p-adic approaches to discretizing holography

Ingmar A. Saberi

Mathematisches Institut der Universität Heidelberg

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This talk is based on work, both completed and ongoing, with many collaborators:

Steve Gubser
Matt Heydeman
Christian Jepsen
Matilde Marcolli
Sarthak Parikh
Bogdan Stoica
Brian Trundy
Przemek Witaszczyk
Menelaos Zidikis

I’m going to try and give a broad picture of what we’ve been thinking about, and fit the (few) answered and (many) unanswered questions into a larger story.

In particular, I’d like to explain why (at least from our perspective) it was natural to begin thinking about the p-adics.

To do this, it helps to start at the beginning, with a question that is almost too large to be meaningful:

What is a quantum field theory?

What is a quantum field theory?

"Lots and lots of harmonic oscillators, coupled together anharmonically, but not too strongly."

—A. M. Polyakov (light paraphrase)

At root, a quantum field theory is a theory of

- many identical local degrees of freedom,
- parameterized by a geometric space,
- coupled together in a local and homogeneous way.

There are lots of additional possible ingredients, but these are the key ones. (This is why spin systems can often be described by field theories, at least in some range of parameters.)

More technical (incomplete) working definition:

A QFT is a quantum theory whose degrees of freedom are functions* on some underlying space X.

These functions represent measurements (observables) that can be made independently everywhere in X.

The interactions of the theory are encoded in an *action functional:*

$$S : \mathscr{F}(X) \to \mathbb{R}.$$

(At this point, X could be anything: a manifold, a lattice, a graph, a set. . .)

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* connections, tensor fields, sections of other bundles, . . .

A central theme in recent interactions between physics and pure mathematics:

How is geometric and topological information about X reflected in the behavior of theories on X?

A somewhat ahistorical example:

X is a smooth four-manifold; the theory is a “twist” of $\mathscr{N} = 2$ supersymmetric SU(2) gauge theory.

The theory leads one to consider the moduli space $\mathscr{M}(X)$ of anti-self-dual instantons on X.

Donaldson:[†] Crude topological invariants of $\mathscr{M}(X)$ are *sophisticated* invariants of the smooth structure of X itself!

[†] Donaldson, *J. Diff. Geom.* 18.2 (1983); Witten, *Comm. Math. Phys.* 117.3 (1988)

Can one carry information across this bridge in reverse?

One thing physicists would really like to understand is how the classical geometry of X is encoded in (e.g.) the Hilbert space of a theory, or the entanglement structure of states therein.

An idea for how to start with this: Vary X, so that different amounts of structure are present in different ways.

What kinds of structures on X are important?

To fit the definition of a field theory, it's probably a basic requirement to be able to make sense of *locality*, *homogeneity*, and *isotropy*.

— *Locality:*
X has a notion of *distance*, *measure*, or *causal structure,* which is respected by the interactions in the theory.

Often, this means something like

$$S[\phi] = \int_X \mathscr{L}[\phi(x)].$$

But locality might be relaxed, to mean that interactions between separated points decay sufficiently fast with distance.

— *Symmetry:*

X may have "spacetime" symmetries (implemented by the action of a group G on X). The theory may or may not respect the action of these symmetries on the fields $\mathscr{F}(X)$.

Often, some G acts transitively on X. The theory is homogeneous when it respects the action of such a spacetime symmetry.

The basic physical example: $X =$ *affine* $\mathbb{R}^n$

- *translations:* Affine $\mathbb{R}^n$ is an $\mathbb{R}^n$-torsor.
- *(Lorentz) rotations.* $\mathrm{SO}(n)$ acts on $\mathbb{R}^n$, preserving the metric. The signature is usually $(0,n)$ or $(1,n-1)$.

Poincaré-invariant theories always preserve these two symmetries. But there are others, which may or may not be preserved in interesting ways:

- *discrete symmetries* (P, T, et cetera. . .)
- *scale invariance.* (Broken scale invariance is renormalization group theory).
- *conformal invariance,* or local scale invariance. Scale-invariant theories are usually conformal.

— *One last piece of structure:*

If X has a notion of (mutually commuting) translation symmetries, I might further ask that there is a complete basis of eigenfunctions $\phi_k \in \mathscr{F}(X)$, diagonalizing those translations. Here k takes values in the joint spectrum of the translation operators, which I'll denote $X^\vee$.

This amounts to saying that there is a notion of mode expansion, or equivalently, of the Fourier transform.

On $\mathbb{R}^n$, $X = X^\vee$, but this isn't necessarily true: in lattice models, for example, $\mathbb{Z}^\vee = S^1$ (the "Brillouin zone").

I might also ask for a notion of “size” on $X^\vee$ (generalizing the length of a vector).

