Non-Archimedean Radial Calculus

Anatoly N. Kochubei
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3, Kiev, 01024 Ukraine,
E-mail: kochubei@imath.kiev.ua

Local fields. Let K be a non-Archimedean local field, that is a non-discrete totally disconnected locally compact topological field. It is well known that K is isomorphic either to a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers (if K has characteristic 0), or to the field of formal Laurent series with coefficients from a finite field, if K has a positive characteristic.
Any local field K is endowed with an absolute value $|\cdot|_K$, such that $|x|_K = 0$ if and only if $x = 0$, $|xy|_K = |x|_K \cdot |y|_K$, $|x+y|_K \leq \max(|x|_K, |y|_K)$. Denote $O = \{x \in K : \ |x|_K \leq 1\}$, $P = \{x \in K : \ |x|_K < 1\}$. O is a subring of K, and P is an ideal in O containing such an element β that $P = \beta O$. The quotient ring O/P is actually a finite field; denote by q its cardinality. We will always assume that the absolute value is normalized, that is $|\beta|_K = q^{-1}$. The normalized absolute value takes the values q^N, $N \in \mathbb{Z}$. Note that for $K = \mathbb{Q}_p$ we have $\beta = p$ and $q = p$; the p-adic absolute value is normalized.

Denote by $S \subset O$ a complete system of representatives of the residue classes from O/P. Any nonzero element $x \in K$ admits the canonical representation in the form of the convergent series

$$x = \beta^{-n}\left(x_0 + x_1\beta + x_2\beta^2 + \cdots\right)$$

where $n \in \mathbb{Z}$, $|x|_K = q^n$, $x_j \in S$, $x_0 \notin P$. For $K = \mathbb{Q}_p$, one may take $S = \{0, 1, \ldots, p-1\}$.

The additive group of any local field is self-dual, that is if χ is a fixed non-constant complex-valued additive character of K, then any other additive character can be written as $\chi_a(x) = \chi(ax)$, $x \in K$, for some $a \in K$. Below we assume that χ is a rank zero character, that is $\chi(x) \equiv 1$ for $x \in O$, while there exists such an element $x_0 \in K$ that $|x_0|_K = q$ and $\chi(x_0) \neq 1$.

Fourier transform:

$$\widetilde{f}(\xi) = \int_K \chi(x\xi) f(x)\, dx, \quad \xi \in K,$$

Inverse transform:

$$f(x) = \int_K \chi(-x\xi) \widetilde{f}(\xi)\, d\xi.$$

Test functions from $\mathcal{D}(K)$: locally constant functions with compact supports.
$\mathcal{D}'(K)$ - Bruhat-Schwartz distributions.

Vladimirov operator:

$$(D^{\alpha}\varphi)(x) = \mathcal{F}^{-1}\left[|\xi|_K^{\alpha}(\mathcal{F}(\varphi))(\xi)\right](x), \alpha > 0,$$

$\varphi \in \mathcal{D}(K)$.
The operator D^{α} can also be represented as a hypersingular integral operator:

$$(D^{\alpha}\varphi)(x) = \frac{1-q^{\alpha}}{1-q^{-\alpha-1}} \int\limits_K |y|_K^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\,dy.$$

This expression makes sense for wider classes of functions.
As an operator in L^2, D^{α} is selfadjoint, nonnegative with pure point spectrum $\{q^{\alpha N},\ N \in \mathbb{Z}\}$ of *infinite* multiplicity.

Right inverse ($\alpha > 0$):

$$\left(D^{-\alpha}\varphi\right)(x) = (f_\alpha * \varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_K |x-y|_K^{\alpha-1}\varphi(y)\,dy, \quad \varphi \in \mathcal{D}(K),$$

and

$$\left(D^{-1}\varphi\right)(x) = \frac{1-q}{q\log q} \int\limits_K \log|x-y|_K \varphi(y)\,dy.$$

Then $D^\alpha D^{-\alpha} = I$ on $\mathcal{D}(K)$, if $\alpha \neq 1$. This property remains valid on $\Phi(K)$ also for $\alpha = 1$. Here

$$\Phi(K) = \left\{\varphi \in \mathcal{D}(K) : \int\limits_K \varphi(x)\,dx = 0\right\}.$$

is the so-called Lizorkin space.

