Dynamical Systems Generated by Mappings with Delay

Livat Tyapaev

National Research Saratov State University

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Automaton transformation of the space of all one-side infinite words over the alphabet $\{0, 1, \dots, p-1\}$, where p is a prime number, is continuous transformation (w.r.t. the p-adic metric) of the ring of p-adic integers $\mathbb{Z}_p$.

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Moreover, a mappings that are realized by (synchronous) automata satisfy the p-adic Lipschitz condition with constant equal 1.
In the p-adic ergodic theory automata are p-adic dynamical systems and automata mappings, in their turn, are a continuous (in particular, 1-Lipschitz) transformations of the space $\mathbb{Z}_p$.

The objects of the study are mappings with delay realized by asynchronous automata in the context of the p-adic dynamics. The ergodic and more generally measure-preserving p-adic dynamical systems are explored.

Outline

Automata

An (synchronous) automaton (or, letter-to-letter transducer) is a 6-tuple $\mathcal{A} = (\mathcal{I}, \mathcal{S}, \mathcal{O}, h, g, s_o)$, where

- $\mathcal{I}$ is a non-empty finite set, the input alphabet,
- $\mathcal{O}$ is a non-empty finite set, the output alphabet,
- $\mathcal{S}$ is a non-empty (possibly, infinite) set of states,
- $h\colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ is a state update function,
- $g\colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}$ is an output function, and
- $s_0 \in \mathcal{S}$ is fixed; s_0 is called the initial state.

Automata

Asynchronous automaton is defined in a similar way, except for output function. Denote the set of finite output words via $\mathcal{O}^*$.

An asynchronous automaton is a 6-tuple $\mathcal{B} = (\mathcal{I}, \mathcal{S}, \mathcal{O}, h, g, s_0)$, where

- $\mathcal{I}$, $\mathcal{O}$ are finite alphabets, $\mathcal{S}$ is a set of states,
- $h\colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ is a state update function,
- $g\colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}^*$ is an output function, and
- s_0 is an initial state.

Note that set of states $\mathcal{S}$ could be an infinite, and in this case an automaton is called infinite.

Automata

Roughly speaking, asynchronous automaton is an letter-to-word transducer that converts an input string of arbitrary length to an output string. The transducer reads one symbol at a time, changing its internal state and outputting a finite sequence of symbols at each step.

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Roughly speaking, asynchronous automaton is an letter-to-word transducer that converts an input string of arbitrary length to an output string. The transducer reads one symbol at a time, changing its internal state and outputting a finite sequence of symbols at each step. Asynchronous transducers are a natural generalization of synchronous transducers, which are required to output exactly one symbol for every symbol read.

Automata

For example, the asynchronous automaton represented by Moor diagram: Starting in initial state, automaton converts any first input symbol to empty word.

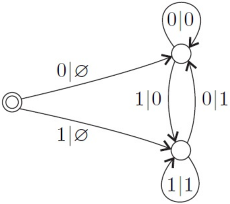

Figure : Example of an asynchronous automaton

Automata

We consider only accessible automata: where any state $s \in \mathcal{S}$ is reachable from initial state s_0; that is, given state $s \in \mathcal{S}$, there exist a finite input word u such that after the word u has been fed to the automaton, the automaton reaches the state $s \in \mathcal{S}$.

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We assume further that both alphabets $\mathcal{I}$ and $\mathcal{O}$ are p-elements: $\mathcal{I} = \mathcal{O} = \mathbb{F}_p = \{0, 1, \dots, p-1\}$.

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We assume further that both alphabets $\mathcal{I}$ and $\mathcal{O}$ are p-elements: $\mathcal{I} = \mathcal{O} = \mathbb{F}_p = \{0, 1, \ldots, p-1\}$. A simple example of an automaton is the 2-adic adding machine: $x \mapsto x+1$, $\mathcal{A} = (\mathcal{I} = \mathbb{F}_2, \mathcal{S} = \{s_0, s_1\}, \mathcal{O} = \mathbb{F}_2, h, g, s_0)$, where

$$h(0, s_0) = s_1; h(1, s_0) = s_0,$$

$$g(0, s_0) = 1; g(1, s_0) = 0,$$

$$h(i, s_1) = s_1; g(i, s_1) = i,$$

for $i \in \mathcal{I} = \mathbb{F}_2$.

