Finite automata models in Quantum Theory

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Goals of the talk

To introduce a formal model of a measurement of a quantum system which is based on the following assumptions:

- **No ‘momentary measurements’**: Every measurement takes some time
- **Discreteness**: The measurement actually is a sequence of ‘impacts’; the ‘impacts’ attain discrete values.
- **Causality**: Each ‘impact’ forces the system to change its ‘state’ and to react with a ‘response’ of some discrete value which depends only on the current state of the system and on the value of current ‘impact’ the system is imposed to.
- **No ‘physical infinity’**: The number of ‘states’ of the system is assumed to be finite.

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Under that assumptions we will show that 'measurement procedure' results in a wave-like picture; and moreover, that the only case when a particle wave function may be ascribed to the procedure is when 'impacts' and 'responses' attain only two values, 0 and 1.

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The model is non-Archimedean; actually it can be described in terms of 1-Lipschitz mappings of the space of 2-adic integers. Nonetheless, standard complex numbers based mathematical formalism of quantum theory, the wave packets, can be derived from the model.

Discrete causality: Formalisation

Speaking of causal discrete system we mean a sort of a **black box** which

1. at every moment $t = 0, 1, 2, \ldots$ is at some state from the set $\mathcal{S} = \{s_0, s_1, \ldots\}$ of all possible states;
2. changes its current state to some new one when affected by some cause from the set of all causes $\mathcal{C} = \{c_0, c_1, c_2, \ldots\}$; and
3. produces an effect from the set of all possible effects $\mathcal{E} = \{e_0, e_1, e_2, \ldots\}$ so that
4. once the cause $c \in \mathcal{C}$ happens,
   - corresponding effect $e \in \mathcal{E}$ depends only on the current state $s \in \mathcal{S}$ and on the cause $c \in \mathcal{C}$, while simultaneously
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This can be formalised by a notion of an (initial) automaton, (or, *transducer, sequential machine*).

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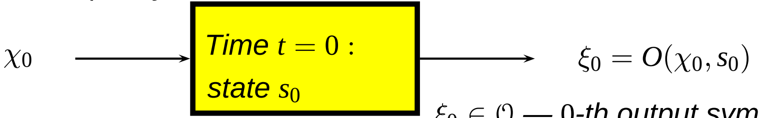
Automaton $\mathfrak{A} = \langle \mathcal{I}, \mathcal{S}, \mathcal{O}, S, O, s_0 \rangle$: $\mathcal{I}$ – *input alphabet;* $\mathcal{O}$ – *output alphabet;* $\mathcal{S}$ – *state set;* $S\colon \mathcal{I} \times \mathcal{S} \to \mathcal{S}$ – *transition function;* $O\colon \mathcal{I} \times \mathcal{S} \to \mathcal{O}$ – *output function;* $s_0 \in \mathcal{S}$ – *initial state*

$\chi_0 \in \mathcal{I}$ — 0-*th input symbol*



$\xi_0 \in \mathcal{O}$ — 0-*th output symbol*

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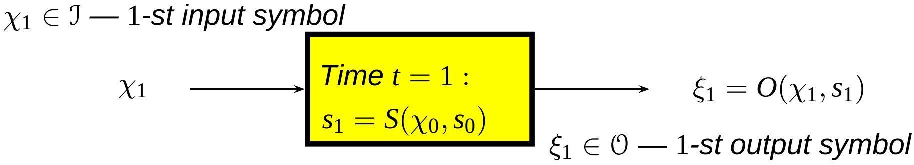

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$\chi_2 \in \mathcal{I}$ — 2-*nd input symbol*

χ_2 → *Time* $t = 2$: $s_2 = S(\chi_1, s_1)$ → $\xi_2 = O(\chi_2, s_2)$

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$\chi_i \in \mathcal{I}$ — *i-th input symbol*

χ_i ⟶ *Time* $t = i$: $s_i = S(\chi_{i-1}, s_{i-1})$ ⟶ $\xi_i = O(\chi_i, s_i)$

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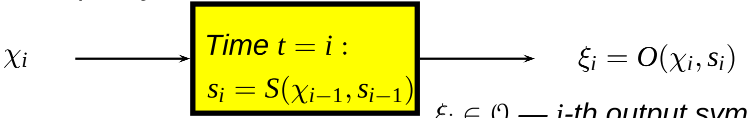

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The automaton $\mathfrak{A}$ determines the automaton function $f_{\mathfrak{A}}$ that maps words over the alphabet $\mathcal{I}$ to words over the alphabet $\mathcal{O}$: $f_{\mathfrak{A}}\colon \ldots \chi_2\chi_1\chi_0 \mapsto \ldots \xi_2\xi_1\xi_0$

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Causality

Every output symbol ξ_i depends only on the first input symbols $\chi_0, \ldots, \chi_i \in \mathcal{I}$; so $\xi_i = \psi_i(\chi_0, \ldots, \chi_i) \in \mathcal{O}$, where $\psi_i\colon \mathcal{I}^{i+1} \to \mathcal{O}$.

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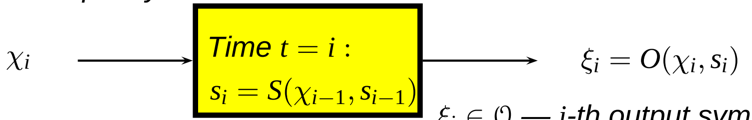

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The automaton function $f_{\mathfrak{A}}\colon \ldots \chi_2\chi_1\chi_0 \mapsto \ldots \xi_2\xi_1\xi_0$ is completely determined by the sequence of maps $\psi_i\colon \mathcal{I}^{i+1} \to \mathcal{O}$, $i \in \mathbb{N}_0$; and vice versa, every such sequence of maps determines an automaton function.