Vladimirov Operator on Radial Functions

a) *Radial eigenfunctions*
Let $u(x)=\psi(|x|)\in L_2(K)$,

$$D^\alpha u=\lambda u,\quad \lambda=q^{\alpha N},\ N\in\mathbb{Z},$$

and u is not identically zero. Then

$$u(x)=\begin{cases} cq^N(1-q^{-1}), & \text{if } |x|\le q^{-N};\\ -cq^{N-1}, & \text{if } |x|=q^{-N+1};\\ 0, & \text{if } |x|>q^{-N+1}.\end{cases}$$

It is shown that $u\in\Phi(K)$.
The only radial eigenfunction u with $u(0)=1$ (an analog of the function $t\to e^{-\lambda t}$, $t\in\mathbb{R}$) corresponds to $c=q^{-N}(1-q^{-1})^{-1}$.
We denote this function as

$$v_N(|x|_K)=\begin{cases} 1, & \text{if } |x|_K\le q^{-N},\\ -\frac{1}{q-1}, & \text{if } |x|_K=q^{-N+1},\\ 0, & \text{if } |x|_K\ge q^{-N+2},\end{cases}$$

Below we interpret this function as an analog of the classical exponential function $x \mapsto e^{-\lambda x}$. Note that $v_N \in \mathcal{D}(K)$; this is a purely non-Archimedean phenomenon reflecting the unusual topological property of K, its total disconnectedness. This function is important for the theory of p-adic wave equation (K., 2008).

b) *Explicit formula*

Lemma

If a function $u = u(|x|_K)$ *is such that*

$$\sum_{k=-\infty}^{m} q^k \left|u(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} q^{-\alpha l} \left|u(q^l)\right| < \infty,$$

for some $m \in \mathbb{Z}$, *then for each* $n \in \mathbb{Z}$ *the hypersingular integral for* $D^\alpha \varphi$ *with* $\varphi(x) = u(|x|_K)$ *exists for* $|x|_K = q^n$, *depends only on* $|x|_K$, *and*

$$(D^\alpha u)(q^n) = d_\alpha \left(1 - \frac{1}{q}\right) q^{-(\alpha+1)n} \sum_{k=-\infty}^{n-1} q^k u(q^k)$$
$$+ q^{-\alpha n-1} \frac{q^\alpha + q - 2}{1 - q^{-\alpha-1}} u(q^n) + d_\alpha \left(1 - \frac{1}{q}\right) \sum_{l=n+1}^{\infty} q^{-\alpha l} u(q^l)$$

where $d_\alpha = \dfrac{1 - q^\alpha}{1 - q^{-\alpha-1}}$.

The regularized integral

$$\boxed{(I^\alpha\varphi)(x) = (D^{-\alpha}\varphi)(x) - (D^{-\alpha}\varphi)(0).}$$

This is defined initially for $\varphi \in \mathcal{D}(K)$. In fact,

$$(I^\alpha\varphi)(x) = \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K \le |x|_K} \left(|x-y|_K^{\alpha-1} - |y|_K^{\alpha-1}\right) \varphi(y)\,dy, \quad \alpha \neq 1,$$

and

$$(I^1\varphi)(x) = \frac{1-q}{q\log q} \int\limits_{|y|_K \le |x|_K} (\log|x-y|_K - \log|y|_K)\,\varphi(y)\,dy.$$

In contrast to $D^{-\alpha}$, the integrals are taken, for each fixed $x \in K$, over bounded sets.

MAIN LEMMA

Suppose that

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right) \left|u(q^k)\right| < \infty, \quad \text{if } \alpha \neq 1,$$

$$\sum_{k=-\infty}^{m} |k| q^k \left|u(q^k)\right| < \infty, \quad \text{if } \alpha = 1,$$

for some $m \in \mathbb{Z}$. Then $I^\alpha u$ exists, it is a radial function, and for any $x \neq 0$,

$$(I^\alpha u)(|x|_K) = q^{-\alpha}|x|_K^\alpha u(|x|_K) + \frac{1-q^{-\alpha}}{1-q^{\alpha-1}} \int\limits_{|y|_K<|x|_K} \left(|x|_K^{\alpha-1} - |y|_K^{\alpha-1}\right) u(|y|_K)\, dy, \quad \alpha \neq 1,$$

and

$$(I^1 u)(|x|_K) = q^{-1}|x|_K u(|x|_K) + \frac{1-q}{q\log q} \int\limits_{|y|_K<|x|_K} (\log|x|_K - \log|y|_K)\, u(|y|_K)\, dy.$$

