Path integrals on real, p-adic, and adelic spaces

Zoran Rakić

Faculty of Mathematics, University of Belgrade, Serbia

joint work with Branko Dragovich

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Formalisms of quantum mechanics:

- Heisenberg: $i\hbar\dfrac{d\hat{A}}{dt} = i\hbar\dfrac{\partial\hat{A}}{\partial t} + [\hat{A}, \hat{H}],$
- Schrödinger: $i\hbar\dfrac{\partial\Psi(x,t)}{\partial t} = H(\hat{k}, x)\Psi(x,t), \quad \hat{k} = -i\hbar\dfrac{\partial}{\partial x}.$
- Feynman: $\Psi(x'', t'') = \int \mathcal{K}(x'', t''; x', t')\,\Psi(x', t')dx'$

  where

  $$\mathcal{K}(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \exp\left(\frac{2\pi i}{h}\int_{t'}^{t''} L(\dot{q}, q, t)dt\right)\mathcal{D}q,$$

  and $\displaystyle\int_{t'}^{t''} L(\dot{q}, q, t)\,dt = S[q]$

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$$\Psi(t'') = U(t'', t')\Psi(t'). \tag{1}$$

$\mathcal{K}(x'', t''; x', t')$ is also called the probability amplitude.

▶ **The probability amplitude** $\mathcal{K}(x'', t''; x', t')$ plays cental role in the theory and has the following properties:

$$\int \mathcal{K}(x'', t''; x, t)\mathcal{K}(x, t; x', t')dx = \mathcal{K}(x'', t''; x', t'),$$

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▶ The Feynman's path integral method is appropriate for the generalizations to the p-adic case,

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where $\chi_p(a)$ is p-adic additive character.

- There is well defined Haar measure and integration, and we have

$$\int_{\mathbb{Q}_p} \chi_p(ayx)\, dx = \delta_p(ay) = |a|_p^{-1}\, \delta_p(y), \quad a \neq 0,$$

$$\int_{\mathbb{Q}_p} \chi_p(\alpha x^2 + \beta x)\, dx = \lambda_p(\alpha)\, |2\alpha|_p^{-\frac{1}{2}}\, \chi_p\left(-\frac{\beta^2}{4\alpha}\right), \quad \alpha \neq 0,$$

where $\delta_p(u)$ is the p-adic Dirac δ function.

- For $x = \sum_{k=m}^{+\infty} x_k p^k, \; x_k = 0, 1, \cdots, p-1, \; x_m \neq 0$, we define $\lambda_p(x)$:

$$\lambda_p(x) = \begin{cases} 1, & m = 2j, \quad p \neq 2, \\ \left(\frac{x_m}{p}\right), & m = 2j+1, \; p \equiv 1 (\mathrm{mod}\ 4), \\ i\left(\frac{x_m}{p}\right), & m = 2j+1, \; p \equiv 3 (\mathrm{mod}\ 4), \end{cases}$$

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$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1 + (-1)^{x_{m+1}} i], & m = 2j, \\ \frac{1}{\sqrt{2}}(-1)^{x_{m+1}+x_{m+2}}[1 + (-1)^{x_{m+1}} i], & m = 2j+1, \end{cases}$$

$\left(\frac{x_m}{p}\right)$ is the Legendre symbol,

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- The functions λ_p satisfies

$$\lambda_p(a^2 x) = \lambda_p(x), \quad \lambda_p(x)\lambda_p(-x) = 1,$$

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Definition 1

Let $\quad \Lambda_p(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} \lambda_p(x_i)$

where subscript $p = \infty, 2, 3, \cdots,$.

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$$\lambda_2(x) = \begin{cases} \frac{1}{\sqrt{2}}[1 + (-1)^{x_{m+1}} i], & m = 2j, \\ \frac{1}{\sqrt{2}}(-1)^{x_{m+1}+x_{m+2}}[1 + (-1)^{x_{m+1}} i], & m = 2j+1, \end{cases}$$

$\left(\frac{x_m}{p}\right)$ is the Legendre symbol,

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The functions $\Lambda_p(x_1\,, x_2\,, \cdots\,, x_n)$ *satisfy the properties*

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Proposition 2

Let $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ be column vectors, and let $B = (B_{kl})$ be a nonsingular $n \times n$ matrix, where $x_k,\ y_k,\ B_{kl} \in \mathbb{Q}_p$. Then

$$\int_{\mathbb{Q}_p^n} \chi_p(y^T B\, x)\, d^n x = |\det B|_p^{-1} \prod_{k=1}^{n} \delta_p(y_k)\,,$$

where y^T denotes transpose map of y.