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In other words, the following theorem is true: Automata functions on the non-Archimedean space of infinite words are exactly non-expansive functions; that is, 1-Lipschitz functions.

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Discrete causality=non-Archimedean 1-Lipschitzness: Causal discrete functions are exactly non-Archimedean 1-Lipschitz functions.

Discrete causal systems: Observations

When observing behavior of a system we are somehow 'measuring' a sequence of causes and effects, in some very loose meaning. That is,

Measurement:

we assign a real value $v(a)$ to the n-tuple $a = \alpha_{n-1} \ldots \alpha_2\alpha_1\alpha_0$ (where α_j, $j = 0, 1, 2, \ldots$, only take p distinct values).

- Without loss of generality we may assume that the values α_j takes are $0, 1, \ldots, p-1$;
- The value $v(a)$ must reflect time ordering of causes/effects We will assume that every $(j+1)$-st position is 'β times heavier' than the j-th one, where $\beta > 1$ is a real number.
- Without loss of generality we may assume that $v(a) \in [0, d]$ for all $n = 1, 2, 3, \ldots$ and all tuples $a = \alpha_{n-1} \ldots \alpha_2\alpha_1\alpha_0$.
- $v(a) = \alpha_{n-1}\beta^{-1} + \alpha_{n-1}\beta^{-2} + \cdots + \alpha_2\beta^{-n+2} + \alpha_1\beta^{-n-1} + \alpha_0\beta^{-n}$

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- Without loss of generality we may assume that the values α_j takes are $0, 1, \ldots, p-1$; that is, we just enumerate distinct values a cause/effect may take by $0, 1, \ldots, p-1$.
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- Without loss of generality we may assume that the values α_j takes are $0, 1, \ldots, p-1$;
- The value $v(a)$ must reflect time ordering of causes/effects Therefore we must assign some 'weight' to every position j of the tuple to make the $(j+1)$-st place 'heavier' than the j-th one. It is clear then the weight must be just a non-decreasing function of j. We will assume that every $(j+1)$-st position is 'β times heavier' than the j-th one, where $\beta > 1$ is a real number.
- Without loss of generality we may assume that $v(a) \in [0, d]$ for all $n = 1, 2, 3, \ldots$ and all tuples $a = \alpha_{n-1} \ldots \alpha_2\alpha_1\alpha_0$.
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- Without loss of generality we may assume that the values α_j takes are $0, 1, \dots, p-1$;
- The value $v(a)$ must reflect time ordering of causes/effects We will assume that every $(j+1)$-st position is 'β times heavier' than the j-th one, where $\beta > 1$ is a real number.
- it is convenient to have all values normalized so that for every tuple a of arbitrary length n its value $v(a)$ lies in some real interval $[c, d]$. Without loss of generality we may assume that $v(a) \in [0, d]$ for all $n = 1, 2, 3, \dots$ and all tuples $a = \alpha_{n-1} \dots \alpha_2 \alpha_1 \alpha_0$.
- $v(a) = \alpha_{n-1}\beta^{-1} + \alpha_{n-1}\beta^{-2} + \dots + \alpha_2\beta^{-n+2} + \alpha_1\beta^{-n-1} + \alpha_0\beta^{-n}$

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- We put therefore
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- $v(a) = \alpha_{n-1}\beta^{-1} + \alpha_{n-1}\beta^{-2} + \cdots + \alpha_2\beta^{-n+2} + \alpha_1\beta^{-n-1} + \alpha_0\beta^{-n}$

Speaking loosely, we just associate a real number $v(a)$ whose base-β expansion is $0.\alpha_{n-1} \ldots \alpha_2\alpha_1\alpha_0$ to the n-tuple $a = \alpha_{n-1} \ldots \alpha_2\alpha_1\alpha_0$.

Discrete causal systems: Observations

When observing behavior of a system we are somehow ‘measuring’ a sequence of causes and effects, in some very loose meaning. That is,

Measurement:

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The above rule is just a model of a standard process of assignment of a numerical value to a physical quantity: For instance, distances can be measured in femtometers, picometers, micrometers, millimeters, decimeters, meters, kilometers, etc., which are linear units in base 10. But if one measures a distance between two milestones, there is no need (and practically impossible) to do this within accuracy up to micrometers, not speaking of femtometers and picometers. So the *rule is just a reasonable model for standard common rule of ‘figuring out’ numerical results of a measurement, after a proper normalization*.

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It easy to see that the range of $v(a)$ is

$$0 \le v(a) < \frac{\lceil\beta\rceil - 1}{\beta - 1}.$$

- In particular, if β is an integer then $v(a) \in [0, 1)$ and $p = \beta$.
- Once $\beta > 1$ is not an integer then $p = \lceil\beta\rceil$. In particular, if $\beta = 1 + \tau$ with $\tau < 1$ then $p = 2$ and the range of $v(a)$ is $[0, 1/\tau)$.

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Measurement and physical laws

- A physical law may be thought of as a mathematical correspondence between quantities of impacts a physical system is exposed to and quantities of responses the system reacts.
- The measured experimental values of physical quantities lie in $\mathbb{Q}$.
- People usually are trying to find a physical law as a correspondence between cluster points w.r.t. the metrics in $\mathbb{R}$ (!) of experimental values.
- An experimental curve is a smooth curve (the C^2-smoothness is common) which is the best approximation of the set of the experimental points.
- A physical law is a curve which can be approximated by the experimental curves with the highest achievable accuracy w.r.t. the metrics in $\mathbb{R}$.

