Nonarchimedean and noncommutative aspects of the interior of the Schwarzschild black hole and signature change

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1. Introduction

- We consider a class of minisuperspace cosmological models, with Lagrangians that describe systems of two oscillators.
- Dynamics of the models is considered on real, p-adic/adelic and noncommutative spaces
- Of a particular interest is to consider the Schwarzschild black hole interior as a homogeneous and anisotropic Kantowski - Sachs minisuperspace cosmological model without a matter field and the cosmological constant.This approach is based on a diffeomorphism between the Schwarzschild solution of the Einstein field equation and the corresponding solution of this cosmological model.

1. Introduction

Motivations:

- Initial/cosmological singularities and local singularities (BH) need quantum approach. Without a complete QG theory, classical and quantum cosmological models are available for consideration.
- Measurements at Planck scale.

p-Adic QM

- Dynamics is described by

$$U(t)\psi_p(x) = \int_{\mathbb{Q}_p} \mathcal{K}_t(x,y)\psi_p(y)\,dy.$$

- For quadratic 1D systems

$$\mathcal{K}_p(x'',t'';x',t') = \lambda_p\left(-\frac{1}{4\pi\hbar}\frac{\partial^2 S_p^{cl}}{\partial x''\partial x'}\right)\left|\frac{1}{2\pi\hbar}\frac{\partial^2 S_p^{cl}}{\partial x''\partial x'}\right|_p^{1/2} \times \chi_p\left(-\frac{1}{2\pi\hbar}S_p^{cl}\right).$$

Adelic QM

Interpretation of p-adic QM?

- Adelic wave function $\Psi^{(adel.)} : \mathcal{A} \to \mathbb{C}$:

$$\Psi^{(adel.)}(x) = \Psi_\infty(x) \prod_{p \in M} \Psi_p(x) \prod_{p \notin M} \Omega(|x|_p),$$

M is a finite set of prime numbers.

- The simplest adelic vacuum state:

$$\Psi_0^{(adel.)}(x) = \Psi_{\infty,0}(x) \prod_p \Omega(|x|_p).$$

- Dynamics in adelic QM:

$$U(t'',t')\Psi^{(adel.)}(x)=\prod_{v=\infty,2,3,\ldots}\int_{\mathbb{Q}_v}\mathcal{K}_v(x_v,t''_v;y_v,t'_v)\Psi_v(y_v)\,dy_v.$$

2. p-Adic and adelic quantum mechanics

- Probability

$$\left|\Psi^{(adel.)}(x)\right|_\infty^2 = |\Psi_\infty(x)|_\infty^2 \prod_{p\in M} |\Psi_p(x)|_\infty^2 \prod_{p\notin M} \Omega(|x|_p),$$

- For the adelic ground state:

$$\left|\Psi_0^{(adel.)}(x)\right|_\infty^2 = \begin{cases} |\Psi_{\infty,0}(x)|_\infty^2, & x \in \mathbb{Z}, \\ 0, & x \in \mathbb{Q} \setminus \mathbb{Z}. \end{cases}$$

3. *p*-Adic and adelic cosmology

- Early universe should be described by $\Psi_0^{(adel.)}(x)$
- Classically

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \frac{8\pi G}{c^4} T_{\mu\nu} + g_{\mu\nu}\Lambda, \quad \mu,\nu = 0,1,2,3.$$

<u>Hamiltonian formulation GTR</u> (3+1) formalism

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + h_{ik}(dx^i + N^i dt)(dx^k + N^k dt),$$

$$L = \frac{1}{16\pi G}\int \left({}^{(3)}\mathcal{R} + K^{ik}K_{ik} - K^2 - 2\Lambda\right) N\sqrt{h}\, d^3x + \int \mathcal{L}_m N\sqrt{h}\, d^3x$$

From above relations one can get π^{ik}, π_Φ, π^0 i π^i conjugate to h_{ik}, Φ, N i N_i, as well as Hamiltonian H.

Hamiltonian formulation GTR $\rightarrow$ canonican quantisation:

- Variables $\rightarrow$ observables:

$$\pi^{ik} \rightarrow \hat{\pi}^{ik} \rightarrow -i\frac{\delta}{\delta h_{ik}}, \qquad \pi_{\Phi} \rightarrow \hat{\pi}_{\Phi} \rightarrow -i\frac{\delta}{\delta \Phi},$$

$$\pi^{0} \rightarrow \hat{\pi}^{0} \rightarrow -i\frac{\delta}{\delta N}, \qquad \pi^{i} \rightarrow \hat{\pi}^{i} \rightarrow -i\frac{\delta}{\delta N_i}.$$

- The primary Dirac constraints:

$$\pi^0 = \frac{\delta L}{\delta \dot{N}} = 0 \to \hat{\pi}^0 \left|\Psi\right\rangle = 0 \to -i\frac{\delta \Psi}{\delta N} = 0,$$

$$\pi^i = \frac{\delta L}{\delta \dot{N}_i} = 0 \to \hat{\pi}^i \left|\Psi\right\rangle = 0 \to -i\frac{\delta \Psi}{\delta N_i} = 0.$$

