On spectral perturbation results of compact self-adjoint operators over a Hilbert-like ultrametric space.

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Outline for section 1

The valued fields $\mathcal{R}$ and $\mathcal{C}$

The **Levi-Civita field** $\mathcal{R}$ and the **Complex Levi-Civita field** $\mathcal{C}$ can be considered as formal power series fields:

$$\mathcal{R} = \left\{ \sum_{k=1}^{\infty} a_k d^{t_k} \,\middle|\, \forall k \in \mathbb{N}, a_k \in \mathbb{R}, t_k \in \mathbb{Q}, t_k < t_{k+1}, \lim t_k = \infty \right\}$$

$$\mathcal{C} = \left\{ \sum_{k=1}^{\infty} a_k d^{t_k} \,\middle|\, \forall k \in \mathbb{N}, a_k \in \mathbb{C}, t_k \in \mathbb{Q}, t_k < t_{k+1}, \lim t_k = \infty \right\}$$

If $a_1 \neq 0$, then the valuation on $\mathcal{R}$ and $\mathcal{C}$ is defined as

$$\left| \sum_{k=1}^{\infty} a_k d^{t_k} \right| := e^{-t_1} \quad \text{and} \quad |0| = 0.$$

The valued fields $\mathcal{R}$ and $\mathcal{C}$

Notice that $\mathcal{C} = \mathcal{R} + i\mathcal{R}$.

For each nonzero $z = x + iy \in \mathcal{C}$ ($x, y \in \mathcal{R}$) the valuation satisfies:

$$|z| = \max\{|x|, |y|\}$$

The involution $x + iy \mapsto \overline{x + iy} := x - iy$ is an automorphism on $\mathcal{C}$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{R}$ for all $z \in \mathcal{C}$.

- $\mathcal{R}$ is real closed.
- $\mathcal{C}$ is algebraically closed.

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Outline for section 2

The normed space c_0

The set

$$c_0 := \left\{ (\lambda_j)_{j\in\mathbb{N}} \,\middle|\, \forall j \in \mathbb{N}, \lambda_j \in \mathcal{C}, \lim_j \lambda_j = 0 \right\}$$

is a vector space over $\mathcal{C}$.

Notice that $c_0 = c_0(\mathcal{R}) \oplus i c_0(\mathcal{R})$, i.e. for each $\mathrm{z} = (z_k) \in c_0$, there are unique $\mathrm{x} = (x_k)$ and $\mathrm{y} = (y_k)$ in $c_0(\mathcal{R})$ such that $\mathrm{z} = \mathrm{x} + i\mathrm{y}$ and the norm on c_0 satisfies:

$$||\mathrm{z}|| := \max_{k\in\mathbb{N}} |z_k| = \max_{k\in\mathbb{N}} \max\{|x_k|, |y_k|\} = \max\{||\mathrm{x}||, ||\mathrm{y}||\}.$$

The space $(c_0, ||\cdot||)$ is Banach.

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The inner product on c_0

Theorem (Narici & Beckenstein (2005) [2, 6.1])

Consider the form $\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathcal{C}$, $\langle \mathrm{z}, \mathrm{w} \rangle = \sum_{k=1}^{\infty} z_k \overline{w_k}$. *The statements below hold for all* $\mathrm{z}, \mathrm{z}', \mathrm{w} \in c_0$ *and* $\alpha, \beta \in \mathcal{C}$.

1. $\langle \cdot, \cdot \rangle$ *is well-defined.*
2. $\langle \mathrm{z}, \mathrm{z} \rangle = 0 \Leftrightarrow \mathrm{z} = 0$
3. $\langle \alpha \mathrm{z} + \beta \mathrm{z}', \mathrm{w} \rangle = \alpha \langle \mathrm{z}, \mathrm{w} \rangle + \beta \langle \mathrm{z}', \mathrm{w} \rangle$
4. $\langle \mathrm{z}, \mathrm{w} \rangle = \overline{\langle \mathrm{w}, \mathrm{z} \rangle}$
5. $|\langle \mathrm{z}, \mathrm{w} \rangle| \leq ||\mathrm{z}|| \, ||\mathrm{w}||$
6. $\langle \mathrm{z}, \mathrm{w} \rangle = 0, \forall \mathrm{w} \in c_0 \Rightarrow \mathrm{z} = 0.$
7. $||\mathrm{z}|| = \sqrt{|\langle \mathrm{z}, \mathrm{z} \rangle|}$

c_0 is not a Hilbert space

- The proper subspace $D := \{(z_k) \in c_0 : \sum_{k=1}^{\infty} z_k = 0\}$ is closed in c_0 such that $D^{\perp} = \{0\}$.

