p-Adic Analogue of the Porous Medium Equation

Anatoly N. Kochubei
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3, Kiev, 01004 Ukraine,
E-mail: kochubei@imath.kiev.ua

Joint work with Andrei Khrennikov

Introduction

The theory of linear partial pseudo-differential equations for complex-valued functions over non-Archimedean fields is a well-established branch of mathematical analysis. By this time, there is a description of various equations whose properties resemble those of classical equations of mathematical physics, there are constructions of fundamental solutions, information on spectral properties of related operators. For equations of evolution type, there are results on initial value problems etc. (Khrennikov, Kochubei, Shelkovich, Vladimirov, Zúñiga-Galindo, ...). Meanwhile very little is known about nonlinear p-adic equations. We can mention only some semilinear evolution equations solved using p-adic wavelets (Khrennikov and Shelkovich, 2010) and a kind of equations of reaction-diffusion type studied by Zúñiga-Galindo, 2016.

In this work, we consider a p-adic analog of one of the most important classical nonlinear equations, the porous medium equation, that is the equation

$$\frac{\partial u}{\partial t} + D^\alpha(\varphi(u)) = 0, \quad u = u(t,x), \quad t > 0, x \in \mathbb{Q}_p, \tag{1}$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, D^α, $\alpha > 0$, is Vladimirov's fractional differentiation operator, φ is a strictly monotone increasing continuous real function, $|\varphi(s)| \le C|s|^m$ for $s \in \mathbb{R}$ ($C > 0$, $m \ge 1$). A typical example of the latter is $\varphi(u) = u|u|^{m-1}$, $m > 1$. We see Eq. (1) as the simplest model example of an equation of this kind for the non-Archimedean situation. Therefore in order to understand specific features of this case, we confine ourselves to the simplest pseudo-differential operator on $\mathbb{Q}_p$ and the simplest kind of nonlinearity. On the other hand, this setting has some common features with recent work on fractional porous medium equation on $\mathbb{R}^n$ (de Pablo, Quiros, Rodríguez and Vázquez, 2012). Another motivation is the p-adic model of a porous medium proposed by Khrennikov, Oleschko and Correa Lopez (2016).

Our strategy for studying Eq. (1) is as follows. There exists an abstract theory of the equations

$$\frac{\partial u}{\partial t} + A(\varphi(u)) = 0. \tag{2}$$

developed by Crandall and Pierre (1982) and based on the theory of stationary equations

$$u + A\varphi(u) = f \tag{3}$$

developed by Brézis and Strauss (1973). In Eq. (2) and (3), A is a linear m-accretive operator in $L^1(\Omega)$ where Ω is a σ-finite measure space. Under some natural assumptions, the nonlinear operator$A\varphi = A \circ \varphi$ is accretive and admits an m-accretive extension A_φ, the generator of a contraction semigroup of nonlinear operators. This result gives information on a kind of generalized solvability of Eq. (2), though the available description of A_φ is not quite explicit.

In order to use this method for Eq. (1), we need an L^1-theory of the Vladimirov operator D^α, which is a subject of independent interest.
In the classical situation where $\Omega = \mathbb{R}^n$, A is the Laplacian, there are stronger results (Bénilan, Brézis and Crandall, 1975) based on the study of Eq. (3), showing that $A\varphi$ is m-accretive itself. This employs some delicate tools of local analysis of solutions, such as imbedding theorems for Marcinkiewicz and Sobolev spaces in bounded domains.
For our p-adic situation, we prove a little weaker result, namely the m-accretivity of the closure of the operator $A\varphi$. Our tool is the L^1-theory of the Vladimirov type operator on a p-adic ball.
Finally, we give an example of an explicit solution of Eq. (1) resembling the "Quadratic Pressure Solution" of the porous medium equation on $\mathbb{R}^n$.

Preliminaries

1. p-Adic numbers and the Vladimirov operator.

Let p be a prime number. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^\nu \frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p. $\mathbb{Q}_p$ is a locally compact topological field.

Note that by Ostrowski's theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$.

We denote $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \le 1\}$. $\mathbb{Z}_p$, as well as all balls in $\mathbb{Q}_p$, is simultaneously open and closed.

