Representation theorem for operators on Free Banach spaces of countable type

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Theorem

Let L be a field with an arquimedean valuation $|\cdot|^{\sim}$. Then, If $\mathbb{L}$ contains the complex numbers field $\mathbb{C}$, then $|\cdot|^{\sim}_{|\mathbb{C}} \neq |\cdot|$ (here, $|\cdot|$ is the absolute valued on $\mathbb{C}$) unless $\mathbb{C} = L$.

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- Therefore, the counterpart of the Gelfand-Mazur Theorem, which asserts that any commutative Banach algebra such that each nonzero element is invertible, then it is isometrically isomorphic to the complex field, doen´t work in this context.
- As a consequence of this fact, one finds a bounded linear operators with empty spectrum (if we define the spectrum in the same way as in the classical situation). For example, if $\alpha \in \mathbb{L} \setminus \mathbb{K}$, then

$$M_\alpha : \mathbb{L} \to \mathbb{L}; \quad l \to \alpha l$$

is such an operator (here, $\mathbb{L}$ is a Banach space over $\mathbb{K}$).

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- As in any kind of operator theory, a central problem is a construction and study the spectral decomposition. In the non-archimedean case, there are several results in this direction.
- The main difficulties here are the absence of nontrivial involution and absence of inner products coordinated with the norms of the Banach spaces.
  Thus, the non-archimedean linear algebra is quite different from the classical one.

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Theorem

The spectrum $Sp(\mathcal{A})$ is nonempty, compact Hausdorff space.

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- A commutative Banach algebra $\mathcal{A}$ with unity is called a C-algebra if there exists a zero-dimensional compact Hausdorff space X such that

$$\mathcal{A} \cong C(X, \mathbb{K}) \text{ (or simply, } C(X)\text{)}.$$

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- This zero-dimensional compact Hausdorff space X is unique up to homeomorphism.

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$$\mathcal{L}_T = \overline{\mathbb{K}[T]}.$$

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- If $\mathcal{L}_T$ satisfies the condition of the theorem, we will say that $\mathcal{L}_T$ is a $C-$Algebra.

Preliminaries

- If $\mathbb{K} = \mathbb{Q}_p$ and
$$T = \begin{pmatrix} p & p & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
then $\mathcal{L}_T$ is not a $C-$algebras. In this case, there exist a polinomial $q \in \mathbb{Q}_p[t]$ such that $\left\| q(T)^2 \right\| \neq \| q(T) \|^2$.

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Definition

A commutative Banach algebra $\mathcal{A}$ is called a C-algebra if there exists a locally compact zero-dimensional Hausdorff space X such that $\mathcal{A}$ is isometrically isomorphic to $C_\infty(X)$, where $C_\infty(X)$ is the space of all continuous functions from X into $\mathbb{K}$ which vanishes at infinity.

Free Banach spaces (Diarra, B)

A non-archimedean Banach space E is said to be Free Banach space if there exists a family $\{e_i\}_{i\in J}$ of non-null vectors of E such that any element x of E can be written in the form of convergent sum

$$x = \sum_{i\in J} x_i e_i, \;\; x_i \in \mathbb{K} \text{ and } \lim_{i\in I} |x_i| \, \|e_i\| = 0$$

and

$$\|x\| = \sup_{i\in J} |x_i| \, \|e_i\| \, .$$

The family $\{e_i\}_{i\in J}$ is called orthogonal basis of E.

Free Banach spaces

Now, any linear continuous operator $u : E \longrightarrow E$ can be represented in a unique way as: for $x = \sum_{i \in I} x_i e_i$

$$u(x) = \sum_{(i,j) \in I \times I} \alpha_{ij} e'_j \otimes e_i(x)\,; \text{ with } \lim_{j \in I} |\alpha_{ij}| \left\| e'_j \otimes e_i(x) \right\| = 0$$

where $\alpha_{ij} \in K$ and $e'_j \otimes e_i(x) = e'_j(x)\, e_i$ and $e'_j(x) = x_j$.
At the same time, each continuous linear operator $u : E \longrightarrow E$ is in correspondence with a infinite matrix