Proposition 3

Let $x = (x_1, \cdots, x_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ be two column vectors, and let $A = (\alpha_{kl})$ be a nonsingular symmetric $n \times n$ matrix, where $x_k, \beta_k, \alpha_{kl} \in \mathbb{Q}_p$. Then

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Corollary 4

$$\Lambda_p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \begin{cases} \frac{1}{\sqrt{i^{n-1}}}\,\lambda_\infty(\det(\alpha_{kl})), & p = \infty, \\ \prod_{i<j\le n}(\alpha_i, \alpha_j)\,\lambda_p(\det(\alpha_{kl})), & p \neq 2\,. \end{cases}$$

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- An **adele** x is an infinite sequence

$$x = (x_\infty, x_2, \cdots, x_p, \cdots),$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set **S** of primes p one has $x_p \in \mathbb{Z}_p$, where $\mathbb{Z}_p = \{a \in \mathbb{Q}_p : |a|_p \leq 1\}$ is the ring of p-adic integers.

- **Properties of the set of all adels $\mathbb{A}$:**
  - $\mathbb{A}$ is a **ring** with componentwise addition and multiplication, and could be regarded as

$$\mathbb{A} = \bigcup_{\mathbf{S}} \mathbb{A}(\mathbf{S}), \qquad \mathbb{A}(\mathbf{S}) = \mathbb{R} \times \prod_{p \in \mathbf{S}} \mathbb{Q}_p \times \prod_{p \notin \mathbf{S}} \mathbb{Z}_p.$$

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22

where $\phi_\infty(x_\infty)$ is an infinitely differentiable function on $\mathbb{R}$ such that $|x_\infty|_\infty^n \phi_\infty(x_\infty) \to 0$ as $|x_\infty|_\infty \to \infty$ for any $n \in \{0, 1, 2, \cdots\}$, and $\phi_p(x_p)$ are locally constant functions with compact support,

- all finite linear combinations of elementary functions make the set $S(\mathbb{A})$ of the **Schwartz-Bruhat adelic functions**. The Fourier transform of $\phi(x) \in S(\mathbb{A})$, which maps $S(\mathbb{A})$ onto $\mathbb{A}$, is

$$\tilde{\phi}(y) = \int_{\mathbb{A}} \phi(x)\chi(xy)dx,$$

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$$(\Psi_1, \Psi_2) = \int_{\mathbb{A}} \bar{\Psi}_1(x)\Psi_2(x)dx$$

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- A **basis of** $L_2(\mathbb{A})$ may be given by the set of orthonormal eigefunctions in spectral problem of the evolution operator $U(t)$, where $t \in \mathbb{A}$. Such eigenfunctions have the form

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Quadratic Lagrangians

- General quadratic Lagrangian on $\mathbb{Q}_p^n$ has the form

$$L(\dot q, q, t) = \frac{1}{2}\,\dot q^T A\,\dot q + \dot q^T B\,q + \frac{1}{2}\,q^T C\,q + D^T\,\dot q + E^T\,q_k + \varepsilon \tag{3}$$

- Euler-Lagrange equations:

$$A\,\ddot q + (\dot A + B - B^T)\,\dot q + (\dot B - C)\,q = E - \dot D,$$ (24)

- The solution of the above system has the form:

$$q = x(t) = F(t)\,\mathcal{C} + \xi(t),$$

where $F(t) = [f_{km}(t)] \in M_{n,2n}$ is a solutions of the corresponding system of homogeneous differential equations, $\mathcal{C} = [C_m] \in M_{2n,1}$ is the vector of constants, and $\xi(t) = [\xi_k(t)] \in M_{n,1}$ is a particular solution of the system.

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where $F(t) = [f_{km}(t)] \in M_{n,2n}$ is a solutions of the corresponding system of homogeneous differential equations, $\mathcal{C} = [C_m] \in M_{2n,1}$ is the vector of constants, and $\xi(t) = [\xi_k(t)] \in M_{n,1}$ is a particular solution of the system.

