Rigorous nonperturbative results related to p-adic AdS/CFT

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1. Introduction
2. The hierarchical continuum
3. The rigorous hierarchical space-dependent renormalization group

QFT basics:

A quantum field theory model on $\mathbb{R}^d$ (example) can be seen as a sequence indexed by $n \geq 0$ of (correlation) functions $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ which depend on n points in $\mathbb{R}^d$. Such functions would be given nonrigorously by

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{\mathcal{Z}} \int_\Omega \phi(x_1) \cdots \phi(x_n) \times$$

$$\exp\left(- \int_{\mathbb{R}^d} \{ (\partial \phi)^2(x) + \mu \phi^2(x) + g \phi^4(x) \} d^d x \right) D\phi$$

where Ω (probability space) is a space of functions from $\mathbb{R}^d$ to $\mathbb{R}$, $D\phi$ is the Lebesgue measure on Ω and $\mathcal{Z}$ (the partition function) is a normalization constant. Namely,

$$\mathcal{Z} = \int_\Omega \exp\left(- \int_{\mathbb{R}^d} \{ (\partial \phi)^2(x) + \mu \phi^2(x) + g \phi^4(x) \} d^d x \right) D\phi$$

Of particular interest is the case where the correlations satisfy conformal invariance, i.e., the QFT is a CFT.

There is no $D\phi$ because Ω is infinite dimensional. To make sense of the wanted probability measure: discretize, work in finite volume, and then take weak limits. Fix some number $L > 1$. For $r, s \in \mathbb{Z}$, replace $\mathbb{R}^d$ by a finite set of points, namely, the points in a lattice of mesh L^r, which fit in a box of linear size L^s. This replaces Ω by a finite dimensional space $\mathbb{R}^{L^{d(s-r)}}$.

The goal is to obtain the wanted probability measure ν, whose moments are the $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ as a (double) weak limit of well defined probability measures $\nu_{r,s}$, namely

$$\nu = \lim_{r \to -\infty} \lim_{s \to \infty} \nu_{r,s} \ .$$

Similar in spirit to numerical approximations by Monte-Carlo methods. . .

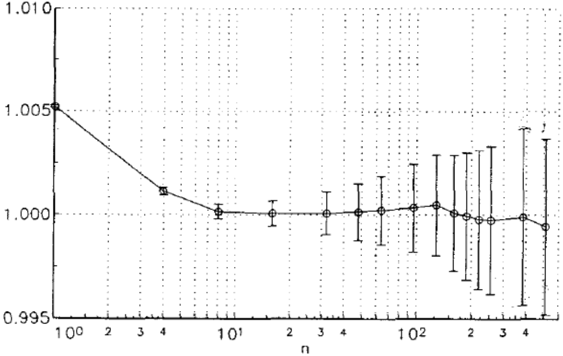


Fig. 17. Ratio of the SPP computed correlation function to the exact continuum limit function for $L = 1024$.

From Talapov et al. IJMP 1993 (Thanks to S. Rychkov for this reference)

A touristic view of AdS/CFT:

Let $\widehat{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\} \simeq \mathbb{S}^d$.
The Möbius group $\mathcal{M}(\mathbb{R}^d)$ is the group of bijective transformations of $\widehat{\mathbb{R}^d}$ generated by isometries, dilations and the unit sphere inversion $J(x) = |x|^{-2}x$.
This is also the invariance group of the absolute cross-ratio

$$CR(x_1, x_2, x_3, x_4) = \frac{|x_1 - x_3|\ |x_2 - x_4|}{|x_1 - x_4|\ |x_2 - x_3|}\ .$$

Conformal ball model: $\widehat{\mathbb{R}^d} \simeq \mathbb{S}^d$ seen as boundary of $\mathbb{B}^{d+1}$ with metric $ds = \frac{2|dx|}{1-|x|^2}$.
Half-space model: $\mathbb{R}^d$ seen as boundary of $\mathbb{H}^{d+1} = \mathbb{R}^d \times (0, \infty)$ with metric $ds = \frac{|dx|}{x_{d+1}}$.
Bijection: $f \in \mathcal{M}(\mathbb{R}^d)$ $\leftrightarrow$ hyperbolic isometry of the interior $\mathbb{B}^{d+1}$ or $\mathbb{H}^{d+1}$, the Euclidean AdS space.

