p-Adic Noncommutative Torus and the Hall Effect

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International Conference
p-adic Mathematical physics and Applications
p-adic.2021

May, 2021

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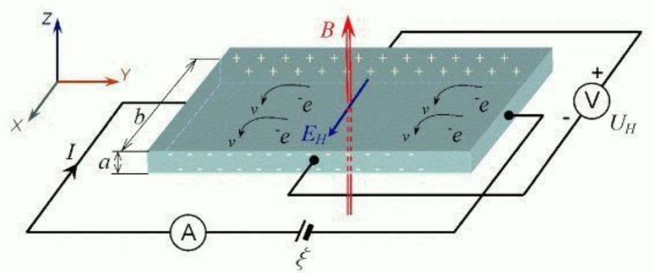
The Classical Hall effect I



The first part of the lecture will be devoted to a brief and very simplified introduction to the theory of the Hall effect. [1].
Consider a particle of mass m and charge $-e$ in a magnetic field B.
Equation of motion:

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{v} \times \mathbf{B}.$$

[1]Edwin Hall (1879). "On a New Action of the Magnet on Electric Currents". American Journal of Mathematics. 2 (3): 287—92.

The Classical Hall effect I

A uniform magnetic field is directed along the axis z, $\mathbf{B} = (0, 0, B)$, the particle moves in the plane $(x, y, 0)$. It is easy to see that the particle moves in circular orbits with a frequency of $\omega_B = \frac{eB}{m}$(cyclotron frequency).
Now we turn on the electric field $\mathbf{E}$ in the direction of the axis x and add "friction"(the Drude model):

$$m\frac{d\mathbf{v}}{dt} == e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \frac{m\mathbf{v}}{\tau}.$$

Here the parameter τ is the scattering time (the average time between collisions).
By entering the notation for the current $\mathbf{J} = -ne\mathbf{v}$, n is the density of charge carriers, from the equilibrium condition we obtain the relation:

$$\mathbf{J} = \sigma\mathbf{E},$$

The Classical Hall effect II

here σ is the conductivity matrix. We will be interested in the inverse matrix $\rho = \sigma^{-1}$ (resistivity matrix), which has the form:

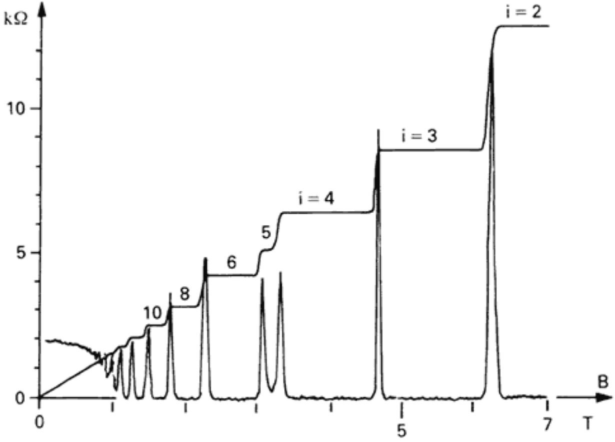
$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$

$$\rho_{xx} = \frac{m}{ne^2\tau}$$

$$\boxed{\rho_{xy} = \frac{1}{ne}B = R_H B}$$

The Integer Quantum Hall effect I

The classical theory of the Hall effect gives a linear dependence of the Hall resistivity on the magnetic field. In reality, the picture is more complex[2].



The Integer Quantum Hall effect II

The Hall resistivity, in contrast to the classical picture, changes in jumps from one plateau to another. In this case, the value of the resistance on the plateau is given by the expression:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{i}, \ \ i = 1, 2, 3, \ldots$$

A plateau with the index i appears when the magnetic field takes values

$$B = \frac{n}{i}\frac{2\pi\hbar}{e} = \frac{n}{i}\Phi_0.$$

[2] Klitzing, K.V., Dorda, G. and Pepper, M. (1980) Phys. Rev. Lett., 45, 494. (Nobel prize 1985)

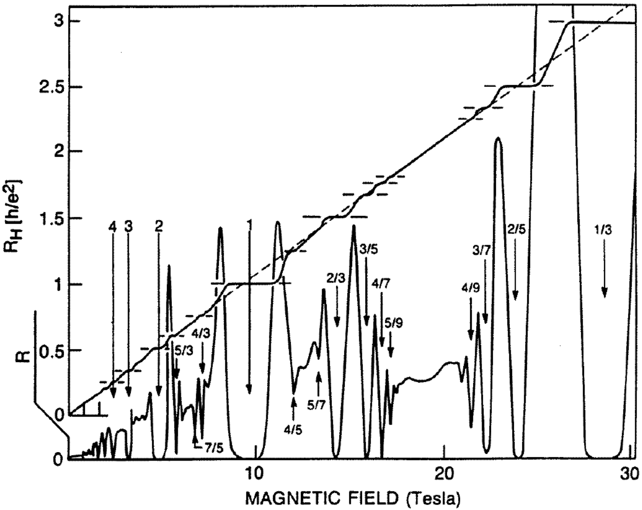
The Fractional Quantum Hall effect I

But this is not the end of the story.
The parameter i can, in fact, take rational values[3] for example

$$i = \frac{5}{2}, \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{3}{7}, \frac{5}{9}, \frac{3}{13}, \ldots$$

Very clean samples were required to observe this effect.