- The secondary Dirac constraints:

$$\dot{\pi}^0 = -\mathcal{H} = 0 \Rightarrow \hat{\mathcal{H}} \left|\Psi\right\rangle = 0, \dot{\pi}^i = -\mathcal{H}^i = 0 \Rightarrow \hat{\mathcal{H}}^i \left|\Psi\right\rangle = 0.$$

Hamilton condition $\rightarrow$ Wheeler-DeWitt equation

- Functional quantisation GTR (Misner, 1957. g.)

$$\langle h''_{ik}, \Phi'', \Sigma'' | h'_{ik}, \Phi', \Sigma' \rangle = \int \exp(iS(g_{\mu\nu}, \Phi)) \mathcal{D}g_{\mu\nu} \, \mathcal{D}\Phi.$$

The factor $\exp(iS)$ is oscillatory $\rightarrow$ integral is divergent $\rightarrow$ $t \rightarrow -i\tau$ and $S \rightarrow i\bar{S}$ (Victor rotation i.e. signature transition)

$$\langle h''_{ik}, \Phi'', \Sigma'' | h'_{ik}, \Phi', \Sigma' \rangle = \int \exp(-\bar{S}(g_{\mu\nu}, \Phi)) \mathcal{D}g_{\mu\nu} \, \mathcal{D}\Phi.$$

Since $\bar{S}$ is not necessarily positive definite, integrations must take into account not only real multiples and complex metrics (the result will then depend on the choice of the contour of integration).

3. *p*-Adic and adelic cosmology

- It implies the application of *p*-adic and adelic quantum mechanics to the early stages of the universe.
- Functional approach - the central object is *p*-adic generalization of the transition amplitude:

$$\left\langle h''_{ik}, \Phi'', \Sigma'' | h'_{ik}, \Phi' \Sigma' \right\rangle_p = \int \chi_p(-S_p(g_{\mu\nu}, \Phi)) \mathcal{D}(g_{\mu\nu})_p \, \mathcal{D}(\Phi)_p.$$

Two directions of development:

- Hartle-Hawking wave functions are determinated as solutions of *p*-adic integrals:

$$\Psi_p^{HH}(q^\alpha) = \int_{|N|_p \leq 1} \mathcal{K}_p(q^\alpha, N; 0, 0)\, dN.$$

- The second approach involves determining and examining the conditions for the existence of vacuum *p*-adic states $\Omega(|x|_p)$, $\Omega(p^\nu |x|_p)$ i $\delta(p^\nu - |x|_p)$, the first of all vacuum state $\Omega(|x|_p)$ whose existence is necessary in terms of adelization.

<u>The issue of adelization</u>

- Adelization implies the construction of the adelic wawe function. Two conditions:
  1. existence of solutions $\Psi_\infty(x, y)$ of VdV equations,
  2. existence of p-adic vacuum states $\Omega(|x|_p)\Omega(|y|_p)$.
- For the Milne model, the first condition is not fulfilled.
- For two-oscillator models both conditions are fulfilled.

Signature change in p-adic space-time

- Sign of signature $\mathbb{Q}_p^4$ can be arbitrary changed only for $p \equiv 1 ($ mod 4) (if and only is $\sqrt{-1} \in \mathbb{Q}_p$).
- After changing of signature $\mathbb{Q}_p^4$ p-adic vacuum state of the model and conditions for their existence will not change.

4. Minisuperspace cosmological models

Minisuperspace models, finite-dimensional minisuper space

- homogeneous and isotropic model $\rightarrow N = N(t)$ and $N^i = 0$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2 dt^2 + h_{ik} dx^i dx^k.$$

For example, model in 2-dim minisuperspace $\{\Phi, R\}$ described by FLRW metrics:

$$ds^2 = -dt^2 + R^2(t)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)).$$

- Model in 3-dim minisuperspace $\{\Phi, c_1, c_2\}$ and homogenous and isotropic Kantowski-Sachs metric:

$$ds^2 = -\frac{N^2(t)}{c_1(t)} dt^2 + c_1(t) dr^2 + c_2^2(t)(d\theta^2 + \sin^2\theta d\varphi^2).$$

4. Minisuperspace cosmological models

Two-oscillatory models in cosmology

- ...are the models that can be represented as algebraic sums (sums or differences) of the Lagrangians of two decoupled linear harmonic oscillators, including suitable substitutions (if necessary),($L = \dot{x}^2 - \omega_x^2 x^2$) or two decoupled linear inverted harmonic oscillators ($L = \dot{x}^2 + \omega_x^2 x^2$).
- General Lagrangian:

$$L = [\dot{x}^2 \pm \omega_x^2 x^2] + sgn\xi[\dot{y}^2 \pm \omega_y^2 y^2],$$

where frequences ω_x i ω_y can be equal or different and $sgn\xi = +$ or $sgn\xi = -$ regardless of the sign in front $\omega_{x,y}$.