Automata

An automaton $\mathcal{A} = (\mathcal{I}, \mathcal{S}, \mathcal{O}, h, g, s_0)$ transforms input words (w.r.t the alphabet $\mathbb{F}_p$) of length n into output words of length n, that is, an automaton $\mathcal{A}$ maps the set W_n of all words of length n into W_n.

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$$x = x_0 + x_1 \cdot p + \ldots + x_{n-1} \cdot p^{n-1} = \sum_{i=0}^{n-1} x_i \cdot p^i.$$

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$$x = x_0 + x_1 \cdot p + \ldots + x_{n-1} \cdot p^{n-1} = \sum_{i=0}^{n-1} x_i \cdot p^i.$$

This number x can be considered as an element of the residue ring $\mathbb{Z}/p^n\mathbb{Z}$ modulo p^n. Thus, every automaton $\mathcal{A}$ corresponds a map f_n from $\mathbb{Z}/p^n\mathbb{Z}$ to $\mathbb{Z}/p^n\mathbb{Z}$, for every $n = 1, 2, 3 \ldots$.

Automata functions

The function $f_n\colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ can be considering as the mapping in the space of infinite words over the alphabet $\mathbb{F}_p$. The latter can be identified with the ring of p-adic integers $\mathbb{Z}_p$.

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Every automaton $\mathcal{A}$ *defines a map $f_{\mathcal{A}}$ from ring of p-adic integers $\mathbb{Z}_p$ to itself*: Given an infinite string $x = \ldots x_{n-1} \ldots x_1 x_0$ over $\mathbb{F}_p$ we consider a p-adic integer $x = x_0 + x_1 \cdot p + \ldots + x_{n-1} \cdot p^{n-1} + \ldots = \sum_{i=0}^{\infty} \delta_i(x) \cdot p^i$, where δ_i are coordinate functions valued in $\mathbb{F}_p$. Here δ_i depends only on the coordinates $x_0, x_1, \ldots, x_i$ of the variable x: $\delta_i = \delta_i(x_0, x_1, \ldots, x_i)$.

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For every $x \in \mathbb{Z}_p$, we put $\delta_i(f_{\mathcal{A}}(x)) = g(\delta_i(x), s_i)$, $i = 0, 1, 2, \ldots$ where $s_i = h(\delta_{i-1}(x), s_{i-1})$, $i = 1, 2, \ldots$.

So, we say, that map $f_{\mathcal{A}}$ is *automaton function* (or, automaton map) of the automaton $\mathcal{A}$.

Automata functions

Similar way, we can consider asynchronous automata: An asynchronous automaton $\mathcal{B} = (\mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, h, g, s_0)$ performs a transformation $f_{\mathcal{B}} \colon \mathbb{Z}_p \to \mathbb{Z}_p$.

Automata functions

Synchronous automaton function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ satisfies 1-Lipschitz condition:

$||f(x) - f(y)||_p \leq ||x - y||_p$ for any $x, y \in \mathbb{Z}_p$, where $||\cdot||_p$ is the p-adic norm.

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For 1-Lipschitz functions the following natural question arises: Can any 1-Lipschitz mapping be generated by some (synchronous) automaton?

Automata functions

The answer is "yes": The class of all (synchronous) automata functions coincides with the class of all 1-Lipschitz mappings from $\mathbb{Z}_p$ to $\mathbb{Z}_p$.

Theorem (V.S. Anashin)

The automaton function $f_{\mathcal{A}}\colon \mathbb{Z}_p \to \mathbb{Z}_p$ of the synchronous automaton $\mathcal{A} = (\mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, S, O, s_0)$ is 1-Lipschitz. Conversely, for every 1-Lipschitz function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ there exists an synchronous automaton $\mathcal{A} = (\mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, S, O, s_0)$ such that $f = f_{\mathcal{A}}$.