The Fractional Quantum Hall effect II



[3] D.C. Tsui; H.L. Stormer; A.C. Gossard (1982). "Two-Dimensional Magnetotransport in the Extreme Quantum Limit". Physical Review Letters. 48 (22): 1559 (Nobel prize with R. Laughlin 1998)

Landau levels I

Assume that the sample is clean enough (i.e. $\tau \to \infty$) and the electrons do not interact. Consider a one-particle Hamiltonian of the form

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2,$$

where $\mathbf{A}$ is the vector potential of the magnetic field, $\nabla \times \mathbf{A} = \mathbf{B}$. We introduce the notation $\boldsymbol{\pi} = \mathbf{p} + e\mathbf{A}$, then

$$[\pi_x, \pi_y] = -ie\hbar B,$$

and for operators

$$a = \frac{1}{\sqrt{2eB}}(\pi_x - i\pi_y), \quad a^\dagger = \frac{1}{\sqrt{2eB}}(\pi_x + i\pi_y)$$

the resulting commutation relations $[a, a^\dagger] = 1$ and

Landau levels II

$$H = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right).$$

This is nothing but the Hamiltonian of the harmonic oscillator. The energy levels are called **Landau levels**. They are degenerate, in the quasi-classical approximation, the degeneracy of the levels is given by the formula $N = \Phi/\Phi_0$, where Φ is the magnetic flux through the sample and Φ_0 is the quantum of the magnetic flux.

TKNN invariants and IQHE I

Where does the topology in this problem come from[4]?
Consider a particle on a two-dimensional square lattice. In this case, the momentum take values on the torus $\mathbb{T}^2$ (Brillouin zone). The wave function of a particle in the Brillouin zone can be represented in the Bloch form

$$\psi_{\mathbf{k}}(\mathbf{x}) = \exp(i\mathbf{k}\mathbf{x}) u_{\mathbf{k}}(\mathbf{x}),$$

the function $u_{\mathbf{k}}(\mathbf{x})$ is periodic. Define $U(1)$ Berry connection on $\mathbb{T}^2$

$$A_i(\mathbf{k}) = -i \langle u_{\mathbf{k}} | \frac{\partial}{\partial k^i} | u_{\mathbf{k}} \rangle.$$

First Chern number is defined by

$$C = -\frac{1}{2\pi} \int_{\mathbb{T}^2} d^2 k \mathcal{F}_{xy},$$

TKNN invariants and IQHE II

where

$$\mathcal{F}_{xy} = \frac{\partial A_x}{\partial k^y} - \frac{\partial A_y}{\partial k^x}.$$

Chern number is an integer, $C \in \mathbb{Z}$.

The surprising fact is as follows. Consider a model of electrons on a lattice with a magnetic field and a periodic potential. Using the Kubo formula, we calculate the Hall resistance.

The answer is as follows:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{C}.$$

The physical meaning of the first Chern number C - it is the number of completely filled Landau levels (filling factor).

In other words – IQHE is a topological effect

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[4] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M.den. Nijs, Phys. Rev. Lett. 49, 405 (1982)

Noncommutative torus $\mathbb{T}^2_\theta$ and IQHE I

In the framework of our lattice model, we consider the unitary operators of magnetic translations:

$$U_j = \exp\left(\frac{i}{\hbar}(p_j + eA_j)\right),\ j = 1, 2.$$

These operators commute with the Hamiltonian, but do not commute with each other:

$$U_1 U_2 = \exp(2\pi i\theta) U_2 U_1.$$

Here θ is the flux of the magnetic field B through the fundamental region of the lattice.
C^* -algebra generated by a pair of unitary operators with such a commutation relation is called the algebra of (irrational) rotations of a circle.