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Let physical quantities which correspond to impacts and responses are quantized; i.e, take only values, say, $0, 1, \ldots, p-1$. Then, once a system is exposed to a sequence $c = \chi_{k-1}\chi_{k-2}\ldots\chi_0$ of k of impacts, it reacts with a sequence $e = \xi_{k-1}\xi_{k-2}\ldots\xi_0$ of k responses.

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Now fix some $\beta > 1$ s.t. $\lceil\beta\rceil = p$ and consider a point

$$(v(c); v(e)) = (0.\chi_{k-1}\chi_{k-2}\ldots\chi_0; 0.\xi_{k-1}\xi_{k-2}\ldots\xi_0)$$

in the square $[0, \frac{p-1}{\beta-1}] \times [0, \frac{p-1}{\beta-1}]$ of $\mathbb{R}^2$, an experimental point.

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Exposing a system $\mathfrak{A}$ to all sequences of impacts and getting respective sequences of responses we this way obtain a point set $E(\mathfrak{A})$ in the square from $\mathbb{R}^2$ and we therefore are trying to derive a physical law in a form of *smooth curves* lying in the set $\mathbf{P}(\mathfrak{A})$ which is a closure of $E(\mathfrak{A})$ in $\mathbb{R}^2$.

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Let us see what sort of smooth curves can be obtained for discrete causal systems; that is, when $\mathfrak{A}$ is an automaton.
We start with the case when $\beta > 1$ is an integer; thus $\beta = p$.

The automata 0-1 law (case $\beta = p > 1$ an integer)

Experimentally it can be observed that when $k \to \infty$ the set $E_k(\mathfrak{A}) = E_k(f_{\mathfrak{A}})$ of experimental points obtained for all impact/response sequences of length k basically exhibits behaviour of two kinds only:

1. $E_k(f_{\mathfrak{A}})$ is getting more and more dense so that at $k \to \infty$ they fill the unit square completely
2. $E_k(f_{\mathfrak{A}})$ is getting less and less dense and with pronounced straight lines that look like *windings of a torus*

We now explain what happens.

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Let $\alpha(f_{\mathfrak{A}})$ be a Lebesgue measure of the plot of $\mathfrak{A}$, i.e., of the closure $\mathbf{P}(f_{\mathfrak{A}}) = \mathbf{P}(\mathfrak{A})$ in the unit square $\mathbb{I}^2 \subset \mathbb{R}^2$.

Theorem (The automata 0-1 law; V. A., 2009)

Given an automaton function $f = f_{\mathfrak{A}}$, either $\alpha(f) = 0$, or $\alpha(f) = 1$.

These alternatives correspond to the cases $\mathbf{P}(f)$ is nowhere dense in $\mathbb{I}^2$ and $\mathbf{P}(f) = \mathbb{I}^2$, respectively.

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Automata with a finite number of states are all of measure 0

Therefore if $\mathfrak{A}$ is a finite automaton then $\mathbf{P}(f_{\mathfrak{A}})$ is nowhere dense in $\mathbb{I}^2$ and thus $\mathbf{P}(f_{\mathfrak{A}})$ cannot contain 'figures', but it may contain 'lines'.

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Therefore if $\mathfrak{A}$ is a finite automaton then $\mathbf{P}(f_{\mathfrak{A}})$ is nowhere dense in $\mathbb{I}^2$ and thus $\mathbf{P}(f_{\mathfrak{A}})$ cannot contain ‘figures’, but it may contain ‘lines’.

We are going to describe these lines starting with the case p a prime.

Plots of finite affine automata (case $\beta = p$ a prime)

- Since $\beta = p$ a prime, an automaton function $f_{\mathfrak{A}}$ of the automaton $\mathfrak{A}$ is a 1-Lipschitz map from the space $\mathbb{Z}_p$ of p-adic integers into itself.
- The automaton $\mathfrak{A}$ is said to be affine if

$$f_{\mathfrak{A}} : z \mapsto az + b \quad (z \in \mathbb{Z}_p)$$

for suitable $a, b \in \mathbb{Z}_p$.

We will consider automata plots on unit torus $\mathbb{T}^2 \subset \mathbb{R}^3$ rather than on unit square $\mathbb{I}^2 \in \mathbb{R}^2$; i.e., we map unit square $\mathbb{I}^2$ on unit torus $\mathbb{T}^2$ in a standard way by 'gluing together' opposite sides of unit square.

Plots of finite affine automata (case $\beta = p$ a prime)

Given an automaton function is $f_{\mathfrak{A}}$, denote via $\mathbf{AP}(\mathfrak{A}) = \mathbf{AP}(f_{\mathfrak{A}})$ the set of all accumulation points of the plot $\mathbf{P}(\mathfrak{A}) \subset \mathbb{T}^2$ on unit torus $\mathbb{T}^2$.

Plots of finite affine automata

Let $\mathfrak{A}$ be a finite automaton, and let $f_{\mathfrak{A}}(z) = f(z) = az + b$ ($a, b \in \mathbb{Z}_p \cap \mathbb{Q}$ then). Considering $\mathbb{I}^2$ as a surface of the torus $\mathbb{T}^2$, we have that

$$\mathbf{AP}(f) = \{(x \bmod 1; (ax+b) \bmod 1) \in \mathbb{T}^2 : x \in \mathbb{R}\}$$

is a *link* of N_f *torus knots* either of which is a *cable(=winding)* with slope a of the unit torus $\mathbb{T}^2$: If $a = q/k, b = r/s$ are irreducible fractions, $d = \gcd(k, s)$ then N_f is multiplicative order of p modulo s/d. Each cable winds q times around the interior of $\mathbb{T}^2$ and k times around Z-axis.