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$
$$|xy|_p = |x|_p \cdot |y|_p;$$
$$|x + y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the ultra-metric inequality (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x - y|_p$, as well as many unusual geometric properties. Note also the following consequence of the ultra-metric inequality:

$$|x + y|_p = \max(|x|_p, |y|_p) \quad \text{if } |x|_p \neq |y|_p.$$

Denote by dx the Haar measure on the additive group of K normalized by the equality $\int_{\mathbb{Z}_p} dx = 1$.

The Fourier transform of a complex-valued function $f \in L^1(\mathbb{Q}_p)$ is again a function on $\mathbb{Q}_p$ defined as

$$\widetilde{f}(\xi) = (\mathcal{F}f)(\xi) = \int\limits_{\mathbb{Q}_p} \chi(x\xi) f(x)\, dx$$

where χ is the canonical additive character.
If $\mathcal{F}f \in L_1(\mathbb{Q}_p)$, then we have the inversion formula

$$f(x) = \int\limits_{K} \chi(-x\xi) \widetilde{f}(\xi)\, d\xi.$$

It is possible to extend $\mathcal{F}$ from $L_1(\mathbb{Q}_p) \cap L_2(\mathbb{Q}_p)$ to a unitary operator on $L_2(\mathbb{Q}_p)$, so that the Plancherel identity holds in this case.

In order to define distributions on $\mathbb{Q}_p$, we need a class of test functions. A function $f: \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if there exists such an integer $l \geq 0$ that for any $x \in \mathbb{Q}_p$

$$f(x + x') = f(x) \quad \text{if } \|x'\| \leq p^{-l}.$$

The smallest number l with this property is called the exponent of local constancy of the function f.
Typical examples of locally constant functions are additive characters, and also cutoff functions like

$$\Omega(x) = \begin{cases} 1, & \text{if } \|x\| \leq 1; \\ 0, & \text{if } \|x\| > 1. \end{cases}$$

In particular, Ω is continuous, which is an expression of the non-Archimedean properties of $\mathbb{Q}_p$.

Denote by $\mathcal{D}(\mathbb{Q}_p)$ the vector space of all locally constant functions with compact supports. Note that $\mathcal{D}(\mathbb{Q}_p)$ is dense in $L^q(\mathbb{Q}_p)$ for each $q \in [1, \infty)$. In order to furnish $\mathcal{D}(\mathbb{Q}_p)$ with a topology, consider first the subspace $D_N^l \subset \mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in a ball

$$B_N = \{x \in \mathbb{Q}_p : \ |x|_p \le p^N\}, \quad N \in \mathbb{Z},$$

and the exponents of local constancy $\le l$. This space is finite-dimensional and possesses a natural direct product topology. Then the topology in $\mathcal{D}(\mathbb{Q}_p)$ is defined as the double inductive limit topology, so that

$$\mathcal{D}(\mathbb{Q}_p) = \varinjlim_{N\to\infty} \varinjlim_{l\to\infty} D_N^l.$$

If $V \subset \mathbb{Q}_p$ is an open set, the space $\mathcal{D}(V)$ of test functions on V is defined as a subspace of $\mathcal{D}(\mathbb{Q}_p)$ consisting of functions with supports in V.

The space $\mathcal{D}'(\mathbb{Q}_p)$ of Bruhat-Schwartz distributions on $\mathbb{Q}_p$ is defined as a strong conjugate space to $\mathcal{D}(\mathbb{Q}_p)$. In contrast to the classical situation, the Fourier transform is a linear automorphism of the space $\mathcal{D}(\mathbb{Q}_p)$. By duality, $\mathcal{F}$ is extended to a linear automorphism of $\mathcal{D}'(\mathbb{Q}_p)$. There exists a detailed theory of convolutions and direct products of distributions on $\mathbb{Q}_p$ closely connected with the theory of their Fourier transforms.