$$A_u = (\alpha_{ij})_{(i,j) \in I \times I}$$

On the other hand, the operator norm of u is very well know as

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|},$$

but, for Free Banach spaces, it can be shown that

$$\|u\| = \sup_{i \in I} \frac{\|u(e_i)\|}{\|e_i\|}$$

Free Banach spaces

If $s : J \longrightarrow (0, \infty)$, then an example of Free Banach space is

$$c_0 (J, \mathbb{K}, s) ,$$

the collection of all $x = (x_i)_{i \in J}$ such that for any $\epsilon > 0$, the set $\{i \in J : |x_i| \, s(i) > \epsilon\}$ is, at most, finite and

$$\|x\| = \sup_{i \in J} |x_i| \, s(i) .$$

We already know that a Free Banach space E is isometrically isomorphic to $c_0 (J, \mathbb{K}, s)$, for some $s : J \longrightarrow (0, \infty)$.

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- In this case, $s : \mathbb{N} \to (0, \infty)$ is precisely the constant function 1 in the previous slice.
- Now, our goal of this talk is to present certain class of non-archimedean fields for which $\mathcal{L}_T$ is a $C-$algebra.

Preliminaries

- We will denote by $\mathbb{k}$ the residue class field of a non-archimedean field $\mathbb{K}$.

  A field F is called formally real if

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$$x \neq \theta \Rightarrow \langle x, x \rangle \neq 0$$
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$$|\langle x, y \rangle| \leq \|x\|_\infty \|y\|_\infty$$

- And it may happen that

$$|\langle x, x \rangle| < \|x\|_\infty^2$$

for some $x \in c_0$.

Theorem (Narici & Beckenstain)

This symetric bilinear form on c_0 is an inner product which induces the original norm if and only if the residue class field of $\mathbb{K}$ is formally real. Moreover,

$$|\langle x, x\rangle| = \|x\|_\infty^2$$

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Theorem (Soler)

Let E be an orthomodular infinite dimensional space and suppose it contains an orthogonal sequence $e_1, e_2, \cdots$ (in the sense of an inner product). Then, the base field is

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- Therefore, we don´t have infinite dimensional Hilbert spaces in the non-archimedean context.

Algebras

- If we define in c_0 the operation

$$\lambda \cdot \mu = (\lambda_i \mu_i)_{i \in \mathbb{N}}\,; \quad \lambda, \mu \in c_0,$$

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- As we know, ${c_0}^+ := \mathbb{K} \oplus c_0$ with the operations

$$(\alpha, \lambda) + (\beta, \mu) = (\alpha + \beta, \lambda + \mu)$$

$$(\alpha, \lambda) \cdot (\beta, \mu) = (\alpha\beta, \beta\lambda + \alpha\mu + \lambda \cdot \mu)\,,$$

and with

$$\|(\alpha, \lambda)\| = \max\{|\alpha|\,, \|\lambda\|\}\,,$$

is a commutative Banach algebra with unity $(1, \theta)$.

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Example

c_0 and ${c_0}^+$ are C-algebras.

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$$T : c_0 \longrightarrow c_0; \quad Tx = \left(\sum_{n=1}^{\infty} x_n \right) e_1$$

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$$T : c_0 \longrightarrow c_0; \quad Tx = \left(\sum_{n=1}^{\infty} x_n \right) e_1$$

- It is possible to prove that a continuous linear operator $T : c_0 \longrightarrow c_0$ admits an adjoint if and only if, for each $y \in c_0$,

$$\lim_{n \to \infty} \langle Te_n, y \rangle = 0.$$

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    1. T is compact
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- that is, for each pair $n, m \in \mathbb{N}$, $n \neq m$,

$$\langle y^n, y^m \rangle = 0 \text{ and } \|y^n\|_\infty = 1.$$

Orthogonal projections and compact operators

- For any $j \in \mathbb{N}$, we define the operator $P_j : c_0 \longrightarrow c_0$ by

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- it is easy to check that $P_j \in \mathcal{L}(c_0)$ and it is an orthonormal projection, which means,

$$P_k \circ P_j = \begin{cases} \theta & if \quad k \neq j \\ P_j & if \quad k = j \end{cases} \; ; \quad \|P_j\| = 1$$

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$$P_k \circ P_j = \begin{cases} \theta & if \quad k \neq j \\ P_j & if \quad k = j \end{cases} \; ; \quad \|P_j\| = 1$$

