|_p^\beta \, d\mathbf{x}$ because

$$\mathcal{Z}_N(\beta) = \left(\sum_{m \in \mathbb{Z}} \rho(p^m) p^{m(N + \binom{N}{2}\beta)} \right) \cdot \left(1 - p^{-(N + \binom{N}{2}\beta)}\right) \cdot \int_{\mathbb{Z}_p^N} \prod_{i<j} |x_i - x_j|_p^\beta \, d\mathbf{x}$$

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- **Nice fact 2:** $V_0 := \{\mathbf{x} \in \mathbb{Z}_p^N : x_i = x_j \text{ for some } i < j\}$ has measure 0, so we only need to do the integral over $\mathbb{Z}_p^N \setminus V_0$.

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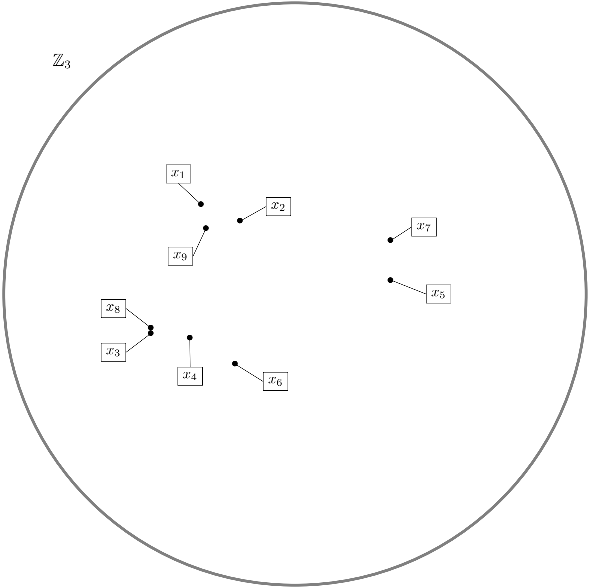
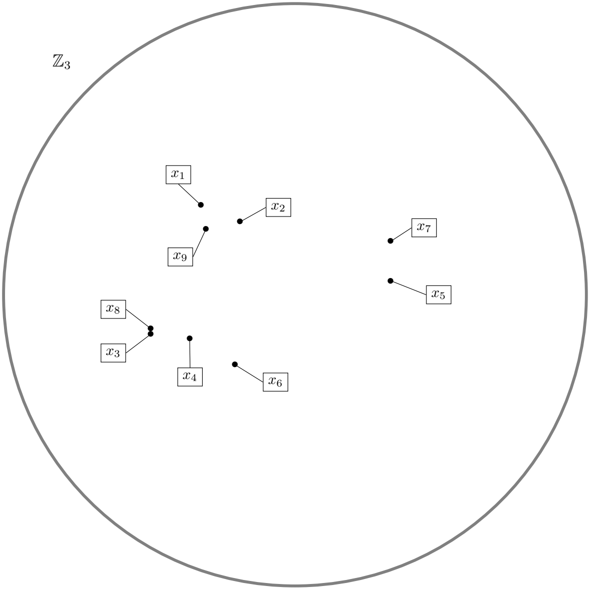
- **Question:** What do microstates $\boldsymbol{x} \in \mathbb{Z}_p^N \setminus V_0$ look like?