4. Minisuperspace cosmological models

Classification of the models:

- "Harmonic" two-oscillatory cosmological models:

$$L_{(1)} = [\dot{x}^2 - \omega^2 x^2] - [\dot{y}^2 - \omega^2 y^2], \quad L_{(2)} = [\dot{x}^2 - \omega^2 x^2] + [\dot{y}^2 - \omega^2 y^2],$$

$$L_{(3)} = [\dot{x}^2 - \omega_x^2 x^2] - [\dot{y}^2 - \omega_y^2 y^2], \quad L_{(4)} = [\dot{x}^2 - \omega_x^2 x^2] + [\dot{y}^2 - \omega_y^2 y^2].$$

- "Inverted harmonic" two-oscillatory cosmological models:

$$L_{(5)} = [\dot{x}^2 + \omega^2 x^2] - [\dot{y}^2 + \omega^2 y^2], \quad L_{(6)} = [\dot{x}^2 + \omega^2 x^2] + [\dot{y}^2 + \omega^2 y^2],$$

$$L_{(7)} = [\dot{x}^2 + \omega_x^2 x^2] - [\dot{y}^2 + \omega_y^2 y^2], \quad L_{(8)} = [\dot{x}^2 + \omega_x^2 x^2] + [\dot{y}^2 + \omega_y^2 y^2].$$

4. Minisuperspace cosmological models

Transformation of Lagrangians:

$$L_{(1)} \leftrightarrow L_{(5)}, \quad L_{(2)} \leftrightarrow L_{(6)}, \quad L_{(3)} \leftrightarrow L_{(7)}, \quad L_{(4)} \leftrightarrow L_{(8)}.$$

can be achieved by two types of relativistic transformations:

- Signature of transformation: $t \rightarrow -i\tau$.
- Changing the sign in front of the square of the oscillator frequency square $\omega_{x,y}^2 \rightarrow -\omega_{x,y}^2$ (which can be a consequence of the change of the parameter's character through which the square of the frequency of the oscillator can be defined in the particular model).

Example 1

- Multidimensional Friedman model with $\tilde{\phi}$ i $U(\tilde{\phi})$
- Metric: $\mathfrak{g} = -\beta d\beta \otimes d\beta + \frac{\bar{R}^2(\beta)}{\left(1+\frac{kr^2}{4}\right)^2} dx^i \otimes dx^i + \bar{a}^2(\beta)\mathfrak{g}^{(d)}$.
- Einstein-Hilbert (EH) action: $S = \frac{1}{2k_D^2} \int_{\mathcal{M}} \mathcal{R}\sqrt{-g}\, d^D x - \frac{1}{2}\int_{\mathcal{M}} \left[-\left(\frac{\partial\tilde{\phi}}{\partial\beta}\right)^2 + 2U(\tilde{\phi})\right] d^D x + S_{YGH}$.
- Lagrangian: $L = -\frac{k_0^2}{\gamma'^2}(\dot{\alpha}_1^2 - \dot{\alpha}_2^2 - \omega_+^2\alpha_1^2 + \omega_-^2\alpha_2^2)$ has form $L_{(3)}$.

Example 2

- Friedman model with ϕ i Λ
- Metric: $ds^2 = -dt^2 + R^2(t)(dr^2 + r^2 d\Omega^2)$.
- EH action: $S = \frac{1}{2k^2} \int_{\mathcal{M}} \mathcal{R}\sqrt{-g}\, d^4x + \int_{\mathcal{M}} \left(-\frac{1}{2} (\nabla\phi)^2 \sqrt{-g}\, d^4x - \Lambda \right) \sqrt{-g}\, d^4x + S_{YGH}$.
- Lagrangian: $L = \dot{x}_1^2 - \dot{x}_2^2 + \frac{3}{4}\Lambda(x_1^2 - x_2^2)$ has form $L_{(5)}$.

Example 3

- Friedman model with ϕ i $U(\phi)$
- Metric: $ds^2 = -dt^2 + R^2(t)(dr^2 + r^2 d\Omega^2)$.
- EH action: $S = \frac{1}{2k^2}\int_{\mathcal{M}} \mathcal{R}\sqrt{-g}\, d^4x + \int_{\mathcal{M}} \left[\frac{1}{2}\partial_0\phi\partial^0\phi - U(\phi)\right]\sqrt{-g}\, d^4x + S_{YGH}$.
- Lagrangian: $L = \left[\dot{Q}_1^2 - \omega_+^2 Q_1^2\right] + \left[\dot{Q}_2^2 - \omega_-^2 Q_2^2\right]$ has form $L_{(4)}$.

4. Minisuperspace cosmological models

Example 4: Models with Lagrangian of the form $L_{(1)}$

- Cosmological minisuper spatial models whose Lagrangian can be reduced to the Lagrangian $L_{(1)}$ - two decoupled harmonic oscillators of equal frequencies whose energy is subtracted in the hamiltonian of the system.
- So far, they are the best-studied class of two-oscillatory models, which includes 6 types of so far discovered cosmological models.
- They can be divided into two groups. One group of models with a cosmological constant in the action, and another group of models without a cosmological constant.