A scalar field $\mathcal{O}$ of scaling dimension Δ in a CFT on $\mathbb{R}^d$ has pointwise correlations which satisfy

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \left(\prod_{i=1}^{n} |J_f(x_i)|^{\frac{\Delta}{d}}\right) \times \langle \mathcal{O}(f(x_1))\cdots\mathcal{O}(f(x_n))\rangle$$

for all $f \in \mathcal{M}(\mathbb{R}^d)$ and all collection of distinct points in $\mathbb{R}^d \backslash \{f^{-1}(\infty)\}$.
Here, $J_f(x)$ denotes the Jacobian of f at x.
The AdS/CFT correspondence, discovered by Maldacena 1997 and made more precise by Gubser, Klebanov, Polyakov and Witten 1998, postulates a relation of the form:

$$\left\langle \ e^{\int_{\mathbb{R}^d} j(x)\mathcal{O}(x)d^dx} \ \right\rangle_{\mathrm{CFT}} = e^{-S[\phi_{\mathrm{ext}}]}$$

where $S[\phi]$ is an action for a field $\phi(x, x_{d+1})$ on AdS space and ϕ_{ext} makes it extremal for a boundary condition $\phi(x, x_{d+1}) \sim (x_{d+1})^{d-\Delta} j(x)$ when $x_{d+1} \to 0$.

AdS/CFT or holographic correspondence not yet known explicitly, i.e., exact $S[\phi]$ still mysterious. However, physicists have been experimenting with toy actions of the form:

$$\int_{\mathbb{R}^d \times (0,\infty)} d^d x \, dx_{d+1} \sqrt{\det g_{\mu\nu}} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \cdots \right\}$$

where m^2 is related to Δ and is allowed to be (not too) negative.

This gives an expansion for connected CFT correlations in terms of tree-level Feynman diagrams (Witten diagrams). The simplest "Mercedes logo" 3-point Witten diagram reproduces the correct CFT prediction

$$\frac{O(1)}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

for $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle$ by a calculation of Freedman, Mathur, Matusis and Rastelli 1999.

The good news:

All of the above makes sense for the hierarchical model, i.e., p-adic analogue.
See in particular:

- Melzer, IJMP 1989.
- Lerner, Missarov, LMP 1991.
- Gubser et al. "p-Adic AdS/CFT", CMP 2017.
- Gubser et al. "$O(N)$ and $O(N)$ and $O(N)$", JHEP 2017.

The calculations of the last reference for scaling dimensions of Φ and Φ^2, for $N = 1$ in hierarchical case were made nonperturbatively rigorous in:

"Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimensions", arXiv 2013, by A.A., Ajay Chandra (Imperial College), Gianluca Guadagni (UVa).

The hierarchical or p-adic continuum:

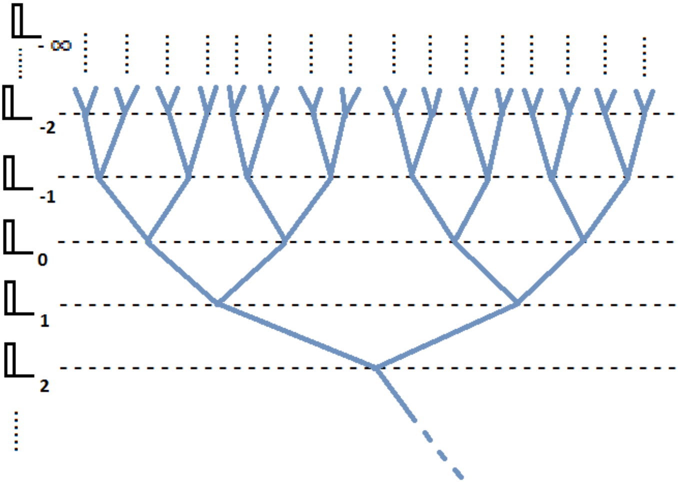
Let p be a prime number. We will replace $\mathbb{R}^d$ by $\mathbb{Q}_p^d$, so now Ω becomes a space of functions from $\mathbb{Q}_p^d$ to $\mathbb{R}$. For $x = (x_1, \ldots, x_d) \in \mathbb{Q}_p^d$ we let

$$|x|_p = \max(|x_1|_p, \ldots, |x_d|_p) \ .$$

We equip $\mathbb{Q}_p^d$ with the distance $|x - y|_p$.