What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like

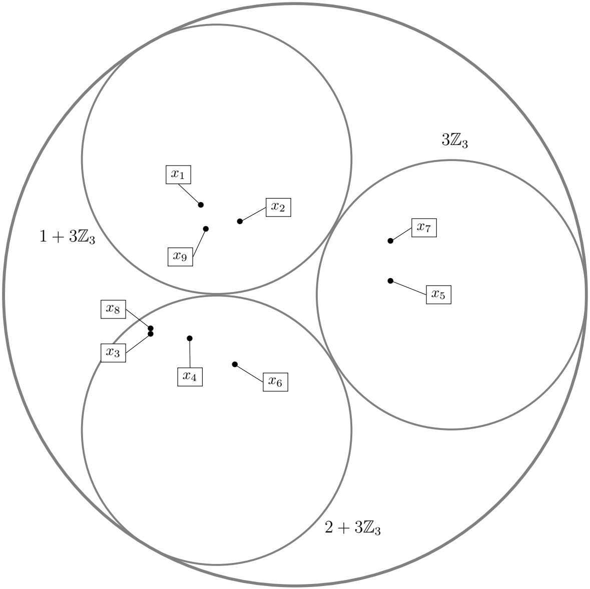


What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^0



$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^1



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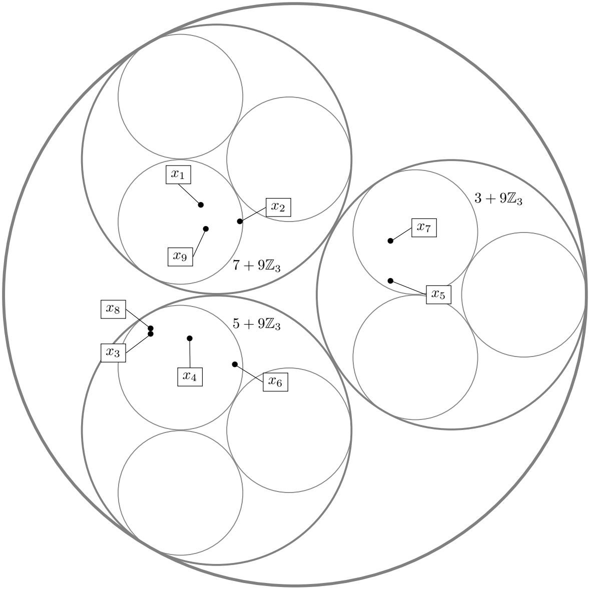

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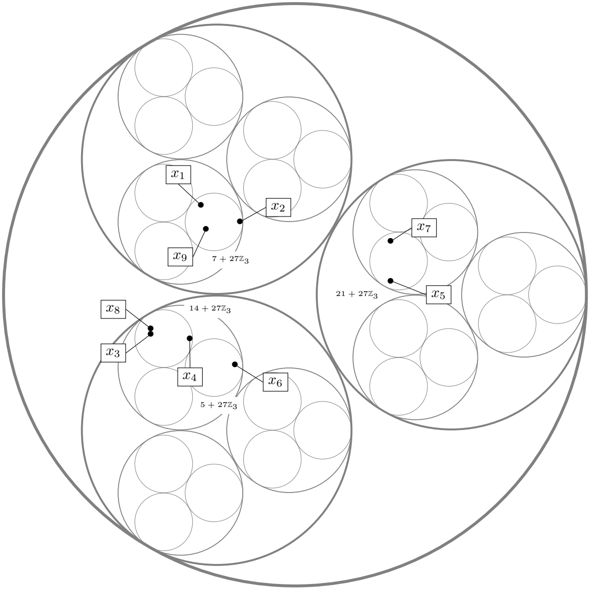
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^3



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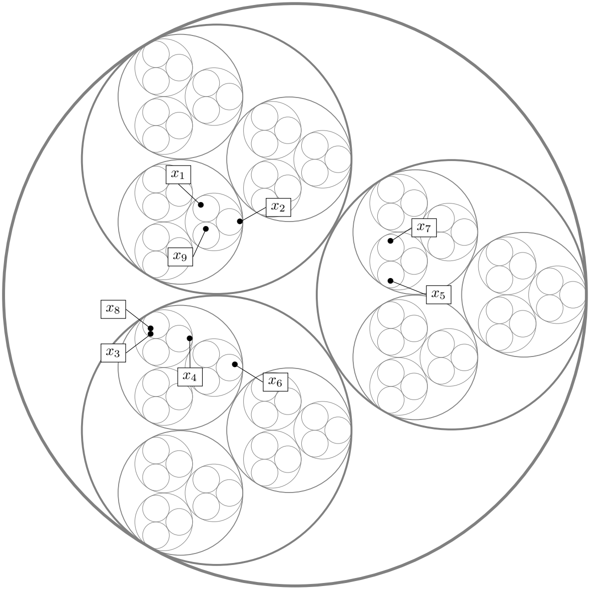
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^4