Proposition ("right inverse")

Suppose that for some $m \in \mathbb{Z}$,

$$\sum_{k=-\infty}^{m} \max\left(q^k, q^{\alpha k}\right) \left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} \left|v(q^l)\right| < \infty,$$

if $\alpha \neq 1$, *and*

$$\sum_{k=-\infty}^{m} |k| q^k \left|v(q^k)\right| < \infty, \quad \sum_{l=m}^{\infty} l \left|v(q^l)\right| < \infty,$$

if $\alpha = 1$. *Then there exists* $\left(D^\alpha I^\alpha v\right)(|x|_K) = v(|x|_K)$ *for any* $x \neq 0$.

Proposition ("left inverse")

Suppose that $u(0)=0$,

$$|u(q^n)| \le Cq^{dn}, \quad n \le 0;$$

$$|u(q^n)| \le Cq^{hn}, \quad n \ge 0,$$

where $d > \max(0, \alpha - 1)$, $0 \le h < \alpha$, *and* $h < \alpha - 1$, *if* $\alpha > 1$. *Then the function* $w = D^\alpha u$ *satisfies, for any* $m \in \mathbb{Z}$, *the inequalities*

$$\sum_{k=-\infty}^{m} \max(q^k, q^{\alpha k}) \left| w(q^k) \right| < \infty, \quad \sum_{l=m}^{\infty} \left| w(q^l) \right| < \infty, \quad \alpha \ne 1;$$

$$\sum_{k=-\infty}^{m} |k| \cdot q^k \left| w(q^k) \right| < \infty, \quad \sum_{l=m}^{\infty} |l| \cdot \left| w(q^l) \right| < \infty, \quad \alpha = 1.$$

Moreover, $I^\alpha D^\alpha u = u$.

Corollary

Let $v = v_0 + u$ where v_0 is a constant, u satisfies the conditions of the above Proposition. Then $I^\alpha D^\alpha v = v - v_0$.

EXAMPLE: the simplest Cauchy problem

$$D^\alpha u(|x|_K) = f(|x|_K), \quad u(0) = 0,$$

where f is a continuous function, such that

$$\sum_{l=m}^{\infty} \left| f(q^l) \right| < \infty, \text{ if } \alpha \neq 1, \text{ or } \sum_{l=m}^{\infty} l \left| f(q^l) \right| < \infty, \text{ if } \alpha = 1.$$

The unique strong solution is $u = I^\alpha f$. Therefore on radial functions, the operators D^α and I^α behave like the Caputo-Dzhrbashyan fractional derivative and the Riemann-Liouville fractional integral of real analysis.

(COUNTER)-EXAMPLE: Let $f(|x|_K) \equiv 1$, $x \in K$. Then $(I^\alpha f)(|x|_K) \equiv 0$.

THE CAUCHY PROBLEM

a) Local solvability. Consider the equation

$$(D^{\alpha}u)(|t|_K) = f(|t|_K, u(|t|_K)), \quad 0 \neq t \in K, \tag{1}$$

with the initial condition

$$u(0) = u_0 \tag{2}$$

where the function $f: \ q^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions

$$|f(|t|_K, x)| \leq M; \tag{3}$$

$$|f(|t|_K, x) - f(|t|_K, y)| \leq F|x - y|, \tag{4}$$

for all $t \in K, x, y \in \mathbb{R}$.

With the problem (1)-(2) we associate the integral equation

$$u(|t|_K) = u_0 + I^\alpha f(|\cdot|_K, u(|\cdot|_K))(|t|_K). \tag{5}$$

Note that, by the definition of I^α, in order to compute $(I^\alpha\varphi)(|t|_K)$ for $|t|_K \le q^m$ ($m \in \mathbb{Z}$), one needs to know the function φ in the same ball $|t|_K \le q^m$. Therefore local solutions of the equation (5) make sense, in contrast to solutions of (1).