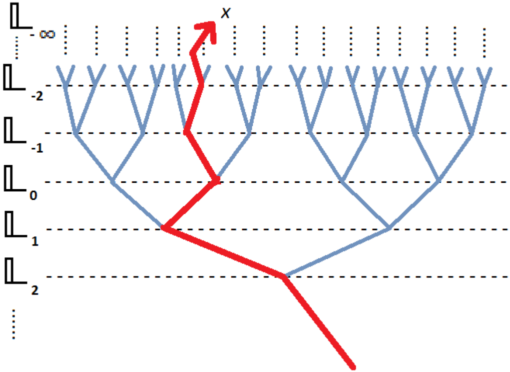
Let $\mathbb{L}_k$, $k \in \mathbb{Z}$, be the set of closed balls of radius p^k in $\mathbb{Q}_p^d$, i.e., $\mathbb{L}_k = \mathbb{Q}_p^d / p^{-k}\mathbb{Z}_p^d$.

Hence $\mathbb{T} = \cup_{k\in\mathbb{Z}}\mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

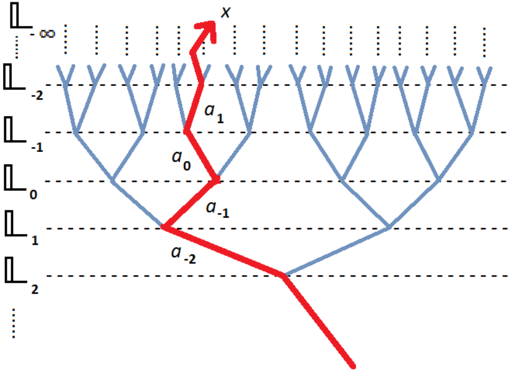
The hierarchical continuum $\mathbb{Q}_p^d$ = leafs at infinity "$\mathbb{L}_{-\infty}$".
More precisely, these leafs at infinity are the infinite bottom-up paths in the tree. $\mathbb{T}$, with the graph distance, will play the role of hyperbolic space $\mathbb{H}^{d+1}$ of AdS bulk space.



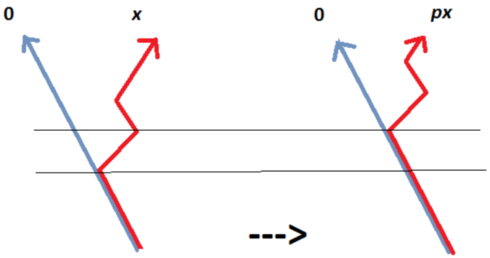
A path representing an element $x \in \mathbb{Q}_p^d$

A point $x = (x_1, \ldots, x_d) \in \mathbb{Q}_p^d$ is encoded by a sequence $(a_n)_{n\in\mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p-1\}^d$. For each n, $a_n = (a_{n,1}, \ldots, a_{n,d})$ and these are the digits of the p-adic expansions $x_i = \sum_{n\in\mathbb{Z}} a_{n,i} p^n$.

a_n represents the local coordinates for a cube of $\mathbb{L}_{-n-1}$ inside a cube of $\mathbb{L}_{-n}$.



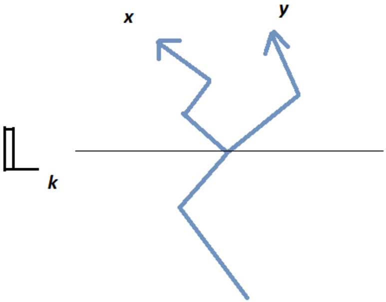
Moreover, rescaling is defined as follows.
If $x = (a_n)_{n\in\mathbb{Z}}$ then $px := (a_{n-1})_{n\in\mathbb{Z}}$, i.e., upward shift.