- General quadratic Lagrangian on $\mathbb{Q}_p^n$ has the form

$$L(\dot{q}, q, t) = \frac{1}{2}\,\dot{q}^T A\,\dot{q} + \dot{q}^T B\,q + \frac{1}{2}\,q^T C\,q + D^T\,\dot{q} + E^T\,q_k + \varepsilon \qquad (3)$$

- Euler-Lagrange equations:

$$A\,\ddot{q} + (\dot{A} + B - B^T)\,\dot{q} + (\dot{B} - C)\,q = E - \dot{D},$$ (24)

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Quadratic Lagrangians

- General quadratic Lagrangian on $\mathbb{Q}_p^n$ has the form

$$L(\dot{q}, q, t) = \frac{1}{2}\, \dot{q}^T A\, \dot{q} + \dot{q}^T B\, q + \frac{1}{2}\, q^T C\, q + D^T\, \dot{q} + E^T\, q_k + \varepsilon \qquad (3)$$

- Euler-Lagrange equations:

$$A\, \ddot{q} + (\dot{A} + B - B^T)\, \dot{q} + (\dot{B} - C)\, q = E - \dot{D},$$

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- The solution of the above system has the form:

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- For the boundary conditions $x'_k = x_k(t')$ and $x''_k = x_k(t'')$, we denote by

$$f^i(t'',t') = [f_{1i}(t''),\dots,f_{ni}(t''),f_{1i}(t')\dots,f_{ni}(t')]^T,$$

$$\mathcal{F} = \mathcal{F}(t'',t') = \begin{bmatrix} F(t'') \\ F(t') \end{bmatrix} = \begin{bmatrix} F'' \\ F' \end{bmatrix} = [f_1(t''),\dots,f_n(t''),f_1(t'),\dots,f_n(t')]^T$$

where $[f_1(t''),..,f_n(t''),f_1(t'),..,f_n(t')]^T$ is a matrix with rows $f_1(t''),..,f_n(t')$

$$\triangle = \triangle(t'',t') = \det\mathcal{F}, \quad x\xi = [x''_1-\xi''_1,\dots,x''_n-\xi''_n,x'_1-\xi'_1,\dots,x'_n-\xi'_n]^T$$

$$\triangle_i = \triangle_i(t'',t') = \det[f^1(t'',t'),\dots,f^{i-1}(t'',t'),x\xi,f^{i+1}(t'',t')\dots,f^{2n}(t'',t')].$$

Proposition 5

Imposing the boundary conditions $x'_k = x_k(t')$ and $x''_k = x_k(t'')$, vector of constants of integration $\mathcal{C}$ become:

$$\mathcal{C} = \mathcal{C}(t'',t') = \frac{1}{\triangle(t'',t')}[\triangle_1(t'',t'),\triangle_2(t'',t'),\dots,\triangle_{2n}(t'',t')]^T$$

- For the boundary conditions $x_k' = x_k(t')$ and $x_k'' = x_k(t'')$, we denote by

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Lemma 7

The solution of the above system take the form

$$x_k(t) = \frac{1}{\triangle(t'',t')} \sum_{i=1}^{2n} \triangle_i(t'',t') f_{ki}(t) + \xi_k(t)\,, \qquad k = 1,2,\ldots,n\,.$$

where $\triangle(t'',t') = \det \mathcal{F}$, *and* $\triangle_i(t'',t')$ *has ordinary meaning.*

Theorem 8

Let $\{f_{1j}\;, j = 1,2,\ldots,2\,n\}$ *be any linearly independent solutions of the resolvent equation for* $x_1(t)$, *then solutions* $f_{km}(t)$ *for* $x_k(t)$, $k \neq 1$, *are determined by the system and the following equality holds*

$$\det \begin{bmatrix} F(t) \\ \dot{F}(t) \end{bmatrix} = \frac{\mathcal{D}}{\det A},$$

where $\mathcal{D}$ *is an non-zero constant, which could be chosen to be equal to 1.*