Given $a, b \in \mathbb{Z}_p$, the mapping $z \mapsto az + b$ is an automaton function of a suitable finite automaton if and only if $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$.

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is a *link* of N_f *torus knots* either of which is a *cable*(=*winding*) with slope a of the unit torus $\mathbb{T}^2$: If $a = q/k, b = r/s$ are irreducible fractions, $d = \gcd(k, s)$ then N_f is multiplicative order of p modulo s/d. By using cylindrical coordinates ($\mathbb{T}^2 \colon (r_0 - R)^2 + z^2 = A^2$; $R = A = 1$ for unit torus) we get:

$$\begin{bmatrix} r_0 \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} R + A \cdot \cos\left(ax - 2\pi b \cdot p^{\ell}\right) \\ x \\ A \cdot \sin\left(ax - 2\pi b \cdot p^{\ell}\right) \end{bmatrix}, \; x \in \mathbb{R}, \ell = 0, 1, 2, \ldots$$

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is a *link* of N_f *torus knots* either of which is a *cable*(=*winding*) with slope a of the unit torus $\mathbb{T}^2$: If $a = q/k, b = r/s$ are irreducible fractions, $d = \gcd(k, s)$ then N_f is multiplicative order of p modulo s/d. Therefore the plot of f (=of the automaton $\mathfrak{A}$) can be described by N_f complex-valued functions:

$$\mathbf{AP}(\mathfrak{A}) = \mathbf{AP}(az + b) \longleftrightarrow e^{i(ax - 2\pi b \cdot p^{\ell})}; \; (x \in \mathbb{R}, \ell \in \mathbb{N}_0)$$

The affinity of smooth finitely computable functions

- **Q:** What *smooth* curves are finitely computable; i.e. what *smooth* curves may lie in the plot $\mathbf{P}(\mathfrak{A})$ of a finite automaton $\mathfrak{A}$?
- **A:** Only straight lines (=cables of torus with rational p-adic slopes and rational p-adic constant terms).

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Theorem (V.A., in pNUAA, 2015, vol. 7, No 3, pp. 169–227)

Given a finite automaton $\mathfrak{A}$, let g be a two times differentiable function (w.r.t. the metric in $\mathbb{R}$) *defined on $[\alpha,\beta]\subset[0,1)$ and valuated in $[0,1)$; let g'' be continuous on $[\alpha,\beta]$. If $(x;g(x))\in\mathbf{P}(\mathfrak{A})$ for all $x\in[\alpha,\beta]$ then there exist $a,b\in\mathbb{Z}_p\cap\mathbb{Q}$ such that $g(x)=ax+b$ for all $x\in[\alpha,\beta]$ and $\mathbf{AP}(az+b)\subset\mathbf{P}(\mathfrak{A})$. Moreover, there are no more than a finite number of $a,b\in\mathbb{Z}_p\cap\mathbb{Q}$ such that $\mathbf{AP}(az+b)\subset\mathbf{P}(\mathfrak{A})$.*

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Actually this means that smooth curves in the plot of a finite automaton constitute a finite union of torus links, and every link consists of a finite number of torus knots with the same slope.

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The theorem holds for automata with m inputs and n outputs: Smooth surfaces in the plot (in multidimensional torus) constitute a finite number of families of multidimensional torus windings, and each family is a finite collection of windings with the same matrix A.

$$\mathbf{AP}(\mathbf{z}A + \mathbf{b}) \longleftrightarrow e^{i(\mathbf{x}A - 2\pi \mathbf{b} \cdot p^{\ell})}; \ (\mathbf{x} \in \mathbb{R}^m; \mathbf{b} \in \mathbb{R}^n; \ell \in \mathbb{N}_0)$$

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Very recently it has been obtained a result which implies that the theorem remains true under a weaker restriction, for C^1-function g rather than for C^2-function g, see Corollary 6.8 in P. Hieronymi and E. Walsberg. *On continuous functions definable in expansions of the ordered real additive group.* (Preprint arXiv:1709.03150)

Automata modeling of quantum systems: What β?

Accumulation points of a plot of a finite affine automaton (whose automaton function is then $f\colon z \mapsto az + b$ for suitable $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) *look like* a finite collection of waves with the same wavenumber a (up to a normalization s.t. $\hbar = 1$), where x stands for position, $2\pi b$ for angular frequency ω and p^ℓ for time t.

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The “time-looking” multiplier p^ℓ is a *proper time* of the automaton.

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The "time-looking" multiplier p^ℓ is a *proper time* of the automaton. Namely, multiplying by p corresponds to one step of the automaton from the current state to a new one: $(p^\ell x) \bmod 1$ is an ℓ-step shift of base-p expansion of $x \in \mathbb{R}$.

Automata modeling of quantum systems: What β?

Accumulation points of a plot of a finite affine automaton (whose automaton function is then $f\colon z \mapsto az + b$ for suitable $a, b \in \mathbb{Z}_p \cap \mathbb{Q}$) *look like* a finite collection of waves with the same wavenumber a (up to a normalization s.t. $\hbar = 1$), where x stands for position, $2\pi b$ for angular frequency ω and p^ℓ for time t.