Likewise $p^{-1}x$ is downward shift, and so on for the definition of $p^k x$, $k \in \mathbb{Z}$.

Distance:

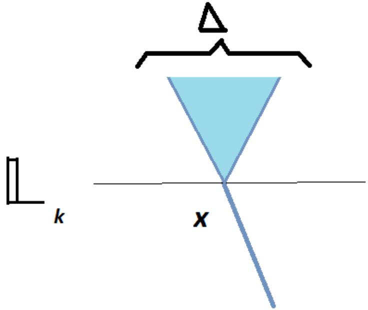
If $x, y \in \mathbb{Q}_p^d$, their distance can be visualized as $|x-y|_p = p^k$ where k is the depth where the two paths merge.



Keep in mind that

$$|px|_p = p^{-1}|x|_p$$

Closed balls Δ of radius p^k correspond to the nodes $\mathbf{x} \in \mathbb{L}_k$



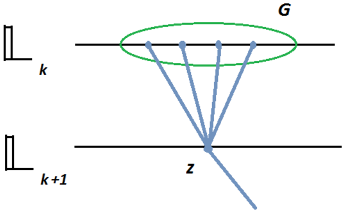
Lebesgue measure:

Metric space $\mathbb{Q}_p^d$ $\to$ Borel σ-algebra $\to$ Lebesgue (or additive Haar) measure $d^d x$ which gives a volume p^{dk} to closed balls of radius p^k.

The hierarchical unit lattice:
Truncate the tree at level zero and take $\mathbb{L} := \mathbb{L}_0$. Using the identification of nodes with balls, define the hierarchical distance as

$$d(\mathbf{x}, \mathbf{y}) = \inf\{|x - y|_p \mid x \in \mathbf{x},\ \ y \in \mathbf{y}\}\ .$$

The massless Gaussian measure:



To every group of offsprings G of a vertex $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x} \in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different groups are independent. We have $\sum_{\mathbf{x} \in G} \zeta_{\mathbf{x}} = 0$ a.s.

The ancestor function: for $k < k'$, $\mathbf{x} \in \mathbb{L}_k$, let $\mathrm{anc}_{k'}(\mathbf{x})$ denote the ancestor in $\mathbb{L}_{k'}$.
Ditto for $\mathrm{anc}_{k'}(x)$ when $x \in \mathbb{Q}_p^d$.
The massless Gaussian field $\phi(x)$, $x \in \mathbb{Q}_p^d$ of scaling dimention $[\phi]$ is given by

$$\phi(x) = \sum_{k \in \mathbb{Z}} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

$$\langle \phi(x)\phi(y) \rangle = \frac{c}{|x-y|^{2[\phi]}}$$

This is heuristic since ϕ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions.
I will now drop the p from $|\cdot|_p$.

Test functions:

$f : \mathbb{Q}_p^d \to \mathbb{R}$ is smooth if it is locally constant.
Define $S(\mathbb{Q}_p^d)$ as the space of compactly supported smooth functions.
Take locally convex topology generated by the set of all semi-norms on $S(\mathbb{Q}_p^d)$.

Distributions:

$S'(\mathbb{Q}_p^d)$ is the dual space with strong topology (happens to be same as weak-$*$).

$$S(\mathbb{Q}_p^d) \simeq \oplus_{\mathbb{N}} \mathbb{R} \ .$$

Thus

$$S'(\mathbb{Q}_p^d) \simeq \mathbb{R}^{\mathbb{N}}$$

with product topology. $\Omega := S'(\mathbb{Q}_p^d)$ is a Polish space.

The p-adic CFT toy model:

$d = 3$, $[\phi] = \frac{3-\epsilon}{4}$, $L = p^{\ell}$ zooming-out factor

$r \in \mathbb{Z}$ UV cut-off, $r \to -\infty$

$s \in \mathbb{Z}$ IR cut-off, $s \to \infty$

The regularized Gaussian measure μ_{C_r} is the law of

$$\phi_r(x) = \sum_{k=\ell r}^{\infty} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

Sample fields are true fonctions that are locally constant on scale L^r. These measures are scaled copies of each other.