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Theorem 9

The general form of the action for classical trajectory $x(t)$ of a quadratic Lagrangian, for a particle being in point x' at the time t' and in position x'' at t'', is

$$\bar{S}(x'',t'';x',t') = \frac{1}{2}x''^T\bar{A}x'' + x''^T\bar{B}x' + \frac{1}{2}x'^T\bar{C}x + \bar{D}^Tx'' + \bar{E}^Tx' + \bar{\varepsilon},$$

where $\bar{A} = [\bar{A}_{kl}]$, $\bar{B} = [\bar{B}_{kl}]$, $\bar{C} = [\bar{C}_{kl}]$, $\bar{D} = [\bar{D}_k]$, *and* $\bar{E} = [\bar{E}_k]$

$$\bar{A}_{kl} = \bar{A}_{kl}(t'',t') = \frac{\partial^2\bar{S}_0}{\partial x''_k\partial x''_l}, \qquad \bar{B}_{kl} = \bar{B}_{kl}(t'',t') = \frac{\partial^2\bar{S}_0}{\partial x''_k\partial x'_l},$$

$$\bar{C}_{kl} = \bar{C}_{kl}(t'',t') = \frac{\partial^2\bar{S}_0}{\partial x'_k\partial x'_l}, \qquad \bar{D}_k = \bar{D}_k(t'',t') = \frac{\partial\bar{S}_0}{\partial x''},$$

$$\bar{E} = \bar{E}_k(t'',t') = \frac{\partial\bar{S}_0}{\partial x'}, \qquad \bar{\varepsilon} = \bar{\varepsilon}(t'',t') = \bar{S}_0.$$

and subscript $_0$ *in the classical action means that after performing derivatives of the* $\bar{S}(x'',t'';x',t')$ *one has to replace* x'' *and* x' *by* $x'' = x' = 0$.

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Theorem 10

The related coefficients from previous theorem are:

$$\bar{A}_{kl} = \bar{A}_{kl}(t'', t') = \frac{1}{2\triangle} \sum_{t=1}^{n} \left(\alpha''_{lt} \dot{\triangle}_k(f''_t) + \alpha''_{kt} \dot{\triangle}_l(f''_t) \right) + \frac{\beta''_{lk} + \beta''_{kl}}{2} ,$$

$$\bar{B}_{kl} = \bar{B}_{kl}(t'', t') = \frac{1}{2\triangle} \sum_{t=1}^{n} \left(\alpha''_{kt} \dot{\triangle}_{n+l}(f''_t) - \alpha'_{lt} \dot{\triangle}_k(f'_t) \right)$$

$$\bar{C}_{kl} = \bar{C}_{kl}(t'', t') = \frac{-1}{2\triangle} \sum_{t=1}^{n} \left(\alpha'_{lt} \dot{\triangle}_{n+k}(f'_t) + \alpha'_{kt} \dot{\triangle}_{n+l}(f'_t) \right) - \frac{\beta'_{lk} + \beta'_{kl}}{2} .$$

where $A(t'') = (\alpha''), A(t') = (\alpha'), B(t'') = (\beta''), B(t') = (\beta'), \triangle = \triangle(t'', t')$ *and*

$$\dot{\triangle}_i(f'_j)(t'', t') = \dot{\triangle}_i(f'_j) = \det[\, f''_1, .., f''_{i-1}, \dot{f}'_j, f''_{i+1}, .., f''_n, f'_1, .., f'_n\,] \,, i, j = 1, .., n,$$

$$\dot{\triangle}_{i+n}(f''_j)(t'', t') = \dot{\triangle}_{i+n}(f''_j) = \det[\, f''_1, .., f''_n, f'_1, .., f'_{i-1}, \dot{f}''_j, f'_{i+1}, .., f'_n], i, j = 1, .., n.$$

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- The corresponding Taylor expansion of the action functional $S[q]$ around classical path $x(t)$ is

$$S[q] = S[x+y] = S[x] + \delta S[x] + \frac{1}{2!}\,\delta^2 S[x] + \dots$$
$$= S[x] + \frac{1}{2}\int_{t'}^{t''}\left(\dot{y}_k\,\frac{\partial}{\partial \dot{q}_k} + y_k\,\frac{\partial}{\partial q_k}\right)^2 L(\dot{q},q,t)dt.$$

- From $\delta S[x] = 0$, for any $p = \infty, 2, 3, \cdots$, we can write

$$\mathcal{K}_p(x'',t'';x',t') = \int \chi_p\left(-\frac{1}{h}\,S[x+y]\right)\mathcal{D}y.$$

- Then the following formula holds

$$\mathcal{K}_p(x'',t'';x',t') = \chi_p\left(-\frac{1}{h}\,\bar{S}(x'',t'';x',t')\right)$$
$$\times \int_{y'\to 0,t'}^{y''\to 0,t''} \chi_p\left(-\frac{1}{2h}\int_{t'}^{t''}\left(\dot{y}_k\,\frac{\partial}{\partial \dot{q}_k} + y_k\,\frac{\partial}{\partial q_k}\right)^2 L(\dot{q},q,t)dt\right)\mathcal{D}y,$$

where we used $y'' = y' = 0, \quad S[x] = \bar{S}(x'',t'';x',t')$.