We call a solution u of (5), if it exists, *a mild solution* of the Cauchy problem (1)-(2). By the above Corollary, a solution u of (1)-(2), such that $u - u_0$ satisfies the conditions of Proposition, is a mild solution.

Theorem

Under the assumptions (3),(4), the problem (1)-(2) has a unique local mild solution, that is the integral equation (5) has a solution $u(|t|_K)$ defined for $|t|_K \le q^N$ where $N \in \mathbb{Z}$ is sufficiently small, and another solution $\overline{u}(|t|_K)$, if it exists, coincides with u for $|t|_K \le q^{\overline{N}}$ where $\overline{N} \le N$.

b) Extension of solutions. Let us study the possibility to continue the local solution constructed in Theorem 1 to a solution of the integral equation (5) defined for all $t \in K$.
Suppose that the conditions of Theorem 1 are satisfied, and we obtained a local solution $u(|t|_K)$, $|t|_K \leq q^N$, $N \in \mathbb{Z}$. Let $\alpha \neq 1$. In order to find a solution for $|t|_K = q^{N+1}$, we have to solve the equation

$$u(q^{N+1}) = u_0 + v_0^{(N)} + q^{\alpha N} f(q^{N+1}, u(q^{N+1})) \tag{6}$$

where

$$v_0^{(N)} = \frac{1 - q^{-\alpha}}{1 - q^{\alpha - 1}} \int\limits_{|y|_K \leq q^N} \left(q^{(N+1)(\alpha-1)} - |y|_K^{\alpha-1} \right) f(|y|_K, u(|y|_K))\, dy. \tag{7}$$

is a known constant. A similar equation can be written for $\alpha = 1$. Then the above procedure, if it is successful, is repeated for all $l > N$.

Theorem

Suppose that the conditions of local existence theorem are satisfied, as well as the following Lipschitz condition:

$$|f(q^l, x) - f(q^l, y)| \leq F_l |x - y|, \quad x, y \in \mathbb{R}, \ l \in \mathbb{Z}, \tag{8}$$

where $0 < F_l < q^{-\alpha l}$ *for each* $l \in \mathbb{Z}$. *Then a local solution of the equation (5) admits a continuation to a global solution defined for all* $t \in K$.

c) From an integral equation to a differential one. Let us study conditions, under which the above continuation procedure leads to a solution of the problem (1)-(2). As before, we assume the conditions (3),(4) and (8). In addition, we will assume that

$$|f(q^l, x)| \leq Cq^{-\beta l}, \quad l \geq 1, \quad \text{for all } x \in \mathbb{R}, \tag{9}$$

where $\beta > \alpha$.

Theorem

Under the assumptions (3),(4), (8) and (9), the mild solution obtained by the iteration process with subsequent continuation, satisfies the equation (1).

Vladimirov operator on the unit ball

The operator D^α_O in the space $L^2(O)$ on the ring of integers (unit ball) O is defined as follows. Extend a function $\varphi \in \mathcal{D}(O)$ (that is a function $\varphi \in \mathcal{D}(K)$ supported in O) onto K by zero. Apply D^α and consider the resulting function on O. After the closure in $L^2(O)$ we obtain a selfadjoint operator D^α_O with a discrete spectrum.
Denote by $\mathcal{H}$ the subspace in $L^2(O)$ consisting of radial functions. The functions

$$e_0(|x|_K) \equiv 1; \quad e_N(|x|_K) = (q-1)^{1/2} q^{N/2} v_N(|x|_K), \quad N \geq 1,$$

form an orthonormal basis in $\mathcal{H}$.

Another (obvious) orthonormal basis in $\mathcal{H}$ is

$$f_n(|x|_K) = \begin{cases} (1-\frac{1}{q})^{-1/2} q^{n/2}, & \text{if } |x|_K = q^{-n}; \\ 0, & \text{elsewhere}, \end{cases} \qquad n = 0, 1, 2, \dots .$$

Integration Operators

Let us study the integral part of the operator I^1, that is

$$\left(I_0^1 u\right)(x) = \frac{1-q}{q\log q} \int\limits_{|y|_K<|x|_K} \left(\log|x|_K - \log|y|_K\right) u(|y|_K)\, dy.$$

Recall that a compact operator is called a Volterra operator, if its spectrum consists of the unique point $\lambda = 0$. An operator A is called simple, if A and A^* have no common nontrivial invariant subspace, on which these operators coincide. It is known that a Volterra operator A is simple, if and only if the equations $Af = 0$ and $A^* f = 0$ have no common nontrivial solutions.