If the law of $\phi(\cdot)$ is μ_{C_0}, then that of $L^{-r[\phi]}\phi(L^r \cdot)$ is μ_{C_r}.

The same Gaussian measures can be defined using (the less intuitive) Fourier representation:

$$\langle \phi(x_1)\phi(x_2)\rangle_{\mu_{C_{-\infty}}} = \int_{\mathbb{Q}^d} \frac{e^{2\pi i\{\xi(x_1-x_2)\}_p}}{|\xi|^{d-2[\phi]}} d^d\xi$$

and

$$\langle \phi(x_1)\phi(x_2)\rangle_{\mu_{C_r}} = \int_{\mathbb{Q}^d} \frac{\chi_r(\xi)e^{2\pi i\{\xi(x_1-x_2)\}_p}}{|\xi|^{d-2[\phi]}} d^d\xi$$

where $\chi_r(\xi) = \mathbb{1}\{|\xi| \le L^{-r}\}$ is a sharp UV cutoff.

Fix the dimensionless parameters g, μ and let $g_r = L^{-(3-4[\phi])r} g$ and $\mu_r = L^{-(3-2[\phi])r} \mu$. Same as strict scaling limit of fixed critical probability measure on unit lattice. Bare/dimensionful couplings g_r, μ_r go to ∞.

Let $\Lambda_s = \overline{B}(0, L^s)$, IR (or volume) cut-off.

Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r (x)\} d^3x$$

where $: \phi^k :_r$ is Wick ordering using $d\mu_{C_r}$.
Define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi) \ .$$

Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = (Z_2)^r \int_{\mathbb{Q}_p^3} \{Y_2 : \phi_{r,s}^2 :_r (x) - Y_0 L^{-2r[\phi]}\} \, j(x) \, d^3x$$

for suitable parameters Z_2, Y_0, Y_2. We also need a Y_1.

Our main result concerns the limit law of the pair $(Y_1 \phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order).

For the precise statement we need the approximate fixed point value

$$\bar{g}_* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})} \ .$$

Theorems:

Theorem 1: A.A.-Chandra-Guadagni 2013

$\exists \rho > 0$, $\exists L_0$, $\forall L \geq L_0$, $\exists \epsilon_0 > 0$, $\forall \epsilon \in (0, \epsilon_0]$, $\exists [\phi^2] > 2[\phi]$, $\exists$ fonctions $\mu(g)$, $Y_0(g)$, $Y_2(g)$ on $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$ such that if one lets $\mu = \mu(g)$, $Y_0 = Y_0(g)$, $Y_2 = Y_2(g)$ and $Z_2 = L^{-([\phi^2]-2[\phi])}$ then the joint law of $(Y_1\phi_{r,s}, N_r[\phi^2_{r,s}])$ converge weakly and in the sense of moments to that of a pair $(\phi, N[\phi^2])$ such that:

1. $\forall k \in \mathbb{Z}$, $(L^{-k[\phi]}\phi(L^k\cdot), L^{-k[\phi^2]}N[\phi^2](L^k\cdot)) \stackrel{d}{=} (\phi, N[\phi^2])$.
2. $\langle \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3}), \phi(\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} < 0$ i.e., ϕ is non-Gaussian. Here, $\mathbf{1}_{\mathbb{Z}_p^3}$ denotes the indicator function of $\overline{B}(0,1)$.
3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3})\rangle^{\mathrm{T}} = 1$.
4. $\langle \phi(\mathbf{1}_{\mathbb{Z}_p^3})^2\rangle = 1$.

The mixed correlation functions satisfy, in the sense of distributions,

$$\langle\ \phi(L^{-k}x_1)\cdots\phi(L^{-k}x_n)\ N[\phi^2](L^{-k}y_1)\cdots N[\phi^2](L^{-k}y_m)\ \rangle$$

$$= L^{-(n[\phi]+m[\phi^2])k}\ \langle\ \phi(x_1)\cdots\phi(x_n)\ N[\phi^2](y_1)\cdots N[\phi^2](y_m)\ \rangle$$

For our hierarchical version of the 3D fractional ϕ^4 model we also proved $[\phi^2]-2[\phi]=\frac{1}{3}\epsilon+o(\epsilon)$.