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where we used $y'' = y' = 0, \quad S[x] = \bar{S}(x'', t''; x', t')$.

Theorem 11

(i1) $\mathcal{K}_p(x'', t''; x', t')$ *has the form*

$$\mathcal{K}_p(x'', t''; x', t') = N_p(t'', t')\,\chi_p\left(-\frac{1}{h}\,\bar{S}(x'', t''; x', t')\right),$$

where $N_p(t'', t')$ *does not depend on end points* x'' *and* x'.

(i2)

$$|N_p(t'', t')|_\infty = \left|\frac{1}{h^n}\det\frac{\partial^2}{\partial x_k''\partial x_l'}\bar{S}_0(x'', t''; x', t')\right|_p^{\frac{1}{2}} = \left|\det\left(\frac{1}{h}\bar{B}(t'', t')\right)\right|_p^{\frac{1}{2}}.$$

- We have now

$$N_p(t'', t') = \left|\det\left(\frac{1}{h}\frac{\partial^2\bar{S}_0(x'', t''; x', t')}{\partial x_k''\partial x_l'}\right)\right|_p^{\frac{1}{2}}\mathcal{A}_p(t'', t'), \qquad (4)$$

where $|\mathcal{A}_p(t'', t')|_\infty = 1$ and $\mathcal{A}_p(t'', t')$ remains to be determined explicitly.

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- We use

$$\int_{\mathbb{Q}_p^n} \mathcal{K}_p(y'',t'';y,t)\mathcal{K}_p(y,t;y',t')\, d^n y = \mathcal{K}_p(y'',t'';y',t').$$

to obtain

Theorem 12

The following relation holds

$$N_p(t'',t') = N_p(t'',t)\, N_p(t,t')\, \Lambda_p(\alpha_1,\alpha_2,\ldots,\alpha_n)\, |\det(2H)|_p^{-\frac{1}{2}},$$

where $\alpha_1,\alpha_2,\cdots,\alpha_n$ *are eigenvalues of matrix* $H = \frac{\bar{C}(t'',t)+\bar{A}(t,t')}{-2h}$.

- We use

$$\int_{\mathbb{Q}_p^n} \mathcal{K}_p(y'', t''; y, t)\mathcal{K}_p(y, t; y', t')\, d^n y = \mathcal{K}_p(y'', t''; y', t').$$

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The following relation holds

$$N_p(t'', t') = N_p(t'', t)\, N_p(t, t')\, \Lambda_p(\alpha_1, \alpha_2, \ldots, \alpha_n)\, |\det(2H)|_p^{-\frac{1}{2}},$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are eigenvalues of matrix* $H = \frac{\bar{C}(t'', t) + \bar{A}(t, t')}{-2h}$.

- We use

$$\int_{\mathbb{Q}_p^n} \mathcal{K}_p(y'', t''; y, t)\mathcal{K}_p(y, t; y', t')\, d^n y = \mathcal{K}_p(y'', t''; y', t').$$

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$$U = (U_{ij}), \qquad U_{ij} = \frac{1}{2}\,\frac{\dot{\triangle}_i(f_j)(t,t')}{\triangle(t,t')} - \frac{1}{2}\,\frac{\dot{\triangle}_{n+i}(f_j)(t'',t)}{\triangle(t'',t)}\,,$$

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where α^i is $i-$th column of matrix A, and U_j is $j-$th row of matrix U. Then we obtain,

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The following relations holds,

(i1) $\det\mathcal{H} = \det A\,\det 2\,U\,,$

(i2) $\det 2\,U = (-1)^n \dfrac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')}\,\det\begin{bmatrix} F(t) \\ \dot{F}(t)\end{bmatrix},$

(i3) $\det\mathcal{H} = (-1)^n\dfrac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')}\,,$ *and* $\det 2H = \dfrac{1}{h^n}\,\dfrac{\triangle(t'',t')}{\triangle(t'',t)\,\triangle(t,t')}$