The Vladimirov operator D^α, $\alpha > 0$, of fractional differentiation, is defined first as a pseudo-differential operator with the symbol $|\xi^\alpha_p$:

$$(D^\alpha u)(x) = \mathcal{F}^{-1}_{\xi \to x}\left[|\xi|^\alpha_p \mathcal{F}_{y \to \xi} u\right], \quad u \in \mathcal{D}(\mathbb{Q}_p),$$

where we show arguments of functions and their direct/inverse Fourier transforms. There is also a hypersingular integral representation giving the same result on $\mathcal{D}(\mathbb{Q}_p)$ but making sense on much wider classes of functions (for example, bounded locally constant functions):

$$(D^\alpha u)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int\limits_{\mathbb{Q}_p} |y|_p^{-\alpha-1}[u(x-y) - u(x)]\, dy.$$

Heat kernel for D^{α}:

$$Z(t,x)=\sum_{k=-\infty}^{\infty}p^{k}c_{k}(t)\Delta_{-k}(x)$$

where $\Delta_{l}(x)$ is the indicator function of the ball B_{l},

$$c_{k}(t)=\exp\left(-p^{k\alpha}t\right)-\exp\left(-p^{(k+1)\alpha}t\right).$$

Another expression for $Z(t,x)$, valid for $x\neq 0$, is

$$Z(t,x)=\sum_{m=1}^{\infty}\frac{(-1)^{m}}{m!}\cdot\frac{1-p^{\alpha m}}{1-p^{-\alpha m-1}}t^{m}|x|_{p}^{-\alpha m-1}.$$

Z is a probability density and

$$0<Z(t,x)\leq Ct(t^{1/\alpha}+|x|_{p})^{-\alpha-1},\quad t>0,x\in\mathbb{Q}_{p}.$$

2. A Heat-Like Equation on a p-Adic Ball

Let us consider the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} + \left(D_N^\alpha u\right)(t,x) - \lambda u(t,x) = 0, \quad x \in B_N,\ t > 0; \tag{4}$$

$$u(0,x) = \psi(x), \quad x \in B_N, \tag{5}$$

where $N \in \mathbb{Z}$, $B_N = \{x \in \mathbb{Q}_p, |x|_p \le p^N\}$, $\psi \in \mathcal{D}(B_N)$, $\lambda = \dfrac{p-1}{p^{\alpha+1}-1} p^{\alpha(1-N)}$, the operator D_N^α is defined by restricting D^α to functions u_N supported in B_N and considering the resulting function $D^\alpha u_N$ only on B_N. Here and below we often identify a function on B_N with its extension by zero onto $\mathbb{Q}_p$. Note that D_N^α defines a positive definite operator on $L^2(B_N)$, λ is its smallest eigenvalue.

The solution:

$$u(x,t) = \int_{B_N} Z_N(t, x-y)\psi(y)\,dy, \quad t>0, x \in B_N,$$

where

$$Z_N(t,x) = e^{\lambda t} Z(x,t) + c(t), \quad x \in B_N,$$
$$c(t) = p^{-N} - p^{-N}(1-p^{-1})e^{\lambda t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \frac{p^{-N\alpha n}}{1-p^{-\alpha n-1}},$$

The kernel Z_N is a transition density of a Markov process on B_N.

Probabilistic interpretation. Let $\xi_\alpha(t)$ be the stochastic process with independent increments corresponding to the generator D^α. Suppose that $\xi_\alpha(0) \in B_N$. Denote by $\xi_\alpha^{(N)}(t)$ the sum of all jumps of the process $\xi_\alpha(\tau),\ \tau \in [0,t]$, whose absolute values exceed p^N. Since ξ_α is right continuous with left limits, $\xi_\alpha^{(N)}(t)$ is finite a.s., $\xi_\alpha^{(N)}(0) = 0$. Let us consider the process

$$\eta_\alpha(t) = \xi_\alpha(t) - \xi_\alpha^{(N)}(t).$$

Since the jumps of η_α never exceed p^N by absolute value, this process remains a.s. in B_N (due to the ultra-metric inequality). The above Cauchy problem corresponds to this process.