Theorem

The operator I_0^1 in $\mathcal{H}$ is a simple Volterra operator with a rank 2 imaginary part $J = \frac{1}{2i}(A - A^*)$, such that $\operatorname{tr} J = 0$.

Characteristic Function

Let us write the imaginary part J in the form

$$\frac{1}{i}\left(I_0^1 - \left(I_0^1\right)^*\right) u = \sum_{\alpha,\beta=1}^{2} \langle u, h_\alpha \rangle j_{\alpha\beta} h_\beta$$

where $h_1(|x|_K) = \dfrac{q-1}{iq\log q}(= \text{const})$, $h_2(|x|_K) = -\log|x|_K$, $x \in O$, $j = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

For the operator I_0^1, we consider the 2×2 characteristic matrix-function of inverse argument

$$W(z^{-1}) = E + izj\left[\left\langle \left(E - zI_0^1\right)^{-1} h_\alpha, h_\beta \right\rangle\right]_{\alpha,\beta=1}^{2}$$

where E denotes both the unit operator in $\mathcal{H}$ and the unit matrix.

For the Volterra operator I_0^1, $W(z^{-1})$ is an entire matrix-function.

Theorem

Matrix elements of $W(z^{-1})$ are entire functions of zero order.

The Laplace Type Transform

We call the function

$$\widetilde{\varphi}(|\xi|_K) = \int\limits_K v_0(|x\xi|_K)\varphi(|x|_K)\,dx$$

the Laplace type transform of a radial function $\varphi \in L^1_{\mathrm{loc}}(K)$. By the dominated convergence theorem, $\widetilde{\varphi}$ is continuous, bounded, and $\widetilde{\varphi}(|\xi|_K) \to 0$, $|\xi|_K \to \infty$.

If $\varphi(|x|_K) \equiv \mathrm{const}$, then $\widetilde{\varphi}(|\xi|_K) \equiv 0$.

We have

$$\widetilde{D^\alpha\varphi}(|\xi|_K) = |\xi|_K^\alpha \widetilde{\varphi}(|\xi|_K), \quad \xi \in K.$$

Theorem (uniqueness)

If $\widetilde{\varphi}(|\xi|_K) \equiv 0$, then $\varphi(|x|_K) \equiv \mathrm{const}$.

Proposition

For all $n \in \mathbb{Z}$,

$$\widetilde{\varphi}(q^n) - \widetilde{\varphi}(q^{n+1}) = q^{-n}\left[\varphi(q^{-n}) - \varphi(q^{-n+1})\right].$$

Corollary

A function φ is (strictly) monotone, if and only if $\widetilde{\varphi}$ is (strictly) monotone.

Theorem (inversion formula)

For each $n = 1, 2, \ldots$,

$$\varphi(q^m) = \varphi(1) + \sum_{j=0}^{m-1} q^{-j} \left[\widetilde{\varphi}(q^{-j+1}) - \widetilde{\varphi}(q^{-j})\right],$$

$$\varphi(q^{-m}) = \varphi(1) + \sum_{j=1}^{m} q^{j} \left[\widetilde{\varphi}(q^{j}) - \widetilde{\varphi}(q^{j+1})\right].$$

Some Publications

1. A. N. Kochubei, A non-Archimedean wave equation, *Pacif. J. Math.* **235** (2008), 245–261.

2. A. N. Kochubei, Radial solutions of non-Archimedean pseudodifferential equations, *Pacif. J. Math.* **269** (2014), 355–369.

3. A. N. Kochubei, Nonlinear pseudo-differential equations for radial real functions on a non-Archimedean field. *J. Math. Anal. Appl.* **483** (2020), no. 1, Article 123609.

4. A. N. Kochubei, Non-Archimedean Radial Calculus: Volterra Operator and Laplace Transform, *Integr. Equ. Oper. Theory*, **92** (2020), Article 44.