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Yes: Just use β-*expansions* (Rényi—Parry) rather than base p-expansions.

$$0.\alpha_1\alpha_2\alpha_3\ldots = \alpha_1\beta^{-1} + \alpha_2\beta^{-2} + \alpha_3\beta^{-3} + \cdots,$$

where $\beta > 1$, $\alpha_1, \alpha_2, \alpha_3, \ldots \in \{0, 1, \ldots, \lceil\beta\rceil - 1\}$.

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For instance, assume that τ is a *Planck time* (=a quant of time) or other time interval which is less then the accuracy of measurements and thus can not be measured.

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Then $p^{\ell} \approx 1+\ell\tau$ and therefore for large ℓ we see that $\ell\tau = t$ is just a time. And here we are:

$$e^{i(ax-2\pi b\cdot p^{\ell})} \approx e^{i(ax-2\pi b\cdot(1+t))} = c\cdot e^{i(ax-2\pi b\cdot t)} \leftarrow \text{the wave!!!}$$

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Note that $\beta = 1+\tau$ with $0<\tau\ll 1$ is the only case when we obtain a wavefunction of a particle. That is, automata over binary alphabet constitute the only class when interpretation of the multiplier p^{ℓ} as a physical time is possible (recall that the alphabet consists of $\lceil\beta\rceil$ symbols).

Automata models of quantum systems imply $\beta = \sqrt[N]{2}$

When constructing a plot of an automaton over a p-letter alphabet we take $\beta > 1$ such that $p = \lceil \beta \rceil$; then to every pair of input/output words

$$\text{input word } \chi_{k-1} \ldots \chi_1 \chi_0 \longrightarrow \text{output word } \xi_{k-1} \ldots \xi_1 \xi_0$$

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we put into the correspondence the following point on the torus:

$$(\chi_{k-1}\beta^{-1} + \cdots \chi_1 \beta^{-k+1} + \chi_0 \beta^{-k}; \xi_{k-1}\beta^{-1} \cdots + \xi_1 \beta^{-k+1} + \xi_0 \beta^{-k})$$

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The only such solution of the equation is $\beta = \sqrt[N]{2}$; i.e., it can be shown that *the above conditions necessarily imply that* $u(x) = x^N$ *and thus* $\beta = \sqrt[N]{2}$.

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Then

- necessarily the input/output alphabets are binary (as $\lceil\beta\rceil = 2$);
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Therefore our main theorem may serve a mathematical evidence in support of J. A. Wheeler's 'it from bit' doctrine since the theorem shows that a specific 'it' — the matter wave, which is a core of quantum theory — is indeed 'from bit'; that is, from sufficiently long binary inputs of an automaton with a relatively small number of states.

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Note that $1, \sqrt[N]{2}, \sqrt[N]{2^2}, \sqrt[N]{2^3}, \ldots$ are linearly independent over $\mathbb{Q}$ (thus, over $\mathbb{Q}_2$); so Numbers $1, \sqrt[N]{2}, \sqrt[N]{2^2}, \sqrt[N]{2^3}, \ldots, \sqrt[N]{2^{N-1}}$ constitute a basis of the module $\mathbb{Z}_2[\sqrt[N]{2}]$ of dimension N over the space of 2-adic integers $\mathbb{Z}_2$.

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Addition in base-$\sqrt[N]{2}$ system is just an addition of base-2 numbers with carry from i-th to $(i+N)$-th position (In example below $N = 2$):

$$
\begin{array}{rccccccccl}
 & \ldots 1 & 1 & 1 & 1 & 1 & 1 & 1 & = & -\sqrt{2} - 1 \\
+ & & & & & & & & & \\
 & \ldots 0 & 0 & 0 & 0 & 0 & 0 & 1 & = & 1 \\
\hline
 & \ldots 0 & 1 & 0 & 1 & 0 & 1 & 0 & = & -\sqrt{2}
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► Numbers $1, \sqrt[N]{2}, \sqrt[N]{2^2}, \sqrt[N]{2^3}, \ldots, \sqrt[N]{2^{N-1}}$ constitute a basis of the module $\mathbb{Z}_2[\sqrt[N]{2}]$ of dimension N over the space of 2-adic integers $\mathbb{Z}_2$. Addition in base-$\sqrt[N]{2}$ system is just an addition of base-2 numbers with carry from i-th to $(i+N)$-th position; hence An automaton which is affine and finite w.r.t. $\sqrt[N]{2}$-expansions is equivalent to binary affine finite automaton with N inputs and N outputs.

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► An automaton which is affine and finite w.r.t. $\sqrt[N]{2}$-expansions is equivalent to binary affine finite automaton with N inputs and N outputs. That is, the plot of the automaton represented via base-$\sqrt[N]{2}$ expansions may be regarded as a point set in $[0, 1/\tau] \times [0, 1/\tau]$ square in $\mathbb{R}^2$

$$(\sqrt[N]{2^{N-1}} A_{N-1} + \sqrt[N]{2^{N-2}} A_{N-2} + \cdots + \sqrt[N]{2} A_1 + A_0; \\ \sqrt[N]{2^{N-1}} B_{N-1} + \sqrt[N]{2^{N-2}} B_{N-2} + \cdots + \sqrt[N]{2} B_1 + B_0)$$

where $A_m, B_n \in [0, 1] \subset \mathbb{R}$ and the mapping $(A_{N-1}, A_{N-2}, \ldots A_1, A_0) \mapsto (B_{N-1}, B_{N-2}, \ldots B_1, B_0)$ is N-dimensional affine transformation $\mod 1$.