- On the other hand, for any $\lambda \in c_0$, the operator T_λ defined by

$$T_\lambda(\cdot) = \sum_{j \in \mathbb{N}} \lambda_j P_j(\cdot)$$

is a compact and self-adjoint operator and

$$\|T_\lambda\| = \|\lambda\|_\infty.$$

C-algebras

- Let us consider the collection

$$\mathcal{S}_{\mathfrak{J}} = \{\alpha Id + T_\lambda \in \mathcal{L}(c_0) : \alpha \in \mathbb{K},\ \lambda \in c_0\}$$

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Now, if $\alpha Id + T_\lambda \in \mathcal{S}_{\mathfrak{J}}$, then $\|\alpha Id + T_\lambda\| = \max\{|\alpha|, \|T_\lambda\|\}$.

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- As a consequence,

$$\|\alpha Id + T_\lambda\| = \sup_{j\in\mathbb{N}} \|(\alpha Id + T_\lambda)(e_j)\|$$
$$= |\alpha| = \max\{|\alpha|, \|T_\lambda\|\}$$

C-algebras

Theorem

The algebras c_0^+ and $\mathcal{S}_{\mathfrak{J}}$ are isometrically isomorphic and, as a consequence, $\mathcal{S}_{\mathfrak{J}}$ is a C-algebra.

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Proof.

The linear mapping $\Lambda : c_0^+ \to \mathcal{S}_{\mathfrak{J}}$, defined by

$$\Lambda(\alpha, \lambda) = \alpha Id + T_\lambda,$$

is an isomorphism of algebras. Now, since $\|\lambda\|_\infty = \|T_\lambda\|$, we have

$$\|(\alpha, \lambda)\| = \max\{|\alpha|, \|\lambda\|_\infty\} = \max\{|\alpha|, \|T_\lambda\|\} = \|\alpha Id + T_\lambda\|.$$

□

Compact and self-adjoint operator

- Let us fix an element $T = T_\lambda \in \mathcal{S}_{\mathfrak{J}}$, where $\lambda = (\lambda_n) \in c_0$ and

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- By properties of C-algebras, $\mathcal{L}_T$ is a C-algebra since it is closed Banach subalgebra of $\mathcal{S}_{\mathfrak{J}}$.
- On the other hand, since $\mathcal{S}_{\mathfrak{J}}$ has the power multiplicative norm property, $\mathcal{L}_T$ inherits such property.
- Under the conditions that $\mathcal{L}_T$ is a **power multiplicative C-algebra** and $Sp(\mathcal{L}_T)$ is compact, we conclude that $\mathcal{L}_T$ is isometrically isomorphic to the space of all continuous functions $C(Sp(\mathcal{L}_T))$ provided by the supremum norm, that is, there exists an isomorphism of algebras
$$\Psi : \mathcal{L}_T \to C(Sp(\mathcal{L}_T))$$
such that, for all $H \in \mathcal{L}_T$, $\|H\| = \|\Psi(H)\|_\infty$.

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- Let us consider the unique homomorphism of algebra

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- We claim that $\{\phi_n : n \in \mathbb{N} \cup \{0\}\}$ can be identified with

$$Sp(\mathcal{L}_T).$$

Compact and self-adjoint operator

- In fact, the next function is well-defined and is injective:

$$\Gamma : \sigma\left(T\right) \rightarrow Sp\left(\mathcal{L}_T\right);\ \lambda_n \longmapsto \Gamma\left(\lambda_n\right) = \phi_n$$

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Proposition

If $z \notin \sigma(T)$, then $zId - T$ is invertible in $\mathcal{S}_{\mathfrak{J}}$.

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Proof.

For $y \in Range(zId - T)$, there exists $x \in c_0$ such that $(zId - T)(x) = y$
Since $z \notin \sigma(T)$, we can solve the above equation for x and get

$$x = \frac{1}{z}y + \frac{1}{z}Tx = \frac{1}{z}y + \frac{1}{z}\sum_{n=1}^{\infty}\lambda_i \frac{\left\langle x, y^{(n)}\right\rangle}{\left\langle y^{(n)}, y^{(n)}\right\rangle} y^{(n)}$$

Applying the continuous functional $\left\langle \cdot, y^{(k)}\right\rangle$ to x, we get the sequence

Compact and self-adjoint operator

Corollary

If $z \notin \sigma(T)$, then $(zId - T)^{-1} \in \mathcal{L}_T$.