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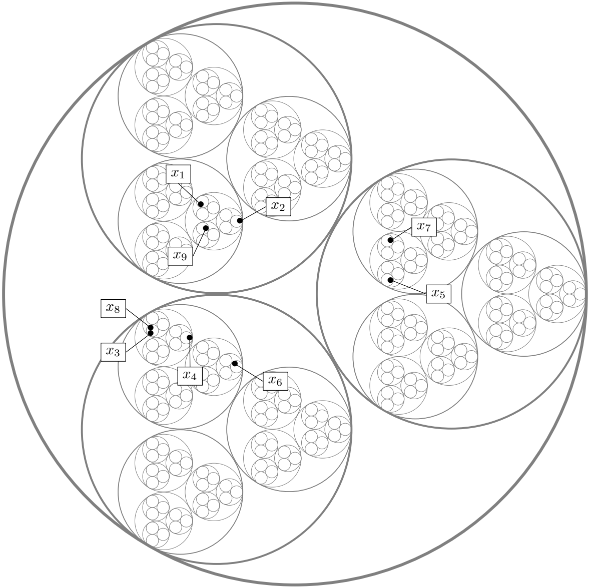
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^5



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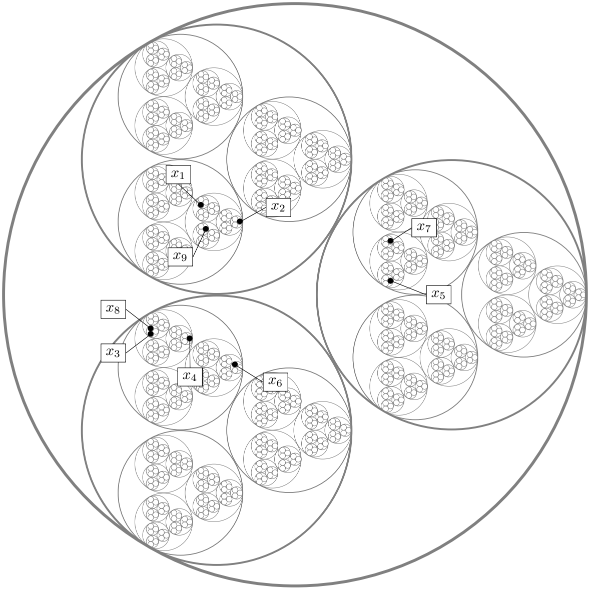

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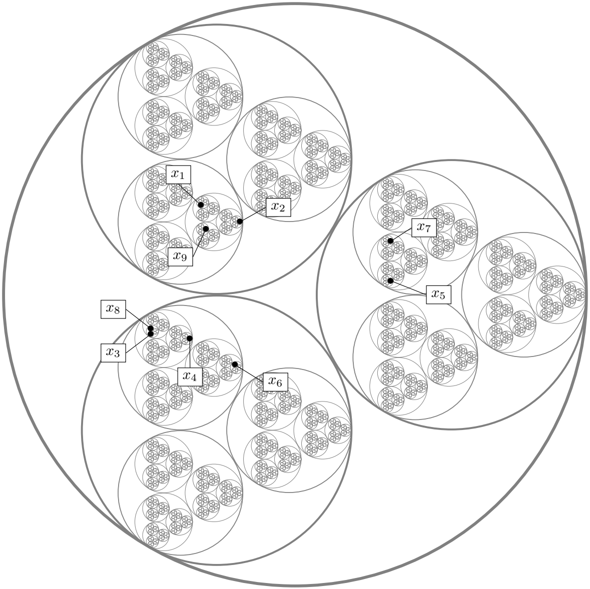

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$\vdots$

What does the diagram tell us?

- The microstate $\boldsymbol{x} \in \mathbb{Z}_3^9 \setminus V_0$ determines a pair of tuples

$$\boldsymbol{\pitchfork} = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \pitchfork_3, \pitchfork_4) \qquad \text{and} \qquad \boldsymbol{n} = (n_0, n_1, n_2, n_3) :$$

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\end{aligned}$$

- **Note:** $p = 3$ is not special here. Many $\boldsymbol{x}$ in $\mathbb{Z}_5^9$, $\mathbb{Z}_7^9$, $\mathbb{Z}_{11}^9$, ..., etc., determine the same pair $(\pitchfork, \boldsymbol{n})$ in the same way.

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- **Note:** $p = 3$ is not special here. Many $\boldsymbol{x}$ in $\mathbb{Z}_5^9$, $\mathbb{Z}_7^9$, $\mathbb{Z}_{11}^9$, ..., etc., determine the same pair $(\boldsymbol{\pitchfork}, \boldsymbol{n})$ in the same way.