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This way to an automaton which is affine and finite w.r.t. $\sqrt[N]{2}$-representations, one may ascribe a 'wave packet' in the $[0, 1/\tau] \times [0, 1/\tau]$ real square (considered as a surface of a torus)

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Note that the smallest currently measured time interval is about 10^{-18} sec. That is why handling time variable as a real number is still possible: 25 orders of magnitude is a too long way to go to see that time is discrete rather than continuous.

Ontology vs Epistemology

- In the automata models, ‘states’ are epistemological rather than ontological: The ‘states’ cannot be observed (at least with the current equipment used in experiments).
- The ontological (observable) states of a quantum system, the pure states, correspond to minimal sub-automata of the automaton.
- Only minimal sub-automata are responsible for the complex functions $c \cdot e^{i(ax-2\pi b \cdot t)}$)
- Mixed states correspond to the automaton epistemological states which lead to more than one minimal sub-automata .
- Helicity (ontological) corresponds to the sign of a in the affine p-adic mapping $f(z) = az + b$.

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Final remarks and conclusions

- Discreteness+Causality+Finiteness $\Rightarrow$ Waves
- Waves may indeed originate from *bits*: 'Standard' wave packets can be produced by automata with binary input/output only.
- Finiteness is essential in the above reasoning; note that no 'truly infinite' physical phenomena are known.
- Usage of β-expansions with $\beta = 1 + \tau$ where $0 < \tau \ll 1$ is also essential:
- The model considered in the talk is a model of measurement in quantum physics rather than a model of evolution of quantum systems.
- The quantum phenomena which we observe has 'wave-like appearance' just because our perception is 'Archimedean' while the Nature seemingly is not: We apply Archimedean metrics to measure entities which are inherently non-Archimedean.

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  Infinity Is a Beautiful Concept — And Its Ruining Physics

  (Max Tegmark. Science Magazine, February 20, 2015)
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- Usage of β-expansions with $\beta = 1 + \tau$ where $0 < \tau \ll 1$ is also essential: It can be shown that when $\tau \to 0$, the 'wave packets' tend to 'bodies'; that is, 'quantum' picture tends to 'classical'.
- The model considered in the talk is a model of measurement in quantum physics rather than a model of evolution of quantum systems.
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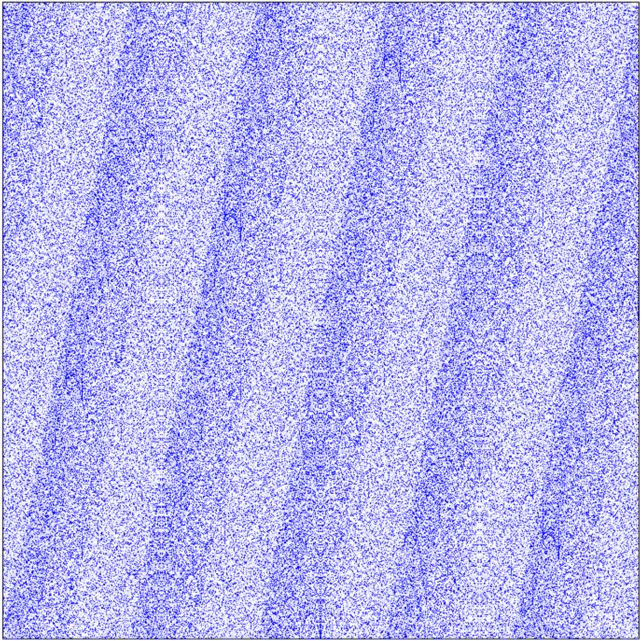
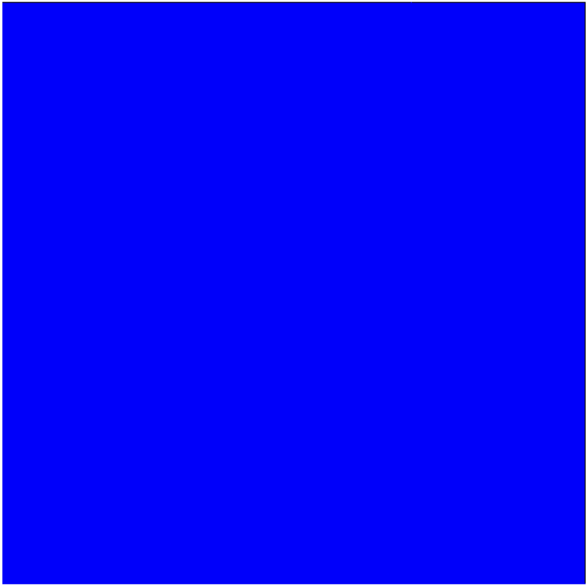
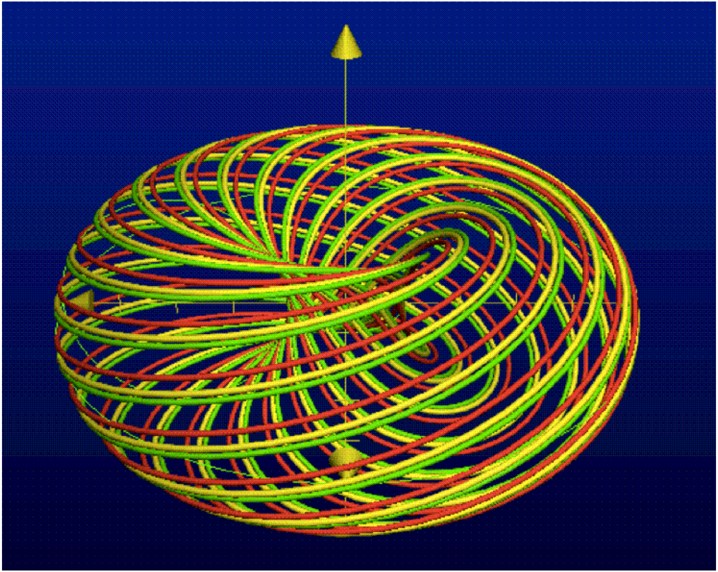
Thank you!