Proof.

We already know that $\mathcal{S}_{\mathfrak{J}}$ is a C-algebra with unity and $zId - T$ is invertible in $\mathcal{S}_{\mathfrak{J}}$. By a theorem of the general theory, we conclude

$$(zId - T)^{-1} \in \overline{\mathbb{K}[zId - T]} \subset \mathcal{L}_T.$$

□

Compact and self-adjoint operator

Proposition

The function Γ is bijective.

Proof.

By above, Γ is injective. If $\phi \in Sp\left(\mathcal{L}_T\right)$, then $\phi\left(T\right) = z$, for some $z \in \mathbb{K}$. Suppose that $z \notin \sigma\left(T\right)$, hence $zId - T$ has an inverse and, by the previous corollary, $\left(zId - T\right)^{-1} \in \mathcal{L}_T$. Since the function ϕ is a homomorphism between algebras with unities, we have

$$1 = \phi\left(Id\right) = \phi\left(\left(zId - T\right)^{-1} \circ \left(zId - T\right)\right) = \phi\left(\left(zId - T\right)^{-1}\right)\phi\left(zId - T\right)$$

but, by the linearity of ϕ, the factor $\phi\left(zId - T\right)$ is null, which is a contradiction. Thus, if $\phi \in Sp\left(\mathcal{L}_T\right)$. □

Compact and self-adjoint operator

Remark

We have already identified $Sp(\mathcal{L}_T)$ with $\sigma(T)$ through the bijective function Γ. Let us consider the induced topology by $\mathbb{K}$ on $\sigma(T)$. Note that $\sigma(T)$ is compact.

Proposition

$\sigma(T)$ is homeomorphic to $Sp(\mathcal{L}_T)$

Proof.

It follows from the fact that

$$\Gamma : \sigma(T) \to Sp(\mathcal{L}_T);\ \lambda_n \longmapsto \Gamma(\lambda_n) = \phi_n$$

is a homeomorphism. □

Compact and self-adjoint operator

- By this proposition and by the uniqueness of X (up to homeomorphism) for which $\mathcal{L}_T \cong C(X)$, we have

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- Since

$$\|f_T\|_\infty = \sup\{|\lambda_n| : n \in \mathbb{N}\} = \|T\|,$$

we define the homomorphism of algebra

$$G : \mathbb{K}[T] \to C(\sigma(T))$$

by $G(T) = f_T$

Isometric Isomorphism

- Clearly, this G sends $\{Id, T, T^2, T^3, \ldots\}$ into $\{1, f_T, f_T^2, f_T^3, \ldots\}$ and it is an isometry.

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- Now, by the compactness of $\sigma(T)$ and a result of the general theory guarantees that $\{1, f_T, f_T^2, f_T^3, \ldots\}$ is also dense in $C(\sigma(T))$.
- $$G : \mathcal{L}_T \to C(\sigma(T)); \quad H \to G(H) = G_H$$
is the isometry isomorphism which is well-known as The Gelfand Transformation.

Boolean ring

Let us denote by $\Omega(\sigma(T))$ the Boolean ring of all clopen subsets of $\sigma(T)$. Of course, η_C is continuous if and only if $C \in \Omega(\sigma(T))$.
Note that the elements of $\Omega(\sigma(T))$ can be classified as follows: the first type are those which are finite subsets of $\sigma(T) \setminus \{0\}$ and the second type are those which are complement on $\sigma(T)$ of the first type.
Denoting by $\Phi = G^{-1} : C(\sigma(T)) \to \mathcal{L}_T$, we have

$$\Phi(\eta_C) = \Phi(\eta_C^2) = \Phi(\eta_C)^2 .$$

In other words, $\Phi(\eta_C)$ is a projection in $\mathcal{L}_T$ and if $C \in \Omega(\sigma(T)) \setminus \{\varnothing\}$, then $\Phi(\eta_C)$ is a non-null.