- $(c_0)' = \ell^{\infty}$.

- The Hanh Banach theorem does not hold on c_0.

Definition

Consider the standard Schauder basis $\{e_1, e_2, \dots\}$ *of* c_0*. The projection map* $e'_j : c_0 \to \mathcal{C}$ *defined by*

$$e'_j(\mathrm{x}) := \langle \mathrm{x}, e_j \rangle$$

is a member of c'_0 *for all* $j \in \mathbb{N}$*.*

Furthermore, $e'_j(e_i) = \delta_{ij}$, $||e'_i|| = 1$ *for all* $i \in \mathbb{N}$, $\mathrm{x} = \sum_{i=1}^{\infty} e'_i(\mathrm{x}) e_i$ *and* $||\mathrm{x}|| = \max_{i \in \mathbb{N}} |e'_i(\mathrm{x})|$ *for all* $\mathrm{x} \in c_0$*.*

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Outline for section 3

Continuous linear operators on c_0

Definition

$L(c_0) := \{T : c_0 \to c_0 : T$ is continuous and linear$\}$ is a Banach space under the norm $||T|| := \sup \left\{ \frac{||T(x)||}{||x||} : x \neq 0 \right\}$

Definition

A linear operator $T : c_0 \to c_0$ is said to be ***self-adjoint*** *if*

$$\langle T\mathrm{x}, \mathrm{y} \rangle = \langle \mathrm{x}, T\mathrm{y} \rangle$$

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Compact operators on c_0

Definition (van Rooij (1978) [3, 4.39, 4.40])

Let E and F be two normed spaces over a non-Archimedean valued field. An operator $T \in L(E, F)$ is said to be compact when satisfies any of the following equivalent conditions:

1. $T(B_E)$ *is compactoid, where* $B_E = \{x \in E : ||x|| \leq 1\}$,
2. *for each* $\varepsilon > 0$, *there exists* $S \in L(E, F)$ *of finite-dimensional range such that* $||T - S|| < \varepsilon$,
3. *there are vectors* $\mathrm{a}_1, \mathrm{a}_2, \cdots \in F$, *and functionals* $g_1, g_2, \cdots \in E'$ *such that* $\lim_k ||g_k|| ||\mathrm{a}_k|| = 0$ *and* $T = \sum_{k=1}^{\infty} g_k \mathrm{a}_k$, *i.e. the sequence* $(\sum_{k=1}^{n} g_k(\cdot)\mathrm{a}_k)_{k \in \mathbb{N}}$ *converges uniformly to* T.

Definition

If $(v_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{C}^n$ (or c_0), then we say that $T \in L(\mathcal{C}^n)$ (or $T \in L(c_0)$) is diagonalizable if $T = \sum_i \alpha_i \langle \cdot, v_i \rangle v_i$.
Notation: $E_i := \langle \cdot, v_i \rangle v_i$.

The proof of the spectral theorem for compact self-adjoint operators in the Archimedean case cannot be adapted to the non-Archimedean case because in the classical case the proof is based on the following facts:

1. the spectrum of a compact self-adjoint operator is non-empty, which is proved by using Liouville's Theorem. In the non-Archimedean case Liouville's Theorem holds for functions $f : K \to K$ that admit a power series expansion. But it is unknown whether a function $f : K \to K$ that is differentiable has a power series expansion. In the classical case this is proved by using the Cauchy's Theorem which heavily depends on the connectedness of $\mathbb{C}$. In our case, any non-Archimedean valued field is totally disconnected.
2. $\sup \left\{ \frac{\langle Tx,x\rangle}{\langle x,x\rangle} : x \neq 0 \right\}$ and $\inf \left\{ \frac{\langle Tx,x\rangle}{\langle x,x\rangle} : x \neq 0 \right\}$ are eigenvalues for T, when T is a compact self-adjoint operator on a Hilbert space. In the non-Archimedean context, an upper bounded set of scalars may not have a supremum. Similarly with infimum.