Automata functions

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The description of finite automata functions was given by Vuillemin, althought only for $p = 2$. V.S. Anashin and T.Smyshlyaeva solved this problem for arbitary p, using a coordinate functions and van der Put series, respectively.

Automata functions

Denote via $\mathcal{I}^\infty$ and $\mathcal{O}^\infty$ the sets of infinite words over input alphabet $\mathcal{I}$ and output alphabet $\mathcal{O}$, respectively.

Theorem (R.I. Grigorchuk, V.V.Nekrashevich, V.I. Sushchanskii)

The mapping $f\colon \mathcal{I}^\infty \to \mathcal{O}^\infty$ is continuous if and only if it is defined by a certain asynchronous automaton.

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Note, in general case, an asynchronous automaton defined a continuous mapping is infinite.
If the mapping $f\colon \mathcal{I}^\infty \to \mathcal{O}^\infty$ is bijective, then this mapping is a homeomorphism, and the inverse mapping f^{-1} is also defined by a certain asynchronous automaton.

Automata functions

So, if the input and output alphabets of automaton coincide (i.e. $\mathcal{I} = \mathcal{O} = \mathbb{F}_p$) and the automaton is initial (i.e., has an initial state s_0), then it induces a transformation of the space of words into itself. These words may be either finite or infinite. In the latter case, we have a continuous (in particular, 1-Lipschitz) transformation of the space of infinite words (i.e., the space of p-adic integers $\mathbb{Z}_p$). Conversely, any continuous transformation is defined by a certain automaton.

Mappings with delay

A mapping $f_{\mathcal{B}}\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is called *n-unit delay* whenever given an asynchronous automaton $\mathcal{B} = (\mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, S, O, s_0)$ traslated infinite input string $\alpha = \ldots \alpha_2\alpha_1\alpha_0$ over $\mathbb{F}_p$ into infinite output string $\beta = \ldots \beta_2\beta_1\beta_0$ over $\mathbb{F}_p$ such that $g(\delta_i(\alpha), s_i) = \emptyset$, where $\emptyset$ is empty word, for $i = 0, 1, 2 \ldots, n-1$, $s_i = h(\delta_{i-1}(\alpha), s_{i-1})$, $i = 1, 2, \ldots, n-1$; and $g(\delta_{n+i}(\alpha), s_{n+i}) = \beta_i$, $i = 0, 1, \ldots$, $s_{n+i} = h(\delta_{n+i-1}(\alpha), s_{n+i-1})$ for $i = 0, 1, 2, \ldots$.

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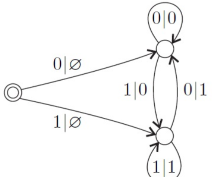
Example of unit-delay map $(n = 1)$:



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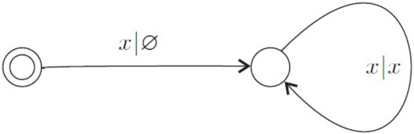
In general case, an n-unit delay mappings form a class of a continuous functions, that in turn, contains a class of shifts. For example, a class of unit-delay mappings contains unilateral shift defined by finite asynchronous automaton, that is irrespective of the first incomming letter $x \in \mathbb{F}_p$, outputs an empty word $\varnothing$; after that, an automaton outputs the incoming word without changes:



Shifts

The p-adic shift $\mathbf{S}\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is defined as follows.

If $x = x_0 + x_1 p + x_2 p^2 + \ldots$, where the $x_i \in \mathbb{F}_p = \{0, 1, \ldots, p-1\}$, we let $\mathbf{S}(x) = x_1 + x_2 p + x_3 p^2 \ldots$.

We see that if $\mathbf{S}^k$ denotes the k-fold iterate of $\mathbf{S}$, then we have that $\mathbf{S}^k(x) = x_k + x_{k+1} p + \ldots$. Moreover, for $x \in \mathbb{Z}$ it is the case that $\mathbf{S}^k(x) = \lfloor \frac{x}{p^k} \rfloor$ where $\lfloor \cdot \rfloor$ is the greatest integer function.