- For any p, let $\mathcal{T}_p(\boldsymbol{\pitchfork}, \boldsymbol{n})$ be the set of all $\boldsymbol{x} \in \mathbb{Z}_p^9$ that determine $(\boldsymbol{\pitchfork}, \boldsymbol{n})$.

The value of $\prod_{i<j} |x_i - x_j|_p$ on $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$

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- If $\mathcal{T}_p(\pitchfork, \boldsymbol{n}) \neq \varnothing$, then for every $\boldsymbol{x} \in \mathcal{T}_p(\pitchfork, \boldsymbol{n})$ we have

$$|x_i - x_j|_p = p^{1-(n_0+n_1+\cdots+n_{\ell_{ij}})} \quad \text{for } 1 \leq i < j \leq 9,$$

where $\ell_{ij} = \max\{\ell : i \text{ and } j \text{ are in a common } \lambda \in \pitchfork_\ell\}$.

The value of $\prod_{i<j} |x_i - x_j|_p$ on $\mathcal{T}_p(⫛, \boldsymbol{n})$

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- This means any function that factors through $\boldsymbol{x} \mapsto (|x_i - x_j|_p)_{i<j}$ is constant on $\mathcal{T}_p(⫛, \boldsymbol{n})$, with value explicitly determined by $(⫛, \boldsymbol{n})$!

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- In particular, the product of the factors $|x_i - x_j|_p$ has a nice form:

Key Fact 1:

Every $\boldsymbol{x} \in \mathcal{T}_p(\pitchfork, \boldsymbol{n})$ satisfies

$$\prod_{i<j} |x_i - x_j|_p = p^{\binom{9}{2}} \prod_{\ell=0}^{3} p^{-\left[\sum_{\lambda \in \pitchfork_\ell} \binom{\#\lambda}{2}\right] n_\ell} = p^{-29}$$

The measure of $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$

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$⋔_0 = \{1,2,3,4,5,6,7,8,9\}$	$(t-1)_{3-1}$
$⋔_1 = \{5,7\}\{1,2,9\}\{3,4,6,8\}$	$(t-1)_{2-1}$
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$⋔_3 = \{7\}\{5\}\{2\}\{1\}\{9\}\{6\}\{4\}\{3,8\}$	$(t-1)_{2-1}$
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We attach a polynomial $M_\pitchfork(t) \in \mathbb{Z}[t]$ to $\pitchfork$ using falling factorials:

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$$M_\pitchfork(t) = (t-1)_2^2 \cdot (t-1)_1^4 = (t-1)^6(t-2)^2$$

The measure of $\mathcal{T}_p(⫛, \boldsymbol{n})$

We attach a polynomial $M_⫛(t) \in \mathbb{Z}[t]$ to ⫛ using falling factorials:

Partitions	Factors of $M_⫛(t)$
$⫛_0 = \{1,2,3,4,5,6,7,8,9\}$	$(t-1)_{3-1}$
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$$M_⫛(t) = (t-1)_2^2 \cdot (t-1)_1^4 = (t-1)^6(t-2)^2$$

Key Fact 2:

The set $\mathcal{T}_p(⫛, \boldsymbol{n})$ is compact and open with Haar measure

$$M_⫛(p) \cdot \prod_{\ell=0}^{3} p^{-\mathrm{rank}(⫛_\ell) n_\ell} = (p-1)^6(p-2)^2 \cdot p^{-27}$$

Putting the Key Facts together

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For each partition $\pitchfork$ and $\beta \in \mathbb{C}$ it is convenient to define

$$E_{\pitchfork}(\beta) := \mathrm{rank}(\pitchfork) + \sum_{\lambda \in \pitchfork} \binom{\#\lambda}{2} \beta,$$

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Key Fact 1 + Key Fact 2 $\Longrightarrow$

$$\sum_{\boldsymbol{n} \in \mathbb{Z}^4_{>0}} \int_{\mathcal{T}_p(\pitchfork, \boldsymbol{n})} \prod_{i<j} |x_i - x_j|_p^{\beta} \, d\boldsymbol{x} = \sum_{\boldsymbol{n} \in \mathbb{Z}^4_{>0}} p^{\binom{9}{2}\beta} \cdot M_{\pitchfork}(p) \cdot \prod_{\ell=0}^{3} p^{-E_{\pitchfork_\ell}(\beta) n_\ell}$$

$$= p^{\binom{9}{2}\beta} \cdot M_{\pitchfork}(p) \cdot \prod_{\ell=0}^{3} \frac{1}{p^{E_{\pitchfork_\ell}(\beta)} - 1}$$

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***Punchline:** Summing over all possible $\pitchfork$ gives $\int_{\mathbb{Z}_p^9} \prod_{i<j} |x_i - x_j|_p^{\beta} \, d\boldsymbol{x}$!