Noncommutative torus $\mathbb{T}^2_\theta$ and IQHE II

On a dense subalgebra $\mathbb{T}^2_\theta$ operators of the form

$$\sum a_{n,m} U_1^n U_2^m,\ (|n|^k + |m|^k)|a_{n,m}|\ bound,\ k > 0$$

define the derivatives $\delta_j(U_k) = 0,\ k \neq j,\ \delta_j(U_j) = 2\pi i U_j$ and two cocycles

$$\tau_0\left(\sum a_{n,m} U_1^n U_2^m\right) = a_{0,0},$$

$$\tau_2(a^0, a^1, a^2) = \tau_0\left(a^0\left(\delta_1(a^1)\delta_2(a^2) - \delta_2(a^1)\delta_1(a^2)\right)\right).$$

Let E be a projector in $\mathbb{T}^2_\theta$, then the equality is valid[5]

$$\frac{1}{2\pi i}\tau_2(E, E, E) = n \in \mathbb{Z},$$

in this case, the number n is uniquely determined from the equality $\tau_0(E) = n\theta(\mod 1)$. This corresponds exactly to the first Chern

Noncommutative torus $\mathbb{T}^2_\theta$ and IQHE III

number. In this language, the formula for Hall conductivity can be expressed as[6]

$$\sigma_H = \frac{e^2}{h}\frac{1}{2\pi i}\tau_2(E_\mu, E_\mu, E_\mu),$$

where E_μ is a projector for energy levels less than the Fermi level. This expression can be "rewritten" as follows:

$$\sigma_H = \frac{e^2}{\hbar}\tau_0\left(E_\mu dE_\mu dE_\mu\right),$$

where $dE_\mu = [F, E_\mu]$ and F – the Hilbert operator, $F = F^*$, $F^2 = 1$.

[5] A, Connes. Noncommutative geometry, 1994

[6] The noncommutative geometry of the quantum Hall effect. J. Bellissard, A. van Elst and H. Schulz-Baldest, 1994)

My idea is to describe FQHE in the same geometric manner. FQHE is a consequence of the interaction of electrons, the modern description is based on the properties of quasiparticles with a fractional charge (anions).[7]

The noncommutative torus $\mathbb{T}^2_\theta$ originated as the C^* - algebra generated by a pair of unitary operators with the commutation relation. There is another representation of this algebra.

On the circle $\mathbb{T}$, the action of the group $\mathbb{Z}$ is given by rotation on the angle θ: $\mathbb{T} \ni t \to \exp(2\pi i\theta)t$. This action is minimal (the orbit of each point is dense).

In the space $L^2(\mathbb{T})$ consider C^*-algebra generated by the operators of multiplication by a continuous function on a circle and the unitary rotation operator $f(t) \to f(e^{2\pi i\theta}t)$. This is exactly the noncommutative torus $\mathbb{T}^2_\theta$,

$$\mathbb{T}^2_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}.$$

[7] Laughlin, R. B. (2 May 1983). "Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations". Physical Review Letters. American Physical Society (APS). 50 (18): 1395–1398.

This algebra has a number of interesting properties.

- separable
- simple
- has unique trace state
- is $A\mathbb{T}$ - algebra, that is, $\mathbb{T}^2_\theta = A \bigotimes C(\mathbb{T})$, where A is an approximative-finite algebra.
- the set of trace values on projectors is its complete system of invariants
- the set of trace values on projectors is $\{m + n\theta, m, n \in \mathbb{Z}\} \cap [0, 1]$
- $K_0(\mathbb{T}^2_\theta) = \mathbb{Z}^2$

Consider the p - adic analog of this algebra. On the set $\mathbb{Z}_p$ of p - adic integers, consider the action of the group $\mathbb{Z}$ of (rational) integers, $\mathbb{Z}_p \ni x \to x + 1$ (odometer). This action is minimal. Consider the C^* -algebra $\mathbb{T}_p^2$ generated by the operators (on the space $L^2(\mathbb{Z}_p)$) multiplication by a continuous function and by the unitary operator generated by the odometer. By construction, this algebra is naturally considered an analog of the noncommutative torus in the p - adic case.

$$\mathbb{T}_p^2 = C(\mathbb{Z}_p) \rtimes \mathbb{Z}.$$

The algebra $\mathbb{T}_p^2$ has the same set of properties as the algebra $\mathbb{T}_\theta^2$:

- separable
- simple
- has unique trace state
- is $A\mathbb{T}$ - algebra, that is, $\mathbb{T}_p^2 = A \bigotimes C(\mathbb{T})$, where A is an approximative-finite algebra.