This was predicted by Wilson in "Renormalization of a scalar field theory in strong coupling", PRD 1972.

This is also what is expected for the Euclidean model on $\mathbb{R}^3$.

Not too far, if one boldly extrapolates to $\epsilon=1$, from the most precise available estimates concerning the short range 3D Ising model: $[\phi^2]-2[\phi]=0.376327\ldots$ (JHEP 2016 by Kos, Poland, Simmons-Duffin and Vichi, using conformal bootstrap).

We also proved the law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$, is independent of g in the interval $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$. This also holds if one also adds ϕ^6, ϕ^8,... terms in the potential, with small couplings. We proved strong local universality for a non-Gaussian scaling limit.

Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of the scaling group $p^{\mathbb{Z}}$ instead of the subgroup $L^{\mathbb{Z}}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the arbitrary factor L.

The two-point correlations are given in the sense of distributions by

$$\langle\phi(x)\phi(y)\rangle = \frac{c_1}{|x-y|^{2[\phi]}}$$

$$\langle N[\phi^2](x)\ N[\phi^2](y)\rangle = \frac{c_2}{|x-y|^{2[\phi^2]}}$$

Note that $2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon) \to$ still $L^{1,\mathrm{loc}}$!

Theorem 3: A.A., May 2015

Use ψ_i to denote the scaling limits ϕ or $N[\phi^2]$. Then, for all mixed correlation $\exists$ a smooth (i.e., locally constant) fonction $\langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle$ on $(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}$ which is locally integrable (on the big diagonal Diag) and such that

$$\mathbb{E}\ \psi_1(f_1)\cdots\psi_n(f_n) = \int_{(\mathbb{Q}_p^3)^n\backslash\mathrm{Diag}} \langle\psi_1(z_1)\cdots\psi_n(z_n)\rangle\ f_1(z_1)\cdots f_n(z_n)\ d^3z_1\cdots d^3z_n$$

for all test functions $f_1,\ldots,f_n \in S(\mathbb{Q}_p^3)$.

This hinges on showing the BNNFB (basic nearest neighbor factorized bound) of A.A., "A Second-Quantized Kolmogorov-Chentsov Theorem via the Operator Product Expansion", CMP 2020. The BNNFB is

$$| \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle | \leq O(1) \times \prod_{i=1}^{n} \frac{1}{|z_i - \mathrm{n.n.}|^{[\psi_i]}}$$

when $z_1, \ldots, z_n$ are confined to a compact set.

This follows from the use of the SDRG (space-dependent renormalization group) to derive an explicit representation of pointwise correlations in terms of very close analogues of tree Witten diagrams. Hence, the emergent connection to the AdS/CFT correspondence.

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3. The rigorous hierarchical space-dependent renormalization group

The renormalization group idea in a nutshell:

Want to study feature $\mathcal{Z}(\vec{V})$ of some object $\vec{V} \in \mathcal{E}$ but too hard!

Find "simplifying" transformation $RG : \mathcal{E} \to \mathcal{E}$, such that $\mathcal{Z}(RG(\vec{V})) = \mathcal{Z}(\vec{V})$, and $\lim_{n\to\infty} RG^n(\vec{V}) = \vec{V}_*$ with $\mathcal{Z}(\vec{V}_*)$ easy.

Example: $\vec{V} = (a, b) \in \mathcal{E} = (0, \infty)^2$

$$\mathcal{Z}(\vec{V}) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \ .$$

Take $RG(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$.
(Landen-Gauss)