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Theorem (B. Diarra (2009) [1])

Every $T \in L(c_0)$ *has a matrix representation* $[T] = (\alpha_{ij})$ *in the sense of* $Tx = [T]x$, *with* $||T|| = \sup\{|\alpha_{ij}| : i, j \in \mathbb{N}\}$. *An infinite matrix* $[T]$ *represents a compact self-adjoint operator* T *if and only if the row and column vectors of* $[T]$ *form a null sequence in* c_0, *and* $[T] = \overline{[T]}^t$ *i.e.*

$$
\begin{array}{cccccccc}
 & ||c_1|| & ||c_2|| & ||c_3|| & \to 0 & & & \\
[T] = & \left(\begin{array}{c} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \\ \vdots \end{array}\right. & \begin{array}{c} \alpha_{12} \\ \alpha_{22} \\ \alpha_{32} \\ \vdots \end{array} & \begin{array}{c} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \\ \vdots \end{array} & \left.\begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \ \end{array}\right) & \begin{array}{c} \to 0 \\ \to 0 \\ \to 0 \\ \ \end{array} & \begin{array}{c} ||r_1|| \\ ||r_2|| \\ ||r_3|| \\ \downarrow \\ 0 \end{array} \\
 & \downarrow & \downarrow & \downarrow & & & \\
 & 0 & 0 & 0 & & &
\end{array}
$$

where $\alpha_{ij} = e'_i(Te_j)$ *for all* $i, j \in \mathbb{N}$, $r_i = (\alpha_{i1}, \alpha_{i2}, \dots) \in c_0$ *is the i-th row vector of* $[T]$, *and* $c_j = (\alpha_{1j}, \alpha_{2j}, \dots) \in c_0$ *is the j-th column vector of* $[T]$, *and* $r_i = \overline{c_i}$.

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 & \downarrow \quad \downarrow \quad \downarrow & & & & \\
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where $\alpha_{ij} = e'_i(Te_j)$ for all $i, j \in \mathbb{N}$, $r_i = (\alpha_{i1}, \alpha_{i2}, \dots) \in c_0$ is the i-th row vector of $[T]$, and $c_j = (\alpha_{1j}, \alpha_{2j}, \dots) \in c_0$ is the j-th column vector of $[T]$, and $r_i = \overline{c_i}$.

In other words, an infinite matrix $[T]$ represents a compact self-adjoint operator T if and only if

$$[T] = \sum_{k=1}^{\infty} d^{t_k} A_k$$

where, for every $k \in \mathbb{N}$, $t_k \in \mathbb{Q}, t_k < t_{k+1}, \lim t_k = \infty$, A_k is an infinite matrix with entries in $\mathbb{C}$ such that $A_k = \overline{A_k}^t$, and its nonzero entries are in the first n_k rows and columns for some n_k.

$$A_k = \begin{bmatrix} & & & 0 & 0 & \cdots \\ & A_k & & 0 & 0 & \cdots \\ & & & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Corollary

Consider a compact self-adjoint operator $T \in L(c_0)$ *with* $||T|| = 1$*. If we define*

$$B_k := A_0 + \sum_{i=1}^{k} d^{\,t_i} A_i$$

and $t_k \in \mathbb{Q}$ *such that* $0 < t_k < t_{k+1}$ *for* $k = 0, 1, \ldots,$ *then*

$$||T - B_k|| = |d^{\,t_{k+1}}| \to 0 \quad \textit{and} \quad T = \lim_{k\to\infty} B_k.$$

Outline for section 4

Spectral Perturbation Theory

Definition (Again)

If $(v_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{C}^n$ (or c_0), then we say that $T \in L(\mathcal{C}^n)$ (or $T \in L(c_0)$) is diagonalizable if $T = \sum_i \alpha_i \langle \cdot, v_i\rangle v_i$.
Notation: $E_i := \langle \cdot, v_i\rangle v_i$.