Harmonic analysis on the additive group of a p-adic ball

Let us consider the p-adic ball B_N as a compact subgroup of $\mathbb{Q}_p$. Any continuous additive character of $\mathbb{Q}_p$ has the form $x \mapsto \chi(\xi x)$, $\xi \in \mathbb{Q}_p$. The annihilator $\{\xi \in \mathbb{Q}_p : \ \chi(\xi x) = 1 \text{ for all } x \in B_N\}$ coincides with the ball B_{-N}. By the Pontryagin duality theorem, the dual group $\widehat{B_N}$ to B_N is isomorphic to the discrete group $\mathbb{Q}_p/B_{-N}$ consisting of the cosets

$$p^m \left(r_0 + r_1 p + \cdots + r_{N-m-1} p^{N-m-1} \right) + B_{-N}, \quad r_j \in \{0, 1, \dots, p-1\},$$

$m \in \mathbb{Z}, m < N$. Analytically, this isomorphism means that any nontrivial continuous character of B_N has the form $\chi(\xi x)$, $x \in B_N$, where $|\xi|_p > p^{-N}$ and $\xi \in \mathbb{Q}_p$ is considered as a representative of the class $\xi + B_{-N}$. Note that $|\xi|_p$ does not depend on the choice of a representative of the class.

The Fourier transform on B_N is given by the formula

$$(\mathcal{F}_N f)(\xi) = p^{-N} \int\limits_{B_N} \chi(x\xi) f(x)\, dx, \quad \xi \in (\mathbb{Q}_p \setminus B_{-N}) \cup \{0\},$$

where the right-hand side, thus also $\mathcal{F}_N f$, can be understood as a function on $\mathbb{Q}_p / B_{-N}$.

The Riesz kernel

$$f_\alpha^{(N)}(x) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-1}} |x|_p^{\alpha-1}, \quad \operatorname{Re}\alpha > 0, \quad \alpha \not\equiv 1 \pmod{\frac{2\pi i}{\log p}\mathbb{Z}}.$$

generates a distribution on B_N extended analytically in α to

$$\left\langle f_{-\alpha}^{(N)}, \varphi \right\rangle = \lambda\varphi(0) + \frac{1 - p^{\alpha}}{1 - p^{-\alpha-1}} \int_{B_N} [\varphi(x) - \varphi(0)] |x|_p^{-\alpha-1} dx.$$

The emergence of λ in the last formula "explains" its role in the probabilistic construction of a process on B_N.

Theorem

The operator D_N^α, $\alpha > 0$, acts from $\mathcal{D}(B_N)$ to $\mathcal{D}(B_N)$ and admits, for each $\varphi \in \mathcal{D}(B_N)$, the representations:

(i) $D_N^\alpha \varphi = f_{-\alpha}^{(N)} * \varphi$ *where the convolution is understood in the sense of harmonic analysis on the additive group of B_N;*

(ii)

$$(D_N^\alpha \varphi)(x) = \lambda\varphi(x) + \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int\limits_{B_N} |y|_p^{-\alpha-1}[\varphi(x-y)-\varphi(x)]\, dy.$$

(iii) *On $\mathcal{D}(B_N)$, $D_N^\alpha - \lambda I$ coincides with the pseudo-differential operator $\varphi \mapsto \mathcal{F}_N^{-1}(P_{N,\alpha}\mathcal{F}_N\varphi)$ where*

$$P_{N,\alpha}(\xi) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int\limits_{B_N} |y|_p^{-\alpha-1}[\chi(y\xi)-1]\, dy.$$

This symbol is extended uniquely from $(\mathbb{Q}_p \setminus B_{-N}) \cup \{0\}$ onto $\mathbb{Q}_p/B_{-N}$.

The Vladimirov Operator in $L^1(\mathbb{Q}_p)$

The Heat-Like Equation and the Corresponding Semigroup of Operators.

Using the fundamental solution Z, we define the operator family

$$(S(t)\psi)(x) = \int_{\mathbb{Q}_p} Z(t, x-\xi)\psi(\xi)\, d\xi, \quad \psi \in L^1(\mathbb{Q}_p),$$

$t > 0$. S is a contraction semigroup in $L^1(\mathbb{Q}_p)$.

Proposition

$S(t)$ has the C_0-property.