1st step: rescale to unit lattice/cut-off

$$\mathcal{S}^{\mathrm{T}}_{r,s}(f) := \log \mathbb{E}_{\nu_{r,s}} e^{\phi(f)} = \log$$

$$\frac{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r :\phi^4:_r(x) + \mu_r :\phi^2:_r\}dx + \int \phi(x)f(x)dx\right)}{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r :\phi^4:_r(x) + \mu_r :\phi^2:_r\}dx\right)}$$

$$= \log \frac{\int d\mu_{C_0}(\phi)\ \mathcal{I}^{(r,r)}[f](\phi)}{\int d\mu_{C_0}(\phi)\ \mathcal{I}^{(r,r)}[0](\phi)} =: \log \frac{\mathcal{Z}(\vec{V}^{(r,r)}[f])}{\mathcal{Z}(\vec{V}^{(r,r)}[0])}$$

with

$$\mathcal{I}^{(r,r)}[f](\phi) = \exp\left(-\int_{\Lambda_{s-r}}\{g :\phi^4:_0(x) + \mu :\phi^2:_0\}d^3x\right.$$

$$\left. + L^{(3-[\phi])r}\int \phi(x)f(L^{-r}x)d^3x\right)$$

2nd step: define inhomogeneous RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Associated Gaussian measure is the law of the fluctuation field

$$\zeta(x) = \sum_{0 \le k < \ell} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

L-blocks (closed balls of radius L) are independent. Hence

$$\int \mathcal{I}^{(r,r)}[f](\phi)\ d\mu_{C_0}(\phi) = \int \int \mathcal{I}^{(r,r)}[f](\zeta + \psi)\ d\mu_\Gamma(\zeta) d\mu_{C_1}(\psi)$$

$$= \int \mathcal{I}^{(r,r+1)}[f](\phi)\ d\mu_{C_0}(\phi)$$

with new integrand

$$\mathcal{I}^{(r,r+1)}[f](\phi) := \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

Need to extract vacuum renormalization $\rightarrow$ better definition is

$$\mathcal{I}^{(r,r+1)}[f](\phi) = e^{-\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

so that we have the fundamental identity

$$\int \mathcal{I}^{(r,r)}[f](\phi)\ d\mu_{C_0}(\phi) = e^{\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r+1)}[f](\phi)\ d\mu_{C_0}(\phi)$$

Repeat: $\mathcal{I}^{(r,r)} \rightarrow \mathcal{I}^{(r,r+1)} \rightarrow \mathcal{I}^{(r,r+2)} \rightarrow \cdots \rightarrow \mathcal{I}^{(r,s)}$

One must control

$$\mathcal{S}^{\mathrm{T}}(f) = \lim_{\substack{r\rightarrow-\infty \\ s\rightarrow\infty}} \sum_{r\le q<s} \left(\delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0])\right)$$

limit of logarithms of characteristic functions.

Use a Brydges-Yau lift

$$\begin{array}{ccc} & RG_{\mathrm{inhom}} & \\ \vec{V}^{(r,q)} & \longrightarrow & \vec{V}^{(r,q+1)} \\ \downarrow & & \downarrow \\ \mathcal{I}^{(r,q)} & \longrightarrow & \mathcal{I}^{(r,q+1)} \end{array}$$

$$\mathcal{I}^{(r,q)}(\phi) = \prod_{\substack{\Delta\in\mathbb{L}_0 \\ \Delta\subset\Lambda_{s-q}}} \Big[e^{f_\Delta \phi_\Delta} \times$$

$$\Big\{ \exp\left(-\beta_{4,\Delta} :\phi_\Delta^4:_{C_0} - \beta_{3,\Delta} :\phi_\Delta^3:_{C_0} - \beta_{2,\Delta} :\phi_\Delta^2:_{C_0} - \beta_{1,\Delta} :\phi_\Delta^1:_{C_0} \right)$$

$$\times \left(1 + W_{5,\Delta} :\phi_\Delta^5:_{C_0} + W_{6,\Delta} :\phi_\Delta^6:_{C_0} \right)$$

$$+ R_\Delta(\phi_\Delta) \Big\} \Big]$$

Dynamical variable is $\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}_0}$ with

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

RG_{inhom} acts on $\mathcal{E}_{\mathrm{inhom}}$, essentially,

$$\prod_{\Delta \in \mathbb{L}_0} \{\mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C})\}$$

Stable subspaces

$\mathcal{E}_{\mathrm{hom}} \subset \mathcal{E}_{\mathrm{inhom}}$: spatially constant data.
$\mathcal{E} \subset \mathcal{E}_{\mathrm{hom}}$: even potential, i.e., g, μ's only and R even function.
Let RG be induced action of RG_{inhom} on $\mathcal{E}$.