Models with cosmological constant:

- Friedman model with a minimally coupled scalar field ϕ and the cosmological constant Λ
- Metric: $ds^2 = -dt^2 + R^2(t)(dr^2 + r^2 d\Omega^2)$.
- Action: $S = \frac{1}{2k^2}\int_{\mathcal{M}} \mathcal{R}\sqrt{-g}\, d^4x +$
  $\int_{\mathcal{M}} \left(-\frac{1}{2}(\nabla\phi)^2 \sqrt{-g}\, d^4x - \Lambda\right)\sqrt{-g}\, d^4x + S_{YGH}$.

4. Minisuperspace cosmological models

- Friedman $D = 4 + d$-dim model with the cosmological constant Λ and inner d-dim space
- Metric:
$$ds^2 = -dt^2 + \frac{R^2(t)}{\left(1+\frac{kr^2}{4}\right)^2}(dr^2 + r^2 d\Omega^2) + a^2(t) g_{ij}^{(d)} dx^i dx^j.$$
- Action: $S = \frac{1}{2k_D^2} \int_{\mathcal{M}} (\mathcal{R} - 2\Lambda)\sqrt{-g}\, d^D x + S_{YGH}.$

- Vacuum Kaluza-Klein (4+1) dim model with cosmological constant Λ
- Metric: $ds^2 = -dt^2 + \frac{R^2(t)dr^i dr^j}{(1+\frac{kr^2}{4})^2} + a^2(t)d\rho^2$.
- Action: $S = \int_{\mathcal{M}} (\mathcal{R} - \Lambda)\sqrt{-g}\,dtd^3rd\rho$.

4. Minisuperspace cosmological models

Models without cosmological constant:

- Friedman model with massless scalar field ϕ that is conformally coupled with gravity
- Metric: $ds^2 = \frac{2G}{3\pi}\left(-a^2(t)\tilde{N}^2 d^2 t + a^2(t) d\Omega_3^2(t)\right)$.
- Lagrangian density: $\mathcal{L}_m = -\frac{3}{8\pi G}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{6}\mathcal{R}\phi^2\right)$.

4. Minisuperspace cosmological models

- Friedman model with massless scalar field ϕ that is minimally coupled to gravity
- Metric: $ds^2 = \frac{2G}{3\pi}\left(-a^2(t)\tilde{N}^2 d^2 t + a^2(t) d\Omega_3^2(t)\right)$.
- Lagrangian density: $\mathcal{L}_m = -\frac{3}{8\pi G} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

4. Minisuperspace cosmological models

- Vacuum Kantowski-Sachs model
- Metric: $ds^2 = \frac{G}{2\pi}\left(-\tilde{N}^2(t)dt^2 + b_1^2(t)d\chi^2 + b_2^2(t)d\Omega_2^2(t)\right).$
- Action: $S = \frac{1}{16\pi G}\int_{\mathcal{M}}\left({}^{(3)}\mathcal{R} + K^{ik}K_{ik} - K^2\right)N\sqrt{h}\,dtd^3x.$
- This model is of particular interest to consider because its dynamics are diffeomorphic to dynamics of the interior of Schwarzschild black hole.

4. Minisuperspace cosmological models

- In all preceding cases, the Lagrangian of the model can be transformed to a Lagrangian two harmonic oscillators of equal frequencies by suitable substitutions $L_{(1)} = [\dot{x}^2 - \omega^2 x^2] - [\dot{y}^2 - \omega^2 y^2]$.

Hamilton formalism

- For Lagrangian of two-oscillator models $L_{(1)},\ldots,L_{(8)}$, we solve Euler-Lagrangian equations: $\frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{q}_\sigma}\right) - \frac{\partial L_i}{\partial q_\sigma} = 0$ of motion along minisuperspace coordinates q_σ. Their further solution yields a general form of finite equations of motion.
- Actions for each of these 8 models can be obtained from: $S_i = \int_{t'}^{t''} L_i \, dt$, and Hamiltonian: $H_i = \sum_\sigma p_\sigma \dot{q}_\sigma - L_i$

4. Minisuperspace cosmological models

- From Hamilton condition in GTR it follows that for these models $H_i = 0$, what imposes the constraints on the constants of integration in the final general solutions and indicates that the total energy of the system is equal to zero (the so-called zero-energy requirement).
- The are two distinct classes of two-oscillator models with Lagrangians $L_{(1)}$ and $L_{(3)}$. Zero energy conditions $H_{(1)} = 0$ and $H_{(3)} = 0$ indicate that the system of two linear harmonic oscillators of the same energies, followed by a corresponding subtraction in the Hamiltonian of the system. Such a type of system is called *oscillator-ghost-oscillator* system or *indefinite oscillator* system.