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The p-adic kernel $\mathcal{K}_p(x'',t'';x',t')$ of the unitary evolution operator and evaluated as the Feynman path integral, for quadratic Lagrangians has the form

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▶ Adelic path integral can be introduced as a generalization of ordinary and p-adic path integrals. As adelic analogue it is related to eigenfunctions in adelic quantum mechanics in the form

$$\psi_{\mathbf{S},\alpha}(x'',t'') = \int_{\mathbb{A}} \mathcal{K}_{\mathbb{A}}(x'',t'';x',t')\psi_{\mathbf{S},\alpha}(x',t')dx',$$

where $\psi_{\mathbf{S},\alpha}(x,t)$ has the form (2) and adelic propagator $\mathcal{K}_{\mathbb{A}}(x'',t'';x',t')$ does not depend on $\mathbf{S}$.

▶ Above equation must be valid for any set $\mathbf{S}$ of primes p, and adelic eigenstate is an infinite product of real and p-adic eigenfunctions, it is natural to consider adelic propagator in the following form:

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \mathcal{K}_{\infty}(x''_{\infty},t''_{\infty};x'_{\infty},t'_{\infty}) \prod_{p} \mathcal{K}_{p}(x''_{p},t''_{p};x'_{p},t'_{p}), \tag{5}$$

where $\mathcal{K}_{\infty}(x''_{\infty},t''_{\infty};x'_{\infty},t'_{\infty})$ and $\mathcal{K}_{p}(x''_{p},t''_{p};x'_{p},t'_{p})$ are propagators in ordinary and p-adic quantum mechanics, respectively.

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- From (5) we see that one can introduce adelic path integral as an infinite product of ordinary and p-adic path integrals for all primes p, and (5) one can rewrite as

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_{\mathbb{A}}\left(-\frac{1}{h}S_{\mathbb{A}}[q]\right)\mathcal{D}_{\mathbb{A}}q, \quad (6)$$

where $\chi_{\mathbb{A}}(x)$ is adelic additive character, $S_{\mathbb{A}}[q]$ and $\mathcal{D}_{\mathbb{A}}q$ are adelic action, and the Haar measure, respectively.

- For practical considerations, we define adelic path integral in the form

$$\mathcal{K}_{\mathbb{A}}(x'',t'';x',t') = \prod_p \int_{x_p',t_p'}^{x_p'',t_p''} \chi_p\left(-\frac{1}{h}\int_{t_p'}^{t_p''} L(\dot{q}_p,q_p,t_p)dt_p\right)\mathcal{D}q_p, \quad (7)$$

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- Adelic Lagrangian is the infinite sequence

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where $|L(\dot{q}_p, q_p, t_p)|_p \leq 1$ for all primes p but a finite set **S** of them.

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- Note that vacuum state $\Omega(|x_p|_p)$ transforms as

$$\Omega(|x_p''|_p) = \int_{\mathbf{Q}_p} \mathcal{K}_p(x_p'', t_p''; x_p', t_p')\Omega(|x_p'|_p)dx_p' = \int_{\mathbf{Z}_p} \mathcal{K}_p(x_p'', t_p''; x_p', t_p')dx_p'. \qquad (10)$$

- As a consequence of (10) one has

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Concluding remarks

- In this work we derived general expressions for propagators $\mathcal{K}(x'', t''; x', t')$ in ordinary, p-adic and adelic quantum mechanics for Lagrangians $L(\dot{q}, q, t)$ which are polynomials at most the second degree in dynamical variables $\dot{q}_k$ and q_k, where $k = 1, 2, \cdots, n$.
- The formalism of ordinary and p-adic path integrals can be regarded as the same at different levels of evaluation, and the obtained results have the same form.
- In fact, this property of number field invariance has to be natural for general mathematical methods in physics and fundamental physical laws (see Volovich, *Number theory as the ultimate physical theory*).

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- In fact, this property of number field invariance has to be natural for general mathematical methods in physics and fundamental physical laws (see Volovich, *Number theory as the ultimate physical theory*).

Concluding remarks

- In this work we derived general expressions for propagators $\mathcal{K}(x'', t''; x', t')$ in ordinary, p-adic and adelic quantum mechanics for Lagrangians $L(\dot{q}, q, t)$ which are polynomials at most the second degree in dynamical variables $\dot{q}_k$ and q_k, where $k = 1, 2, \cdots, n$.
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THANK YOU FOR

YOUR ATTENTION !!!