Shifts

The p-adic shift is continuous as a function of $\mathbb{Z}_p$: if $||x - y||_p < p^{-(k+1)}$ then $||\mathbf{S}(x) - \mathbf{S}(y)||_p < p^{-k}$.

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By Mahler's Theorem, any continuous function $T\colon \mathbb{Z}_p \to \mathbb{Z}_p$ can be expressed in the form of a uniformly convergent series, called its *Mahler Expansion*:

$$T(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m}$$

where

$$a_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} T(m-i) \in \mathbb{Z}_p$$

Shifts

We let $a_m^{(k)}$ be the m^{th} Mahler coefficient of $\mathbf{S}^k$:

$$\mathbf{S}^k(x) = \sum_{m=0}^{\infty} a_m^{(k)} \binom{x}{m}.$$

Theorem. (J.Kingsbery, A. Levin, A. Preygel, C.E. Silva)

The coefficients $a_m^{(k)}$ satisfy the following properties:

1. $a_m^{(k)} = 0$ for $0 \leq m < p^k$;
2. $a_m^{(k)} = 1$ for $m = p^k$;
3. Suppose $j \geq 0$. Then p^j divides $a_m^{(k)}$ for $m > jp^k - j + 1$ (and so, $||a_m^{(k)}||_p \leq p^{-j}$).

1-Lipschitz functions

This theorem describes synchronous automata (in other words, 1-Lipschitz functions) in terms of Mahler expansion.

Theorem. (A.S. Anashin)

A function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ represented by Mahler expansion is 1-Lipschitz if and only if

$$||a_i||_p \leq p^{-\lfloor \log_p i \rfloor}$$

for all $i = 1, 2, \ldots$.

Recall that for $i \in \mathbb{N}$ a number $\lfloor \log_p i \rfloor$ is reduced by 1 a number of digits in a base-p expansion for i.

n-unit delay functions

For n-unit delay mapping, $n \in \mathbb{N}$, we gets next theorem.

Theorem 1

A function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ represented by Mahler expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m},$$

where $a_m \in \mathbb{Z}_p$, $m = 0, 1, 2 \ldots$, is an n-unit delay if and only if

$$||a_i||_p \le p^{-\lfloor \log_{p^n} i \rfloor + 1}$$

for all $i \ge 1$.

Dynamics

Dynamical system on a measuarable spase $\mathbb{S}$ is understood as a triple $(\mathbb{S}, \mu, f)$, where $\mathbb{S}$ is a set endowed with a measure μ, and $f : \mathbb{S} \to \mathbb{S}$ is a measurable function.

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Measure-preservation and ergodicity

A mapping $F\colon \mathbb{S} \to \mathbb{S}$ of measurable space $\mathbb{S}$ onto $\mathbb{S}$ endowed with probabilistic measure μ, is said to be *measure-preserving* whenever $\mu(F^{-1}(S)) = \mu(S)$ for each measurable subset $S \subseteq \mathbb{S}$.

A measure-preserving map $F\colon \mathbb{S} \to \mathbb{S}$ is said to be *ergodic* if for each measurable subset S such that $F^{-1}(S) = S$ holds either $\mu(S) = 1$ or $\mu(S) = 0$.

Automata as p-adic dynamical systems

We study dynamical system $(\mathbb{Z}_p, \mu, f)$ on $\mathbb{Z}_p$, where map $f \colon \mathbb{Z}_p \to \mathbb{Z}_p$ defined by some asynchronous automaton $\mathcal{B} = (\mathbb{F}_p, \mathcal{S}, \mathbb{F}_p, S, O, s_0)$. The ring $\mathbb{Z}_p$ can be endowed with a probability measure μ_p. The measure μ_p is a normalized Haar measure. The base of elementary measurable subsets are all balls $B_{p^{-k}}(a)$ of non-zero radii p^{-k}; and we put

$$\mu_p(B_{p^{-k}}(a)) = p^{-k}.$$

Measure-preservation and ergodicity of 1-Lipschitz functions in terms of Mahler expansion

Theorem. (V.S. Anashin)

The function f defines a 1-Lipschitz measure-preserving transformation on $\mathbb{Z}_p$ whenever the following conditions hold simultaneously:

1. $a_1 \not\equiv 0 \pmod{p}$;
2. $a_i \equiv 0 \pmod{p^{\lfloor \log_p i \rfloor + 1}}$, $i = 2, 3, \ldots$.