4. Minisuperspace cosmological models

Canonical Quantization

- Quantization of Hamilton constraint H_i for a given model leads to VdW equation, an analogue of Schroedinger equation in standard QM.
- For *oscillator-ghost-oscillator* system with Lagrangians L_1 i L_3:

$$\left[-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\tilde{\omega}^2(x^2-y^2)\right]\Psi(x,y)=0,$$

$$\left[-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\tilde{\omega}_x^2x^2-\tilde{\omega}_y^2y^2\right]\Psi(x,y)=0,.$$

4. Minisuperspace cosmological models

- Solutions of VdW equations for *oscillator-ghost -oscillator* systems:

$$\Psi_{n_1,n_2}(x,y) = \left(\frac{\tilde{\omega}}{\pi}\right)^{\frac{1}{4}} \left[\frac{H_{n_1}(\sqrt{\tilde{\omega}}x)}{\sqrt{2^{n_1} n_1!}}\right] e^{-\frac{\tilde{\omega}x^2}{2}} \left(\frac{\tilde{\omega}}{\pi}\right)^{\frac{1}{4}} \left[\frac{H_{n_2}(\sqrt{\tilde{\omega}}y)}{\sqrt{2^{n_2} n_2!}}\right] e^{-\frac{\tilde{\omega}y^2}{2}},$$

$$\Psi_{n_1,n_2}(x,y) = \left(\frac{\tilde{\omega}_x}{\pi}\right)^{\frac{1}{4}} \left[\frac{H_{n_1}(\sqrt{\tilde{\omega}_x}x)}{\sqrt{2^{n_1} n_1!}}\right] e^{-\frac{\tilde{\omega}_x x^2}{2}} \left(\frac{\tilde{\omega}_y}{\pi}\right)^{\frac{1}{4}} \left[\frac{H_{n_2}(\sqrt{\tilde{\omega}_y}y)}{\sqrt{2^{n_2} n_2!}}\right] e^{-\frac{\tilde{\omega}_y y^2}{2}}$$

Signature change

- Signature transition: Lorentzian $\rightarrow$ Euclidean, $(-,+,+,+) \rightarrow (+,+,+,+)$, using Vick rotation $t \rightarrow -i\tau$.
- Quantum gravity (QG) $\rightarrow$ at the Planck scale; the concept of measurement and the structure of space-time loses meaning $\rightarrow$ structure of space-time is: nonarchimedean + Volovich conjecture $\rightarrow$ p-adic/adelic $\rightarrow$ discreetness of space-time (in noncommutative approach is also assumed)

4. Minisuperspace cosmological models

- *Loop* QG tell us that this discreteness can also be connected with a signature transition mechanism.
- This signature change in regard to functional formalism as mathematical justification (convergence issue).
- In the case of two-oscillatory models classical signature transition translate one model to another. Then, additional constraints are imposed on the parameters of the model.

5. Ultrametricity vs noncommutativity in cosmology

- Many similarities and connections between nonarchimedean and noncommutative (and q-deformed) quantum theory have been noticed and considered [5]. It will not be discussed here in details.
- Noncommuatative coordinates: used for the very first time by Heisenberg, then by Wigner and Snyder
- Connes and Woronowich: noncommutative geometry, Seiberg: string theory.
- The string theory predicts that Plank's length is a minimum length that can still be measured, what supports appearance of discreteness, as well as, in general, a non-commutative structure of space-time at the Plank scale $\rightarrow$ noncommutative cosmology.

5. Ultrametricity vs noncommutativity in cosmology

- The study of the non-commutativity of functions in the classical phase space is based on the replacement of their usual product with the so-called star-product („$\star$").
- Generally speaking, the star-product is any associative, complex-bilinear product of complex-valued smooth functions defined on a manifold $\mathcal{M}$ presented in a formal form as the row of bilinear operators beginning with a commutative product of functions.
- On a flat euclidean manifold $\mathcal{M} = \mathbb{R}^{2n}$ such product is well known. It's about Moyal's star-product.

5. Ultrametricity vs noncommutativity in cosmology

- On $\mathcal{M} = \mathbb{R}^{2n}$ all acceptable star-products are c-equivalent to Moyal product.
- Let $f_1(x_1, ..., x_n; p_1, ..., p_n)$ and $f_2(x_1, ..., x_n; p_1, ..., p_n)$ are two functions on phase space $\mathbb{R}^{2n}$. Moyal product is defined as

$$f_1 \star_M f_2 = f_1 e^{\frac{1}{2} \overleftarrow{\partial}_a \alpha^{ab} \overrightarrow{\partial}_b} f_2.$$

Deformed Poisson bracket of f_1 and f_2 is defined through Moyal product (also called Moyal bracket):

$$\{f_1, f_2\}_M = f_1 \star_M f_2 - f_2 \star_M f_1.$$

5. Ultrametricity vs noncommutativity in cosmology

- Phase coordinates satisfy

$$\{x_i, x_j\}_M = \theta_{ij}, \quad \{x_i, p_j\}_M = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\}_M = \bar{\theta}_{ij}.$$