- the set of trace values on projectors is its complete system of invariants
- the set of trace values on projectors is $\{\frac{m}{p^n}, m, n \in \mathbb{Z}\} \cap [0, 1]$
- $K_0(\mathbb{T}_p^2) = \mathbb{Z}\left[\frac{1}{p}\right]$

The algebra A in the decomposition $\mathbb{T}_p^2 = A \bigotimes C(\mathbb{T})$ is the UHF (uniformly hyperfinite) algebra with the Glimm invariant (p^∞), that is, is the direct limit of matrix algebras M_{p^k} with respect to the natural diagonal embedding.

$$M_p \to M_{p^2} \to M_{p^3} \to \cdots \to M_{p^k} \to \cdots$$

This is a well-known algebra (the Bunce-Deddens algebra). Another representation of this algebra will be useful in the future. Consider the group $\hat{\mathbb{Z}}_p$ of characters $\mathbb{Z}_p$. This group has the form

$$\hat{\mathbb{Z}}_p = \mathbb{Z}\left[\frac{1}{p}\right] / \mathbb{Z} = \mathbb{Z}(p^\infty).$$

This is the Prufer group. It is a direct limit of finite cyclic groups of order p^n.

$$\mathbb{Z}/p\mathbb{Z}_p \to \mathbb{Z}/p^2\mathbb{Z}_p \to \cdots \to \mathbb{Z}/p^n\mathbb{Z}_p \to \cdots$$

The finite cyclic group acts on the circle $\mathbb{T}$ by rotations, and thus one can define the C^* - algebra

$$C(\mathbb{T}) \rtimes \mathbb{Z}/p^n\mathbb{Z} \equiv M_{p^n} \bigotimes C(\mathbb{T}).$$

Thus:

$$\mathbb{T}_p^2 = C(\mathbb{T}) \rtimes \mathbb{Z}(p^\infty).$$

Recall that the group $\mathbb{Z}(p^\infty)$ is the character group of the group $\mathbb{Z}_p$, that is, it parametrizes the vectors of the orthonormal basis of characters in the space $L^2(\mathbb{Z}_p)$. Define the mapping $\mathbb{Z}(p^\infty) \to \{-1, 1\}$:

$$\mathbb{Z}(p^\infty) \ni \sum \frac{a_k}{p^k} \to \left(\frac{a_k}{p}\right),$$

$\left(\frac{a_k}{p}\right)$ – Legendre symbol of the first nonzero term in the canonical representation.
Space $L^2(\mathbb{Z}_p)$ is represented as a direct sum of the subspaces H_- and H_+ spanned by characters with the corresponding value of the mapping above. The Hilbert operator is defined by the formula $F = P_+ - P_-$, where $P_\pm$ are orthogonal projectors on subspaces $H_\pm$, respectively. We define the differentiation operator on the algebra of bounded operators $du = [F, u]$.

Theorem

$p \neq 2$

$$\tau\left(EdEdE\right) \in \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$$

for any projector $E \in \mathbb{T}_p^2$.

It remains to combine the results for real and p-adic case. Let θ be the irrational number, $\theta \in [0, 1]$.

Let $G = \mathbb{R} \times \mathbb{Z}_p$. The rational integers $\mathbb{Z}$ form a subgroup in G of the form $\{(n, n),\ n \in \mathbb{Z}\}$. A solenoid is the following group

$$\mathbb{S} = G/\mathbb{Z}.$$

This is the inverse limit of the sequence

$$\mathbb{T} \leftarrow \mathbb{T} \leftarrow \cdots \leftarrow \mathbb{T} \leftarrow \cdots$$

with respect to p-winding homomorphisms: $t \to t^p$. $\hat{\mathbb{S}} = \mathbb{Z}\left[\frac{1}{p}\right]$.
Let's build the embedding of the group $\hat{\mathbb{S}}$ in $\mathbb{T}$,
$\hat{\mathbb{S}} \ni s \to \exp(2\pi i\theta s) \in \mathbb{T}$ and the corresponding algebra

$$\mathbb{S}_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}\left[\frac{1}{p}\right].$$

This algebra is called a noncommutative solenoid and has the same nice set of properties.
Combining the statements for the algebras $\mathbb{T}^2_\theta$ and $\mathbb{T}^2_p$, we get the statement:

Theorem

$$\tau(EdEdE) \in \mathbb{Z}\left[\frac{1}{p}\right].$$

Thus, the Hall conductivity, which in the framework of the proposed approach has the form

$$\sigma_H = \frac{e^2}{h}\frac{1}{2\pi i}\tau(E_\mu dE_\mu dE_\mu),$$

(where E_μ is a projector for energy levels less than the Fermi level) can take fractional values (in dimensionless units).

Conclusions:

- (Noncommutative) Brillouin zone $\equiv$ noncommutative solenoid.
- In the proposed theory, the Hall conductivity values can take (in dimensionless units) both integer and fractional values (inverse powers of a prime number $p \neq 2$).
- p-windings of the circle correspond to quasiparticles with a fractional charge of $1/p$.
- The theory can be generalized to more general solenoids and obtain a wider range of conductivity values. Instead of p-windings at each stage, we can consider an arbitrary positive integer instead of p.
- It is open question for $p = 2$ (i.e. $\frac{5}{2}$ filling factor).

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