Definition

We define the realization A of D^α in $L^1(\mathbb{Q}_p)$ as the generator of the semigroup $S(t)$.

Let $D(A)$ be the domain of the operator A.

Proposition

If $u \in \mathcal{D}(\mathbb{Q}_p)$, then $u \in D(A)$ and $Au = D^\alpha u$ where the right-hand side is understood as usual in terms of the Fourier transform or the hypersingular integral representation.

The proof is based on the detailed analysis of actions of D^α and $S(t)$ on characteristic functions of open-closed sets.

The Green function.

Since the operator A in $L^1(\mathbb{Q}_p)$ is defined as the generator of the contraction semigroup $S(t) = e^{-tA}$, then by the Hille-Yosida theorem, we can find the resolvent $R_\mu(A) = (A + \mu I)^{-1}$, $\mu > 0$, by the formula

$$R_\mu(A)\psi = -\int\limits_0^\infty e^{-\mu t} S(t)\psi \, dt, \quad \psi \in L^1(\mathbb{Q}_p).$$

We will consider below the case where $\alpha > 1$, in which the resolvent is an integral operator with a kernel possessing some smoothness properties. Thus, from now on,

$$\alpha > 1.$$

In this case, R_μ is a convolution operator with the continuous integral kernel $E_\mu(x-\xi)$, such that $E_\mu(x) \sim \text{const} \cdot |x|_p^{-\alpha-1}$, $|x|_p \to \infty$. The function E_μ is represented by the uniformly convergent series

$$E_\mu(x) = \sum_{N=-\infty}^{\infty} e_\mu^{(N)}(x),$$

$$e_\mu^{(N)}(x) = \int\limits_{|\xi|_p = p^N} \frac{\chi(-x\xi)}{|\xi|_p^\alpha + \mu} \, d\xi.$$

Description of A in the distribution sense.
Let $u \in L^1(\mathbb{Q}_p)$. Then $D^\alpha u$ can be defined as a distribution from $\mathcal{D}'(\mathbb{Q}_p)$, a convolution $u * f_{-\alpha}$, $f_{-\alpha}(x) = \dfrac{|x|_p^{-\alpha-1}}{\Gamma_p(-\alpha)}$,

$$\Gamma_p(z) = \frac{1-p^{z-1}}{1-p^{-z}}.$$

$f_{-\alpha}$ defines a distribution by analytic continuation.

Proposition

The operator A defined as a semigroup generator has the domain $D(A) = \{u \in L^1(\mathbb{Q}_p) : \ D^\alpha u \in L^1(\mathbb{Q}_p)\}$ where $Au = D^\alpha u$ (understood in the distribution sense).

L^1-Theory of the Vladimirov Type Operator on a p-Adic Ball

The Heat-Like Semigroup.

On a ball B_N, $N \in \mathbb{Z}$, we consider the Cauchy problem (4)-(5). Its fundamental solution Z_N defines a contraction semigroup

$$(T_N(t)u)(x) = \int\limits_{B_N} Z_N(t, x - \xi)u(\xi)\, d\xi$$

on $L^1(B_N)$.

Proposition

The semigroup T_N is strongly continuous.

The Generator.

Denote by A_N the generator of the contraction semigroup T_N on $L^1(B_N)$. By the Hille-Yosida theorem, A_N has a bounded resolvent $(A_N + \mu I)^{-1}$ for each $\mu > 0$. In order to study the domain $D(A_N)$, we need the following auxiliary result.

Proposition

Let the support of a function $u \in L^1(\mathbb{Q}_p)$ be contained in $\mathbb{Q}_p \setminus B_N$. Then the restriction to B_N of the distribution $D^\alpha u \in \mathcal{D}'(\mathbb{Q}_p)$ coincides with the constant

$$R_N = R_N(u) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int\limits_{|x|_p > p^N} |x|_p^{-\alpha-1} u(x)\, dx.$$

The following main result of this section is based on this property. As before, A denotes the generator of the semigroup $S(t)$ on $L^1(\mathbb{Q}_p)$.