- One can introduce coordinate transformation (deformations)

$$x_i' = x_i - \frac{1}{2}\theta_{ij} p^j, \quad p_i' = p_i + \frac{1}{2}\bar{\theta}_{ij} x^j,$$

keeping in mind that for x_i and p_i:

$$\{x_i, y_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0.$$

5. Ultrametricity vs noncommutativity in cosmology

- For new x_i' and p_i' one has

$$\{x_i', x_j'\} = \theta_{ij}, \quad \{x_i', p_j'\} = \delta_{ij} + \sigma_{ij}, \quad \{p_i', p_j'\} = \bar{\theta}_{ij}.$$

- The obtained terms have the same form as in the case of non-commutative coordinates, but this time it is given through the usual Poisson brackets (the so-called non-commutative approach through deformation).
- In a two-dimensional case, with the presence of a noncommutative purely spatial type of deformation, transformation of the coordinates in the phase (configuration) space are

$$x' = x - \frac{1}{2}\theta p_y, \quad y' = y + \frac{1}{2}\theta p_x, \quad p_x' = p_x, \quad p_y' = p_y.$$

After these transformations Lagrangians of two-oscillatory models are

- For "harmonic" models:

$$L^{\theta}_{(1)}=[(1+\omega^2\theta^2)\dot{x}^2-\omega^2x^2]-[(1+\omega^2\theta^2)\dot{y}^2-\omega^2y^2]+2\omega^2\theta[\dot{x}y-\dot{y}x],$$
$$L^{\theta}_{(2)}=[(1-\omega^2\theta^2)\dot{x}^2-\omega^2x^2]+[(1-\omega^2\theta^2)\dot{y}^2-\omega^2y^2]-2\omega^2\theta[\dot{x}y-\dot{y}x],$$
$$L^{\theta}_{(3)}=[(1+\omega_y^2\theta^2)\dot{x}^2-\omega_x^2x^2]-[(1+\omega_x^2\theta^2)\dot{y}^2-\omega_y^2y^2]+2\theta[\dot{x}y\omega_y^2-\dot{y}x\omega_x^2],$$
$$L^{\theta}_{(4)}=[(1-\omega_y^2\theta^2)\dot{x}^2-\omega_x^2x^2]+[(1-\omega_x^2\theta^2)\dot{y}^2-\omega_y^2y^2]-2\theta[\dot{x}y\omega_y^2-\dot{y}x\omega_x^2],$$

- For "inverted harmonic" models:

$$L^{\theta}_{(5)}=[(1-\omega^2\theta^2)\dot{x}^2+\omega^2x^2]-[(1-\omega^2\theta^2)\dot{y}^2+\omega^2y^2]-2\omega^2\theta[\dot{x}y-\dot{y}x],$$
$$L^{\theta}_{(6)}=[(1+\omega^2\theta^2)\dot{x}^2+\omega^2x^2]+[(1+\omega^2\theta^2)\dot{y}^2+\omega^2y^2]+2\omega^2\theta[\dot{x}y-\dot{y}x],$$
$$L^{\theta}_{(7)}=[(1-\omega_y^2\theta^2)\dot{x}^2+\omega_x^2x^2]-[(1-\omega_x^2\theta^2)\dot{y}^2+\omega_y^2y^2]-2\theta[\dot{x}y\omega_y^2-\dot{y}x\omega_x^2],$$
$$L^{\theta}_{(8)}=[(1+\omega_y^2\theta^2)\dot{x}^2+\omega_x^2x^2]+[(1+\omega_x^2\theta^2)\dot{y}^2+\omega_y^2y^2]+2\theta[\dot{x}y\omega_y^2-\dot{y}x\omega_x^2].$$

5. Ultrametricity vs noncommutativity in cosmology

- There exist two types of trigonometric solutions (whether $\omega_{x,y}^2$ is equal to $\frac{1}{\theta^2}$ or not):

$$x(t) = C_1 \cos(\omega_\theta t) + C_2 \sin(\omega_\theta t), \quad y(t) = D_1 \cos(\omega_\theta t) + D_2 \sin(\omega_\theta t),$$

$$x(t) = C_1 \cos(\Omega_1 t) + C_2 \sin(\Omega_1 t) + C_3 \cos(\Omega_2 t) + C_4 \sin(\Omega_2 t),$$

$$\begin{aligned} y(t) = & - \frac{\alpha_x C_2 \Omega_1}{\beta_y - \Omega_1^2} \cos(\Omega_1 t) + \frac{\alpha_x C_1 \Omega_1}{\beta_y - \Omega_1^2} \sin(\Omega_1 t) \\ & - \frac{\alpha_x C_4 \Omega_2}{\beta_y - \Omega_2^2} \cos(\Omega_2 t) + \frac{\alpha_x C_3 \Omega_2}{\beta_y - \Omega_2^2} \sin(\Omega_2 t). \end{aligned}$$