Density

On the other hand, by the fact that the linear hull of $\{\eta_C : C \in \Omega(\sigma(T))\}$ is dense in $C(\sigma(T))$, for any fixed $f \in C(\sigma(T))$ and $\epsilon > 0$, there exists a finite clopen partition $\{C_k : k = 1, \ldots, s\}$ of $\sigma(T)$ and a finite collection of scalars $\{\alpha_k : k = 1, \ldots, s\}$ such that

$$\left\| f - \sum_{k=1}^{s} \alpha_k \eta_{C_k} \right\|_\infty = \sup_{x \in \sigma(T)} \left| f(x) - \sum_{k=1}^{s} \alpha_k \eta_{C_k}(x) \right| < \epsilon$$

Since $\{C_k : k = 1, \ldots, s\}$ is a clopen partition of $\sigma(T)$, only one of these sets is of second type, say $C_1 = \sigma(T) \setminus \{\lambda_{m_1}, \lambda_{m_2}, \ldots, \lambda_{m_n}\}$.

From this, $\cup_{k=2}^{s} C_k = \{\lambda_{m_1}, \lambda_{m_2}, \ldots, \lambda_{m_n}\}$. Using the characteristic functions properties and the fact that the single subsets $\{\lambda_k\}$ belong to $\Omega(\sigma(T))$, we can rewrite as follows:

$$\left\| f - \left[\alpha_1 \eta_{\sigma(T)} + \sum_{l=1}^{n} (\alpha_l - \alpha_1) \eta_{\{\lambda_{m_l}\}} \right] \right\|_\infty$$
$$= \sup_{x \in \sigma(T)} \left| f(x) - \left[f(\lambda_{n_0}) \eta_{\sigma(T)}(x) + \sum_{l=1}^{n} [f(\lambda_{m_l}) - f(\lambda_{n_0})] \eta_{\{\lambda_{m_l}\}}(x) \right] \right|$$
$$< \epsilon$$

Isometric Isomorphism

Using the isometry Φ, we have

$$\left\| \Phi(f) - \left[f(\lambda_{n_0}) Id + \sum_{l=1}^{n} \left[f(\lambda_{m_l}) - f(\lambda_{n_0}) \right] E_l \right] \right\| < \epsilon,$$

where E_l is the corresponding projection $\Phi\left(\eta_{\{\lambda_{m_l}\}}\right)$. At the same time, shows that the space generated by $\left\{ E \in \mathcal{L}_T : E^2 = E \right\}$ is dense in $\mathcal{L}_T$.

Spectral measure

Let us consider the following set-function:

$$m_T : \Omega(\sigma(T)) \to \mathcal{L}_T; \quad C \longmapsto m_T(C) = \Phi(\eta_C) = E_C.$$

m_T is a finite additive measure valued-projection which is known as spectral measure associated to $\mathcal{L}_T$.

Integral

For $f \in C(\sigma(T))$ and $\alpha = \{C_1, C_2, \ldots, C_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}$, where $\sigma(T) = \sqcup_{k=1}^{n} C_k$, we define

$$\omega_\alpha(f, m_T, \sigma(T)) = \sum_{k=1}^{n} f(x_k)\, m_T(C_k) = \sum_{k=1}^{n} f(x_k)\, E_{C_k}.$$

and since the function f can be reached by a net

$$\left\{ \sum_{k=1}^{n} f(x_k)\, \eta_{C_k} \right\}_{\alpha \in \mathcal{D}}$$

in $C(\sigma(T))$, the isometry of Φ allows us to get

$$\lim_{\alpha \in \mathcal{D}} \omega_\alpha(f, m_T, \sigma(T)) = \Phi(f)$$

Integral

Therefore, the operator $\Phi(f)$ is interpreted as an integral, that is,

$$\Phi(f) = \int_{\sigma(T)} f dm_T = \lim_{\alpha \in \mathcal{D}} \omega_\alpha(f, m_T, \sigma(T))$$

Integral

For example, for f_T, $\eta_{\{\lambda_n\}}$ or $f \equiv 1$, their respective integral are

$$T = \Phi\left(f_T\right) = \int_{\sigma(T)} f_T \, dm_T;$$

$$E_n = \int_{\sigma(T)} \eta_{\{\lambda_n\}} dm_T$$

$$Id = \Phi\left(1\right) = \int_{\sigma(T)} dm_T.$$

Remark

If we suppose that, for $\lambda, \mu \in c_0$, the corresponding sets $\{\lambda_n : n \in \mathbb{N}\}$ and $\{\mu_n : n \in \mathbb{N}\}$ are infinite, then $\sigma\left(T_\lambda\right)$ and $\sigma\left(T_\mu\right)$ are homeomorphic and therefore we conclude that $\mathcal{L}_{T_\lambda} \cong \mathcal{L}_{T_\mu}$.