Proposition

If $\psi \in D(A)$, then the restriction ψ_N of the function ψ to B_N belongs to $D(A_N)$, and $A_N \psi_N = (D_N^\alpha - \lambda)\, \psi_N$ where $D_N^\alpha \psi_N$ is understood in the sense of $\mathcal{D}'(B_N)$, that is ψ_N is extended by zero to a function on $\mathbb{Q}_p$, D^α is applied to it in the distribution sense, and the resulting distribution is restricted to B_N.

In the study of nonlinear equations, this result makes it possible to use the operator A_N in the investigation of local properties of functions. This is a substitute for the local Sobolev and Marcinkiewicz spaces used in the classical literature.

Nonlinear Equations: the Main Result

Let us return to Eq. (1) interpreted as Eq. (2) on $L^1(\mathbb{Q}_p)$, where the linear operator A is a generator of the semigroup $S(t)$, φ is a strictly monotone increasing continuous real function, $|\varphi(s)| \leq C|s|^m$, $m \geq 1$. Below we re-interpret Eq. (1) as the equation

$$\frac{\partial u}{\partial t} + \overline{A\varphi}(u) = 0 \tag{6}$$

where $\overline{A\varphi}$ is the closure of $A\varphi$.

Recall that a mild solution of the Cauchy problem for a nonlinear equation with the initial condition $u(0,x) = u_0(x)$ is defined as a function given by a limit, uniformly on compact time intervals, of solutions of the problem for the difference equations approximating the differential one. This the usual “nonlinear version” of the notion of a generalized solution.

Theorem

The operator $\overline{A\varphi}$ *is m-accretive, so that, for any initial function* $u_0 \in L^1(\mathbb{Q}_p)$*, the Cauchy problem for Eq. (6) has a unique mild solution.*

Idea of Proof. By the general results of Crandall and Pierre, the operator $A\varphi$ is accretive. Therefore it is sufficient to show that $I + A\varphi$ has a dense range. This property is proved using a priori estimates by Brézis and Strauss, relative local compactness criterion for subsets of $L^1(\mathbb{Q}_p)$ and the local compactness considerations based on properties of the operator A_N.

Explicit Solution: an Example

Let us consider Eq. (1) with $\alpha > 0$, $\varphi(u) = |u|^m$, $m > 1$. We look for a solution of the form

$$u(t,x) = \rho \left(\frac{|x|_p^\gamma}{t_0 - t} \right)^\nu, \quad 0 < t < t_0, x \in \mathbb{Q}_p,$$

where $t_0 > 0$, $\gamma > 0$, $\nu > 0$, $0 \neq \rho \in \mathbb{R}$.

After investigating possible values of parameters, we come to the solution

$$u(t,x) = \rho (t_0 - t)^{-\frac{1}{m-1}} |x|_p^{\frac{\alpha}{m-1}}$$

where

$$\rho = - \left[\frac{\Gamma_p(1 + \frac{\alpha}{m-1})}{(m-1)\Gamma_p(1 + \frac{\alpha m}{m-1})} \right]^{\frac{1}{m-1}}.$$

In a similar way, we can obtain another solution

$$u(t,x)=\mu(t_0+t)^{-\frac{1}{m-1}}|x|_p^{\frac{\alpha}{m-1}},\quad t>0, x\in\mathbb{Q}_p,$$

where $\mu=-\rho$.

Publications

1. A. Yu. Khrennikov and A. N. Kochubei. p-Adic Analogue of the Porous Medium Equation, *J. Fourier Anal. Appl.* (published online), ArXiv 1611.08863.

2. A. Khrennikov, K. Oleschko and M. J. Correa Lopez, Application of p-adic wavelets to model reaction-diffusion dynamics in random porous media, *J. Fourier Anal. Appl.* **22** (2016), 809 – 822.

3. A. Khrennikov, K. Oleschko and M. J. Correa Lopez, Modeling fluid's dynamics with master equations in ultrametric spaces representing the treelike structure of capillary networks, *Entropy* **18** (2016), art. 249, 28 pp.

4. A. N. Kochubei, Linear and nonlinear heat equations on a p-adic ball, ArXiv 1708.03261.