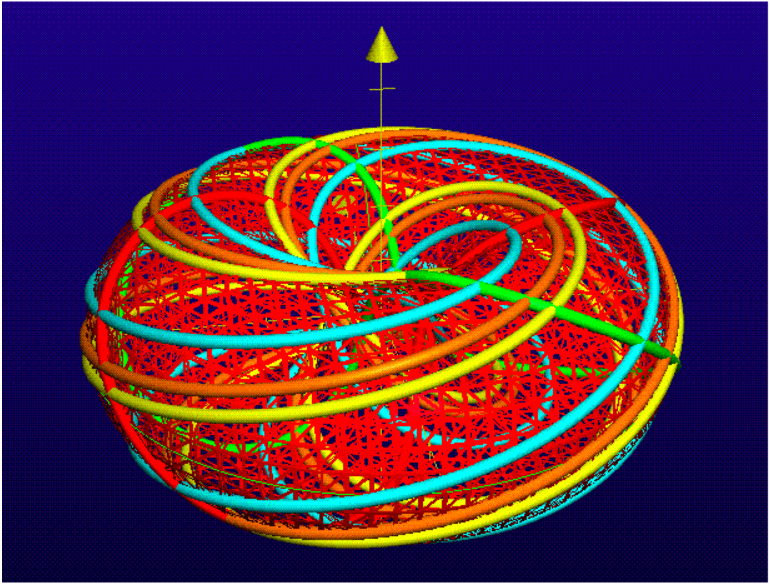




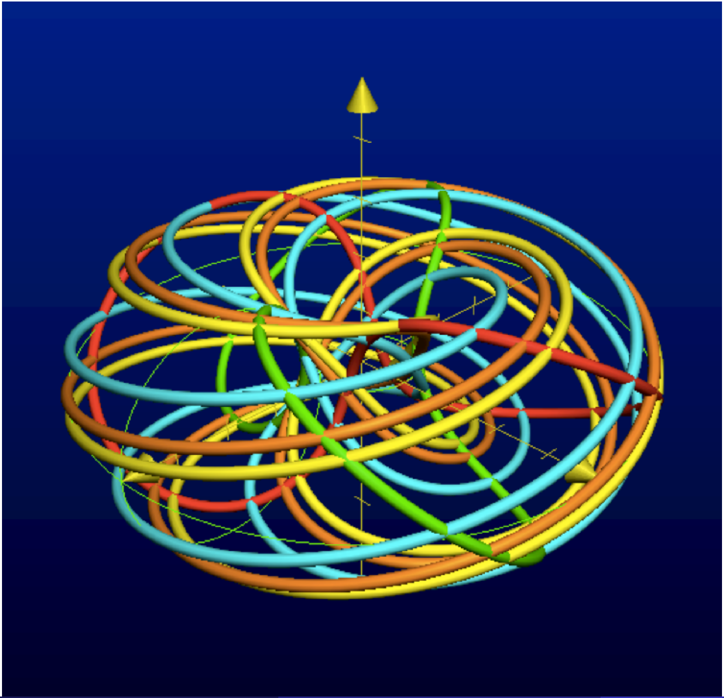
$p = 2$: $f(x) = 1 + x + 4x^2$; $\alpha(f) = 1$.