Definition: Splitting chains

A tuple $\boldsymbol{\pitchfork} = (\pitchfork_0, \ldots, \pitchfork_L)$ of partitions of $\{1, 2, \ldots, N\}$ is called a **splitting chain** of order N and length $L(\boldsymbol{\pitchfork}) = L$ if

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- an associated rational expression

$$J_{\pitchfork,t}(\beta) := M_{\pitchfork}(t) \cdot \prod_{\ell=0}^{L(\pitchfork)-1} \frac{1}{t^{E_{\pitchfork_\ell}(\beta)} - 1} \in \mathbb{Q}(t, t^{\beta})$$

The value of the p-adic Mehta integral

Theorem (W., 2020)

The p-adic Mehta integral converges for $\mathrm{Re}(\beta) > -2/N$ *with value*

$$\mathcal{Z}_N(\beta) = \left(\sum_{m\in\mathbb{Z}} \rho(p^m) p^{m(N+\binom{N}{2}\beta)}\right) \cdot (p^{\binom{N}{2}\beta} - p^{-N}) \cdot \sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,p}(\beta)$$

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Note: The same strategy yields a more general formula for

$$\int_{K^N} \rho(\|\mathbf{x}\|)\Big(\max_{i<j}|x_i - x_j|\Big)^a \Big(\min_{i<j}|x_i - x_j|\Big)^b \prod_{i<j}|x_i - x_j|^{s_{ij}}\, d\mathbf{x}$$

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where K is an arbitrary nonarchimedean local field. This provides the canonical partition function for mixed-charge gases and joint moments of the diameter $\max_{i<j} |x_i - x_j|$ and minimal particle spacing $\min_{i<j} |x_i - x_j|$.

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$$\begin{aligned}\sum_{\pitchfork \in \mathcal{S}_4} J_{\pitchfork,t}(\beta) = {} & \frac{(t-1)(t-2)(t-3)}{t^{3+6\beta}-1} + 4 \cdot \frac{(t-1)^2(t-2)}{(t^{3+6\beta}-1)(t^{2+3\beta}-1)} + 6 \cdot \frac{(t-1)^2(t-2)}{(t^{3+6\beta}-1)(t^{1+\beta}-1)} \\ & + 3 \cdot \frac{(t-1)^3}{(t^{3+6\beta}-1)(t^{2+2\beta}-1)} + 6 \cdot \frac{(t-1)^3}{(t^{3+6\beta}-1)(t^{2+2\beta}-1)(t^{1+\beta}-1)} \\ & + 12 \cdot \frac{(t-1)^3}{(t^{3+6\beta}-1)(t^{2+3\beta}-1)(t^{1+\beta}-1)}\end{aligned}$$

A scary theorem and a quadratic recurrence

Theorem (Lengyel, 1984)

$$\#\mathcal{S}_N = \Omega\left(\frac{(N!)^2}{(2\ln(2))^N \cdot N^{1+\ln(2)/3}}\right) \quad \text{as} \quad N \to \infty$$

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- Define $F_0(r,\beta) := 1$ and $F_1(r,\beta) := 1$ for all $\beta \in \mathbb{C}$ and all $r \in \mathbb{R}$

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- Note: If N and β are fixed, $r \mapsto F_N(r,\beta)$ is even and smooth

An efficient formula

Recall: The p-adic Mehta Integral with $N \geq 2$ and $\rho = \mathbf{1}_{[0,1]}$ has the form

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Theorem (Sinclair and W., 2021)

The value of the integral can be computed efficiently via

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Corollary (The $p \to 1$ Limit and $p \mapsto p^{-1}$ Functional Equation)