Scalar measures and matrix representation

- For any pair $x, y \in c_0$, we define $m_{x,y} : \Omega(\sigma(T)) \to \mathbb{K}$ by

$$m_{x,y}(C) = \langle m(C)x, y \rangle .$$

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- Clearly, $m_{x,y}$ is a scalar measure
- Since the inner product $\langle \cdot, \cdot \rangle$ is continuous, we have

$$\Phi_{x,y}(f) = \langle \Phi(f)\, x, y \rangle$$

and since

$$\lim_{\alpha \in \mathcal{D}} \omega_\alpha(f, m_T, \sigma(T)) = \Phi(f)$$

we conclude that

$$\int_{\sigma(T)} f dm_{x,y} = \left\langle \left(\int_{\sigma(T)} f dm \right)(x), y \right\rangle .$$

Scalar measures and matrix representation

- A particular, when $x = e_i$ and $y = e_j$, the measure m_{e_i,e_j} will be denoted by m_{ij} and

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- Let us denote by $\mathcal{M}$ the space of all infinite matrices of the form $(\Phi_{ij}(f))_{i,j\in\mathbb{N}}$, i.e.,

$$\mathcal{M} = \left\{ A(f) = (\Phi_{ij}(f))_{i,j\in\mathbb{N}} : f \in C((T)) \right\} .$$

- A particular, when $x = e_i$ and $y = e_j$, the measure m_{e_i,e_j} will be denoted by m_{ij} and

$$\Phi_{ij}(f) = \int_{\sigma(T)} fdm_{ij} = \left\langle \left(\int_{\sigma(T)} fdm \right) (e_i), e_j \right\rangle .$$

- Let us denote by $\mathcal{M}$ the space of all infinite matrices of the form $(\Phi_{ij}(f))_{i,j \in \mathbb{N}}$, i.e.,

$$\mathcal{M} = \left\{ A(f) = (\Phi_{ij}(f))_{i,j \in \mathbb{N}} : f \in C((T)) \right\} .$$

- $\mathcal{M}$ is a Banach space with the norm

$$\|A(f)\| = \sup_{i,j \in \mathbb{N}} |\Phi_{ij}(f)|$$

Scalar measures and matrix representation

- Since

$$\sup_{i,j\in\mathbb{N}} |\Phi_{ij}(f)| = \sup_{i,j\in\mathbb{N}} |\langle H(e_i), e_j\rangle| = \|H\| = \|f\|,$$

we get that

$$\|A(f)\| = \|H\|.$$

Theorem

Each operator in $\mathcal{L}_T$ is represented as a matrix whose entries are integrals defined by scalar measures.

Scalar measures and matrix representation

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- Therefore,

$$\mathcal{M} \cong \mathcal{L}_T$$

Theorem

Each operator in $\mathcal{L}_T$ is represented as a matrix whose entries are integrals defined by scalar measures.

J. Aguayo, M. Nova and K. Shamseddine, *Inner product on B*-algebras of operators on a free Banach space over the Levi-Civita field*, Indag. Math. (N.S.) **26** (2015), no. 1, 191-205.

J. Aguayo and M. Nova, *Compact and Self-adjoint operators on Free Banach Spaces of countable type*, Advances in non-Archimedean analysis, 1-17, Contemp. Math., **665**, Amer. Math. Soc., Providence, RI, 2016.

V. Berkovich, *Spectral Theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monograph, **33** AMS (1990).

B. Diarra, *Bounded linear operators on ultrametric Hilbert spaces*, Afr. Diaspora J. Math., **8** (2009), no. 2, 173-181.

A. K. Katsaras, *On p-adic vector measure space*, Journal of Mathematical Analysis and Applications, **365** (2010), no. 1, 342-357.

A. Khochubei, Non-Archimedean normal operator, Journal of Mathematical Physics, 51, http://dx.doi.org/10.1063/1.3293980 (2010).

L. Narici and E. Beckenstein, *A non-Archimedean Inner Product*, Contemp. Math., **384**, 187-202, Amer. Math. Soc., 2005.

A. van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York, 1978.

M. Vishik, *Non-Archimedean spectral Theory*, J. Soviet Math., **30** (1985), 2513-2554.

Thank you

Gracias