The function f defines a 1-Lipschitz ergodic transformation on $\mathbb{Z}_p$ whenever the following conditions hold simultaneously:

1. $a_0 \not\equiv 0 \pmod{p}$;
2. $a_1 \equiv 1 \pmod{p}$, for p odd;
3. $a_1 \equiv 1 \pmod{4}$, for $p = 2$;
4. $a_i \equiv 0 \pmod{p^{\lfloor \log_p (i+1) \rfloor + 1}}$, $i = 2, 3, \ldots$.

Moreover, in the case $p = 2$ these conditions are necessary.

Preserve the measure for n-unit delay mappings

Let F_k be a reduction of function $f \bmod p^{n\cdot(k-1)}$ on the elements of the ring $\mathbb{Z}/p^{n\cdot k}\mathbb{Z}$ for $k = 2, 3, \ldots$.

Theorem 2

A n-unit delay mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving if and only if the number $\#F_k^{-1}(x)$ of F_k-pre-images of the point $x \in \mathbb{Z}/p^{n\cdot(k-1)}\mathbb{Z}$ is equal p^n, $k = 2, 3, \ldots$.

Ergodicity

A point $x_0 \in \mathbb{Z}_p$ is said to be a *periodic point* if there exists $r \in \mathbb{N}$ such that $f^r(x_0) = x_0$. The least r with this property is called the *length* of period of x_0. If x_0 has period r, it is called an r-*periodic point*. The orbit of an r-periodic point x_0 is $\{x_0, x_1, \ldots, x_{r-1}\}$, where $x_j = f^j(x_0)$, $0 \le j \le r-1$. This orbit is called an r-*cycle*.

Ergodicity

Let $\gamma(k)$ be an $r(k)$-cycle $\{x_0, x_1, \ldots, x_{r(k)-1}\}$, where

$$x_j = (f \bmod p^{k \cdot n})^j(x_0),\ 0 \leq j \leq r(k) - 1,$$

$k = 1, 2, 3, \ldots$.

Theorem 3

A measure-preserving a n-unit delay mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is ergodic if a $\gamma(k)$ is an unique cycle, for all $k \in \mathbb{N}$.

Measure-preservation and ergodicity in terms of Mahler expansion

Let n-unit delay function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ be represented by Mahler expansion

$$f(x) = \sum_{m=0}^{\infty} a_m \binom{x}{m},$$

where $a_m \in \mathbb{Z}_p$, $m = 0, 1, 2 \ldots$.

Theorem 4

A n-unit delay mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is measure-preserving whenever the following conditions hold simultaneously:

1. $a_i \not\equiv 0 \pmod p$ *for* $i = p^n$;
2. $a_i \equiv 0 \pmod{p^{\lfloor \log_{p^n} i \rfloor}}$, $i > p^n$.

Measure-preservation and ergodicity in terms of Mahler expansion

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where $a_m \in \mathbb{Z}_p$, $m = 0, 1, 2 \ldots$.

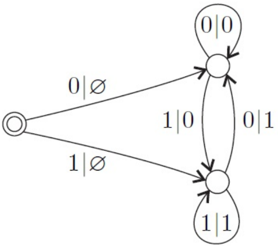
Theorem 5

Let $p = 3$. Then a n-unit delay mapping $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$ is ergodic on $\mathbb{Z}_p$ whenever the following conditions hold simultaneously:

1. $a_1 + a_2 + \ldots + a_{p^n-1} \equiv 0 \pmod p$;
2. $a_i \equiv 1 \pmod p$ *for* $i = p^n$;
3. $a_i \equiv 0 \pmod{p^{\lfloor \log_{p^n} i \rfloor}}$, $i > p^n$.

Thank you!

Asynchronous automaton



Unilateral shift