The value of $\mathcal{Z}_N(\beta)$ extends to a smooth function of $p \in (0, \infty)$ satisfying

$$\lim_{p \to 1} \mathcal{Z}_N(\beta) = N! \cdot F_N(0, \beta) \quad \text{and} \quad \mathcal{Z}_N(\beta)\Big|_{p \mapsto p^{-1}} = p^{-\binom{N}{2}\beta} \cdot \mathcal{Z}_N(\beta)$$

Grand canonical partition functions

- Suppose the gas also exchanges particles with the reservoir with "fugacity parameter" f

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- Similarly, for log-Coulomb gases in $p\mathbb{Z}_p$ we define

$$\mathcal{Z}^{\circ}(\beta, f) := \sum_{N=0}^{\infty} \mathcal{Z}_N^{\circ}(\beta) \frac{f^N}{N!} \quad \text{where} \quad \mathcal{Z}_N^{\circ}(\beta) := \int_{(p\mathbb{Z}_p)^N} \prod_{i<j} |x_i - x_j|_p^{\beta} \, d\boldsymbol{x}$$

The pth Power Law

Theorem (Sinclair, 2020)

The grand canonical partition function for log-Coulomb gas in $\mathbb{Z}_p$ satisfies

$$\mathcal{Z}(\beta, f) = (\mathcal{Z}^\circ(\beta, f))^p$$

for all $\beta \geq 0$ and $f \in \mathbb{C}$.

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Definition (The projective *p*-adic Mehta Integral)

The canonical partition function for an N-particle log-Coulomb gas in $\mathbb{P}^1(\mathbb{Q}_p)$ is given by

$$\mathcal{Z}_N^*(\beta) := \int_{(\mathbb{P}^1(\mathbb{Q}_p))^N} \prod_{i<j} \delta(x_i, x_j)^\beta \, d\mu^N$$

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Abbreviated Theorem (W., 2021)

The integral $\mathcal{Z}_N^*(\beta)$ converges absolutely if and only if $\mathrm{Re}(\beta) > -2/N$. Like $\mathcal{Z}_N(\beta)$, it is a finite sum over splitting chains of order N. Each summand is a rational function of p and $p^{-\beta}$ closely resembling $J_{\pitchfork,p}(\beta)$.

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Corollary (The $p \to 1$ Limit and $p \mapsto p^{-1}$ Functional Equation)

The value of $\mathcal{Z}_N^*(\beta)$ extends to a smooth function of $p \in (0,\infty)$ satisfying

$$\lim_{p\to 1} \mathcal{Z}_N^*(\beta) = N! \sum_{k=0}^{N} F_k(0,\beta) F_{N-k}(0,\beta)$$

and $\mathcal{Z}_N^*(\beta)\big|_{p\mapsto p^{-1}} = \mathcal{Z}_N^*(\beta)$.

The $(p+1)$th Power Law

There is also a grand canonical partition function for log-Coulomb gas in $\mathbb{P}^1(\mathbb{Q}_p)$:

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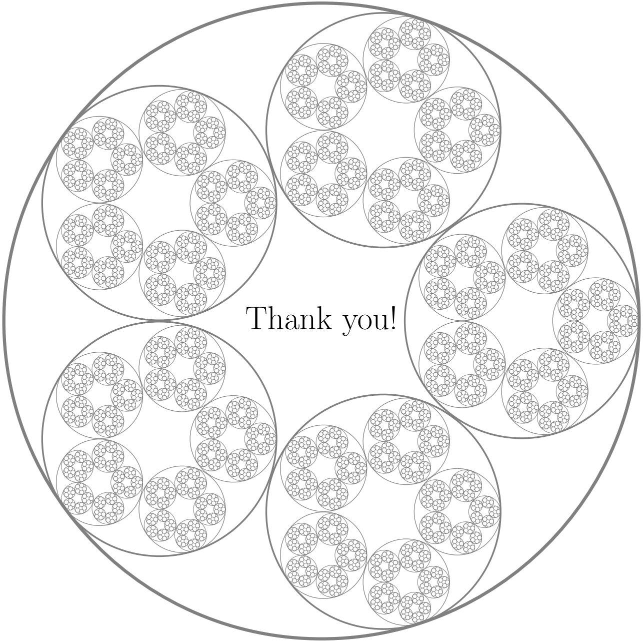

Thank you!