Lemma

If $\lambda \in \mathcal{C}$ is not an eigenvalue for $G \in L(\mathcal{C}^n)$ and $G = \sum_i \lambda_i E_i$, then

$$(\lambda I_n - G)^{-1} = \sum_i \frac{1}{\lambda - \lambda_i} E_i$$

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Spectral Perturbation Theory

Lemma

Let $F, G \in L(\mathcal{C}^n)$ *and let* $\lambda \in \mathcal{C}$ *an eigenvalue of* F. *Then either* λ *is an eigenvalue for* G *or* $1 \leq ||(\lambda I_n - G)^{-1}|| \, ||F - G||$.

Spectral Perturbation Theory

Theorem

Let $F, G \in L(\mathcal{C}^n)$ and let $\lambda \in \mathcal{C}$ be an eigenvalue of F. If λ is not an eigenvalue of G and $G = \sum_i \lambda_i E_i$, then

$$\min_i |\lambda - \lambda_i| \leq ||F - G||.$$

Spectral Perturbation Theory

Corollary (Again)

Consider a compact self-adjoint operator $T \in L(c_0)$ with $||T|| = 1$. If we define

$$B_k := A_0 + \sum_{i=1}^{k} d^{t_i} A_i$$

and $t_k \in \mathbb{Q}$ such that $0 < t_k < t_{k+1}$ for $k = 0, 1, \ldots,$ then

$$||T - B_k|| = |d^{t_{k+1}}| \to 0 \quad \text{and} \quad T = \lim_{k \to \infty} B_k.$$

Corollary

If $\lambda \in \mathcal{R}$ is an eigenvalue of B_k, then there exists an eigenvalue $\tau \in \mathcal{R}$ of B_{k+1} such that

$$|\lambda - \tau| \leq |d^{t_{k+1}}|.$$

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$$||T - B_k|| = |d^{t_{k+1}}| \to 0 \quad \text{and} \quad T = \lim_{k \to \infty} B_k.$$

Theorem

For every eigenvalue $\lambda_0 \in \mathcal{R}$ *of* A_0*, there exists a convergent sequence* $(\lambda_k)_{k \in \mathbb{N}}$ *in* $\mathcal{R}$*, where* λ_k *is an eigenvalue of* B_k*. Moreover, its limit is an approximate eigenvalue of* T*.*

Spectral Perturbation Theory

Theorem

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Spectral Perturbation Theory

Theorem

Consider $B_1 = A_0 + d^{t_1} A_1$ *where* A_0, A_1 *are nonzero Hermitian complex matrices and* $t_1 \in \mathbb{Q}$, $t_1 > 0$. *Let* $\lambda_0 \in \mathbb{R}$ *and* $\lambda_1 \in \mathcal{R}$ *be eigenvalues of* A_0 *and* B_1 *respectively such that* $|\lambda_0 - \lambda_1| \leq |d^{t_1}|$. *If* $u \in \ker(B_1 - \lambda_1 I)$, *then there exists* $v \in \ker(A_0 - \lambda_0 I)$ *such that* $||u - v|| \leq |d^{t_1}|$.

Example of Compact self-adjoint operator

Example (A. Barria Comicheo, 2018)

Let $T \in L(c_0)$ *be defined by*

$$T(x_1, x_2, x_3, \dots) = \left(\sum_{i=2}^{\infty} x_i d^{i-2}, x_1, dx_1, d^2 x_1 \dots \right)$$

The matrix that defines this operator relative to the canonical basis of c_0 *is:*

$$[T] = \begin{pmatrix} 0 & 1 & d & d^2 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ d & 0 & 0 & 0 & \cdots \\ d^2 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Example of Compact self-adjoint operator

There exists a base for c_0 such that T has the following matrix representation:

$$[T] = \begin{pmatrix} \sigma & 0 & 0 & 0 & \cdots \\ 0 & -\sigma & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

where $\sigma := \sqrt{\sum_{i=0}^{\infty} d^{2i}}$.

Thank you!

Outline for section 5

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