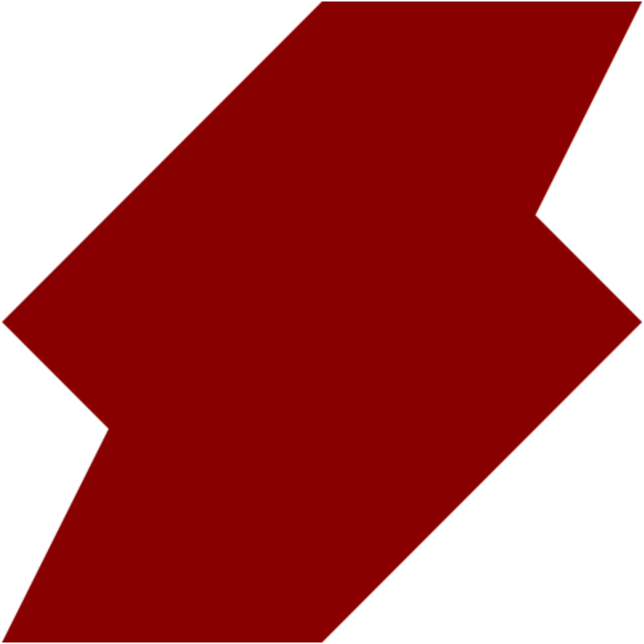


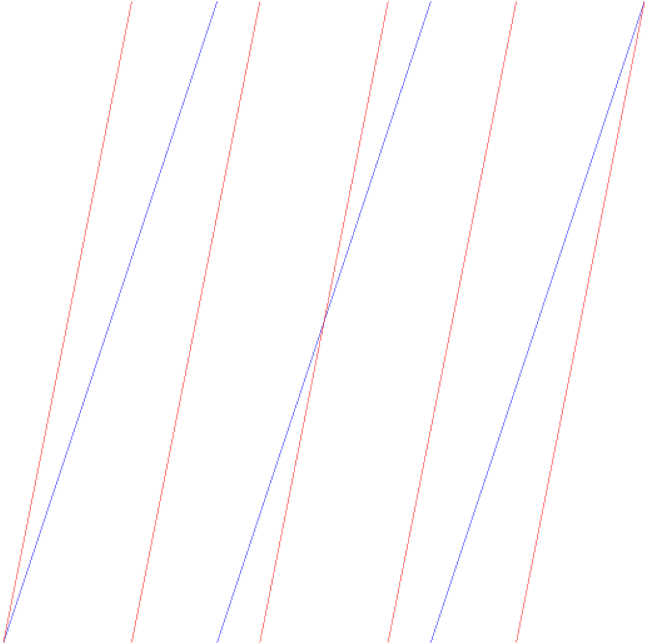
$f(z) = \frac{11}{15}z + \frac{1}{21}$, $p = 2$. (Therefore $N_f = \mathrm{mult}_7\, 2 = 3$)



$f_1(z) = -2z + \frac{1}{3}$; $f_2(z) = \frac{3}{5}z + \frac{2}{7}$, $(p = 2)$.







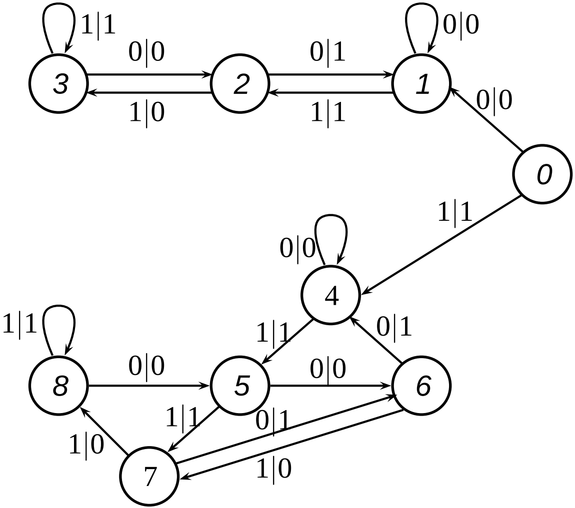
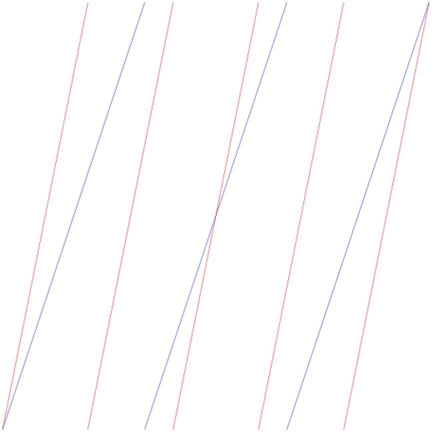


Figure: Example state diagram of an automaton with two minimal sub-automata (whose automata functions are $z \mapsto 3z$ and $z \mapsto 5z$,resp.) Initial state is 0

Tending β to 1.



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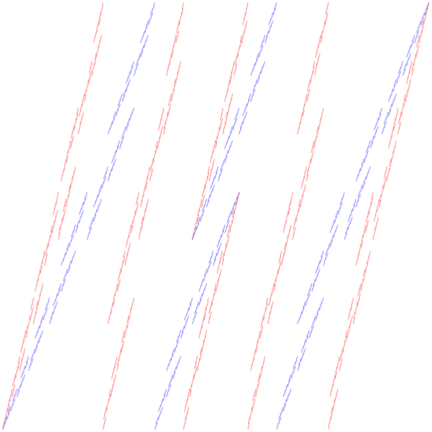

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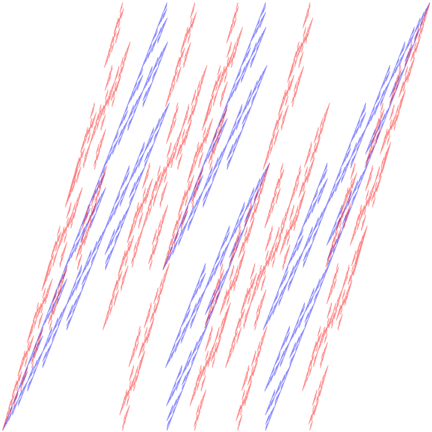

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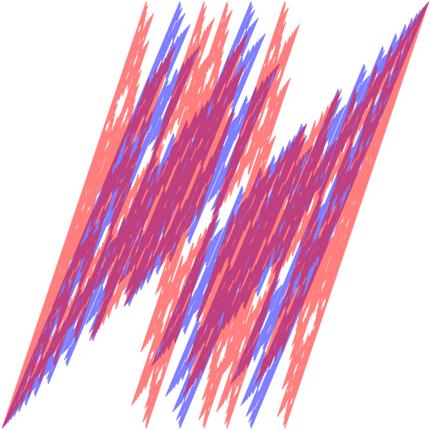

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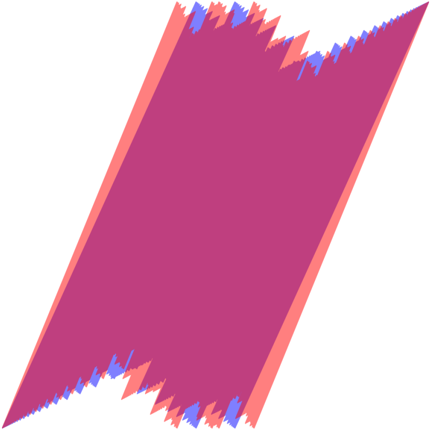

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