- Quadratic action:

$$
\begin{aligned}
S_{(4)}^{cl,\theta}(x'',y'',T;x',y',0) &= \gamma_{i,x}({x'}^2+{x''}^2)+\gamma_{j,y}({y'}^2+{y''}^2)\\
&+ \gamma_{k,x}x'x''+\gamma_{l,y}y'y''+\gamma_5(x'y'-x''y'')+\gamma_6(x'y''-y'x''),\\
S_{(4)}^{cl,\theta}(x'',y'',T;x',y',0) &= \frac{1}{2}\gamma_{11}{x'}^2+\frac{1}{2}\gamma_{22}{x''}^2+\frac{1}{2}\gamma_{33}{y'}^2+\frac{1}{2}\gamma_{44}{y''}^2\\
&+ \gamma_{12}x'x''+\gamma_{13}x'y'+\gamma_{14}x'y''+\gamma_{23}y'x''+\gamma_{24}x''y''+\gamma_{34}y'y''.
\end{aligned}
$$

5. Ultrametricity vs noncommutativity in cosmology

Feynman propagator for $S^{cl,\theta}_{(4)}$:

$$\begin{aligned}\mathcal{K}^{\theta}(x'',y'',T;x',y',0) &= \frac{1}{2\pi i}\left[\det\begin{pmatrix} -\frac{\partial^2 S^{cl,\theta}}{\partial x''\partial x'} & -\frac{\partial^2 S^{cl,\theta}}{\partial x''\partial y'} \\ -\frac{\partial^2 S^{cl,\theta}}{\partial y''\partial x'} & -\frac{\partial^2 S^{cl,\theta}}{\partial y''\partial y'}\end{pmatrix}\right]^{\frac{1}{2}} \\ &\times \chi_\infty\left(-\frac{1}{2\pi}S^{cl,\theta}(x'',y'',T;x',y',0)\right),\end{aligned}$$

$$\mathcal{K}^{\theta}(x'',y'',T;x',y',0)=\frac{1}{2\pi i}\sqrt{\gamma_{k,x}\gamma_{l,y}+\gamma_6^2}\,\chi_\infty\left(-\frac{1}{2\pi}S^{cl,\theta}_{(4)}(x'',y'',T;x',y',0)\right),$$

$$\mathcal{K}^{\theta}(x'',y'',T;x',y',0)=\frac{1}{2\pi i}\sqrt{\gamma_{12}\gamma_{34}-\gamma_{14}\gamma_{23}}\,\chi_\infty\left(-\frac{1}{2\pi}S^{cl,\theta}_{(4)}(x'',y'',T;x',y',0)\right).$$

5. Ultrametricity vs noncommutativity in cosmology

- For the vacuum Kantowski-Sachs model in noncommutative case the two classical actions are obtained. Further research will better reveal quantum dynamics of the model.
- Importance of the model lies in existence of diffeomorphism with Schwarzschild solution.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- The Schwarzschild metric element for centrally-symmetric gravitational field:

$$ds^2 = -(1 - \frac{r_g}{r})dt^2 + (1 - \frac{r_g}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

- For $r < r_g$, inside the horizon, one has $g_{00} = g_{tt}$ and $g_{11} = g_{rr}$.
- It suggests that space and time (coordinates) change roles, in a way, once the event horizon is passed.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- In this sense for both observers there is only one direction, for the outside observer the only direction is time, however for the inner observer the only direction is straight to the singularity.
- It is possible to consider the possibility that inside of a Schwarzschild black hole can be described by a quadratic metric form which can be obtained by substitution $t \leftrightarrow r$, i.e. by:

$$ds^2 = -(\frac{r_g}{t} - 1)^{-1} dt^2 + (\frac{r_g}{t} - 1) dr^2 + t^2 (d\theta^2 + \sin^2\theta d\varphi^2),$$

where components of metric explicitly depend on time.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- The previous expression has the form of homogeneous and anisotropic Kantowski-Sachs metrics:
  $ds^2 = -\frac{N^2(t)}{c_1(t)}dt^2 + c_1(t)dr^2 + c_2^2(t)(d\theta^2 + \sin^2\theta d\varphi^2)$.
- An example of an interesting feature of General Theory of Relativity is that it generates solutions for cosmologycal models from static solutions of Einstein equations of gravitational field.
- Therefore: interior of a Schwarzschild black hole $\leftrightarrow$ Vacuum K-S model $\leftrightarrow$ Two oscilator *oscillator-ghost-oscillator* model.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- The Lagrangian:

$$L = -\frac{M_{pl}^2 V_0}{\tilde{N}} \left[(\dot{x}^2 - \tilde{N}^2 x^2) - (\dot{y}^2 - \tilde{N}^2 y^2) \right],$$

- The Hamiltionian:

$$H = -\frac{\tilde{N}}{4 M_{pl}^2 V_0} (\pi_x^2 - \pi_y^2) - V_0 M_{pl}^2 \tilde{N} (x^2 - y^2),$$

- Wheeler-DeWitt equation:

$$\hat{\mathcal{H}} \Psi(x, y) = \left[-\frac{1}{4 M_{pl}^2 V_0} (\hat{\pi}_x^2 - \hat{\pi}_y^2) - V_0 M_{pl}^2 (\hat{x}^2 - \hat{y}^2) \right] \Psi(x, y) = 0.$$