3rd step: stabilize bulk (homogeneous) evolution

Show that $\forall q \in \mathbb{Z}$, $\lim_{r\to-\infty} \vec{V}^{(r,q)}[0]$ exists, i.e.,

$$\lim_{r\to-\infty} RG^{q-r}\left(\vec{V}^{(r,r)}[0]\right)$$

exists.

$$RG \left\{ \begin{array}{lllllllll} g' & = & L^{\epsilon} g & - & A_1 g^2 & + & \cdots & & \\ \mu' & = & L^{\frac{3+\epsilon}{2}} \mu & - & A_2 g^2 & - & A_3 g\mu & + & \cdots \\ R' & = & \mathcal{L}^{(g,\mu)}(R) & + & \cdots & & & & \end{array} \right.$$

Tadpole graph with mass insertion

$$A_3 = 12 L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(0,x)^2 \ d^3x$$

is main culprit for anomalous scaling $[\phi^2] - 2[\phi] > 0$.

Irwin's proof $\to$ stable manifold W^{s}

Restriction to W^{s} $\to$ contraction $\to$ IR fixed point v_*.

Construct unstable manifold W^{u}, intersect with W^{s}, transverse at v_*.

Here, $\vec{V}^{(r,r)}[0]$ is independent of r: strict scaling limit of fixed model on unit lattice. (We can also do the Gaussian to non-Gaussian crossover continuum limit).

$\vec{V}^{(r,r)}[0]$ must be chosen in W^{s} $\to$ $\mu(g)$ critical mass.

Thus

$$\forall q \in \mathbb{Z}, \qquad \lim_{r \to -\infty} \vec{V}^{(r,q)}[0] = v_*$$

Tangent spaces at fixed point: E^{s} and E^{u}.

$E^{\mathrm{u}} = \mathbb{C} e_{\mathrm{u}}$, with e_{u} eigenvector of $D_{v_*}RG$ for eigenvalue $\alpha_{\mathrm{u}} = L^{3-2[\phi]} \times Z_2 =: L^{3-[\phi^2]}$.

4th step: control deviation from homogeneous evolution $\vec{V}^{(r,q)}[f] - \vec{V}^{(r,q)}[0]$, for all effective scale q, uniformly in r.

1) $\sum_{\mathbf{x}\in G} \zeta_{\mathbf{x}} = 0$ a.s. $\rightarrow$ deviation is 0 for $q <$local constancy scale of test function f.

2) Deviation resides in closed unit ball containing origin for $q >$ radius of support of f $\rightarrow$ exponential decay for large q.

For source term with ϕ^2 add

$$Y_2 Z_2^r \int :\phi^2:_{C_r}(x) j(x) d^3x$$

to potential. $\mathcal{S}^{\mathrm{T}}_{r,s}(f,j)$ now involves two test functions. After rescaling to unit lattice/cut-off

$$Y_2 \alpha_{\mathrm{u}}^r \int :\phi^2:_{C_0}(x) j(L^{-r}x) d^3x$$

to be combined with μ into $(\beta_{2,\Delta})_{\Delta\in\mathbb{L}_0}$ space-dependent mass.

5th step: partial linearization

In order to replay same sequence of moves with j present, construct

$$\Psi(v,w) = \lim_{n\to\infty} RG^n(v + \alpha_{\mathrm{u}}^{-n} w)$$

for $v \in W^{\mathrm{s}}$ and all direction w (especially $\int :\phi^2:$).

For v fixed, $\Psi(v,\cdot)$ is parametrization of W^{u} satisfying $\Psi(v,\alpha_{\mathrm{u}} w) = RG(\Psi(v,w))$.

If there were no W^{s} directions (1D dynamics) then Ψ would be conjugation $\to$ Poincaré-Kœnigs Theorem.

$\Psi(v,w)$ is holomorphic in v and w.

This is essential for probabilistic interpretation of $(\phi, N[\phi^2])$ as pair of random variables in $S'(\mathbb{Q}_p^3)$.

Thank you for your attention.