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- Substituting $\Psi_{n_1,n_2}(x,y) = \mu_{n_1}(x)\tau_{n_2}(y)$, Wheeler-DeWitt equation takes the form:

$$\left[-\frac{\hbar^3 \tilde{N}}{4V_0 c^2 M_{pl}^2}\frac{\partial^2}{\partial x^2} + \frac{\tilde{N} V_0 c^2 M_{pl}^2}{\hbar} x^2\right] \mu_{n_1}(x) = E_{n_1}\mu(x),$$

$$\left[-\frac{\hbar^3 \tilde{N}}{4V_0 c^2 M_{pl}^2}\frac{\partial^2}{\partial y^2} + \frac{\tilde{N} V_0 c^2 M_{pl}^2}{\hbar} y^2\right] \tau_{n_2}(y) = E_{n_2}\tau(y).$$

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- A general adelic solution is:

$$\begin{aligned}\Psi^{(adel.)}(x,y) &= \left(\frac{\tilde{\omega}}{\pi}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{C_n}{2^n n!} e^{-\frac{\tilde{\omega}}{2}(x^2+y^2)} H_n(\sqrt{\tilde{\omega}}x) H_n(\sqrt{\tilde{\omega}}y) \\ &\times \prod_{p\in M} \Psi_p(x,y) \prod_{p\notin M} \Omega(|x|_p)\Omega(|y|_p),\end{aligned}$$

- The preceding, decoupled, equations are the equations of two quantum harmonic oscillators whose frequencies are: $\omega^2 = \frac{1}{\tilde{m}}\frac{\partial^2 E_p(x)}{\partial x^2} = \frac{1}{\tilde{m}}\frac{\partial^2 E_p(y)}{\partial y^2} = \frac{\tilde{N}}{\tilde{m}}\frac{2V_0 c^2 M_{pl}^2}{\hbar}$ and energies are E_{n_1} and E_{n_2}.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

- We assume that these oscillators relate to two gravitational degrees of freedom, namely, one on a black hole and the other on a corresponding white hole.
- Freedom of choice of laps function $\tilde{N}$ allows the possibility of fixing gauge $\frac{\tilde{N}}{\tilde{m}} = \frac{6c^2}{V_0\hbar}$.
- Frequency of oscilator is: $\hbar\omega = \sqrt{\frac{3}{2\pi}} E_{pl}$.
- The correction partition function is: $Z_{approx.} = \sqrt{\frac{2\pi}{3}} \frac{e^{-\frac{\beta^2 E_{pl}^2}{16\pi}}}{\beta E_{pl}}$.

6. The Schwarzschild black hole interior and two-oscillator Kantowski-Sachs model

Starting from partition function $Z_{approx.}$ one can determine:

- Internal energy of a black hole:
  $E_{in.} = -\frac{\partial}{\partial\beta}(\ln Z_{approx}) = \frac{E_{pl}^2}{8\pi}\beta + \frac{1}{\beta} = mc^2,$
- Hawking temperature with quantum corre (for $E_{pl} \ll mc^2$):
  $\beta = \beta_H \left[1 - \frac{1}{\beta_H mc^2}\right],$
- Entropy of a black hole (for $E_{pl} \ll mc^2$):
  $\frac{S}{k_B} = \ln Z_{approx.} + \beta E_{in.} = \frac{S_{BH}}{k_B} - \frac{1}{2}\ln\left(\frac{S_{BH}}{k_B}\right) + \mathcal{O}(S_{BH}^{-1}).$

7. Conclusion

- We present a classification of (cosmological) two-oscillator models.
- Classical and standard quantum forms of a class of these models are presented. Signature transition in standard case is considered.

7. Conclusion

- In p-adic case all vacuum states are found as well as conditions for their existence. They have the same for all two-oscillator models! Signature transition in p-adic space-time $\mathbb{Q}_p^4$ are possible under certain restrictions.
- We calculated the exact forms of action and propagator on noncommutative configuration space.

7. Conclusion

- Starting from a diffeomorphism between the Schwarzschild solution of the Einstein field equation and the corresponding solution of this cosmological model we presented the dynamics of the interior of the non-rotating and non-charged, Schwarzschild, black hole
- By applying the Feynman - Hibbs procedure, Hawking temperature was determined as well as entropy with the corresponding quantum corrections. The expression for the entropy of the Schwarzschild black hole obtained in this approach is equivalent to the expression for the entropy obtained in other approaches.
- Consideration of General Uncertainty Principle (GUP) and its application of the presented class of models is our work in progress and will be presented in details elsewhere.

Work in progress

- The effects of the Generalized Uncertainty Principle (GUP) on the one-dimensional minisuperspace FLRW cosmological model, with mixture of noninteracting dust and radiation is considered.
- In the classical case the Lagrangian of the model is reduced by a suitable coordinate transformation to Lagrangian of a linear harmonic oscillator.
- The model can be presented in a classical, *p*-adic and standard quantum case. Within the standard quantum approach Wheeler-DeWitt equation and its general solutions are obtained, i.e. a wave function of the model is found, and then an adelic wave function is constructed.
- The effects of the modified standard commutation relations give interesting results in both classical and quantum case.

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