Once I have this, together with a notion of measure, translation symmetry, and a mode expansion, I have enough to write down a free theory of a real field on X:

$$S[\phi] = \int_{X^\vee} \phi(-k)\left(|k|^2 + m^2\right)\phi(k) + \cdots$$

And once I can do this, I’m really in familiar territory...

Key point: The more of this structure X has, the more a theory on it looks like your favorite typical QFT.

Most of these structures on $\mathbb{R}^n$ exist because it's an affine space over a field.

At least as far as algebraic structures are concerned, affine spaces over fields all behave similarly. We can simply replace one field by another. . .

Hierarchical (or p-adic) models can be thought of as replacing the real line by an analogue, that has even more powerful geometric and algebraic structures—at the expense of its one-dimensional structures (ordering, path connectedness, . . .)[‡]

[‡] T. Tao, "Dyadic models," *What's new* (July 27, 2007).

$\mathbb{Q}_p$ has many of the structures I catalogued before:

- There is an obvious translation symmetry on the affine space $\mathbb{Q}_p^n$.
- There are scaling symmetries as well.
- There is a unique translation-invariant integration measure dx on $\mathbb{Q}_p$ (additive Haar measure).
- The space of well-behaved (locally constant) functions on $\mathbb{Q}_p$ is spanned by eigenfunctions of translation, which take the form

$$\chi_p(kx) = e^{2\pi i \{kx\}_p}.$$

- $\mathbb{Q}_p$ is Fourier-self-dual: $k \in \mathbb{Q}_p^\vee \cong \mathbb{Q}_p$.
- There's a notion of size, namely $|\cdot|_p$.

So I can start to write down free and interacting field theories on $X = \mathbb{Q}_p$—and I'll do some of this later. If I were just interested in motivating the study of such field theories, I could stop here.

But what about conformal transformations?

How (and why) did AdS/CFT enter the story?

What is AdS/CFT?[§]

[§] Maldacena, in *AIP Conf. Proc.* CONF-981170, 484.1 (1999); Witten, *ATMP* 2 (1998); Gubser, Klebanov, & Polyakov, *Phys. Lett. B* 428.1 (1998)

Two ways to make $\mathrm{Conf}(n)$*-invariant Euclidean theories:*

- Pick $X = S^n$ (or $\mathbb{R}^n$), and look for conformal theories: fixed points of renormalization group flow.
- Pick $X = H^{n+1}$ (hyperbolic space one dimension higher), and take any field theory!

Isometries of H^{n+1} (analogues of Poincaré symmetry) are given by the group $G = \mathrm{Conf}(n) = \mathrm{SO}(n+1,1)$.

In fact, $S^n = \partial H^{n+1}$, and a metric on H induces a conformal structure on its boundary. . .

The AdS/CFT correspondence states that (certain) conformally invariant theories on S^n are *equivalent* to gravity theories on H^{n+1}.

- Fields in the bulk correspond to operators in the boundary theory.
- Both transform in representations of the same group.
- In particular, the mass of the bulk field corresponds to the conformal dimension of the boundary operator.

A precise ansatz for the relationship was given by Witten:

$$\left\langle \exp \int \phi_0 \mathscr{O} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{bulk}}(\phi_0) \cong \exp\left(-S_{\mathrm{cl}}(\phi_0)\right).$$

So a crucial ingredient is the ability to solve for the classical solution extending a given boundary field configuration ϕ_0—in other words, a solution to the Dirichlet problem. Such a solution is usually expressed in terms of a Green's function.

In the half-space model of H^{n+1}, i.e. $\mathbb{R}_+ \times \mathbb{R}^n$ with metric

$$ds^2 = \frac{1}{x_0^2} \sum_i dx_i^2,$$

such a Green's function is given by

$$K(x) = \frac{x_0^n}{(x_0^2 + x_1^2 + \cdots + x_n^2)^n}.$$

From it, one can extract the two-point function of the corresponding operator; it has dimension n. More generally, the asymptotics of the Green's function determine the scaling dimension of the operator:

$$\Delta = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right).$$

In low dimensions, the whole story of conformal invariance can be formulated algebraically:¶

$$S^2 = P^1(\mathbb{C}), \quad H^3 = \mathrm{SL}(2,\mathbb{C})/K \quad (K = \mathrm{SU}(2)),$$

$$\mathrm{Conf}(2) \cong \mathrm{PGL}(2,\mathbb{C})$$

$$S^1 = P^1(\mathbb{R}), \quad H^2 = \mathrm{SL}(2,\mathbb{R})/K \quad (K = \mathrm{SO}(2)),$$

$$\mathrm{Conf}(1) \cong \mathrm{PGL}(2,\mathbb{R})$$

¶Manin & Marcolli, *ATMP* 5 (2001)

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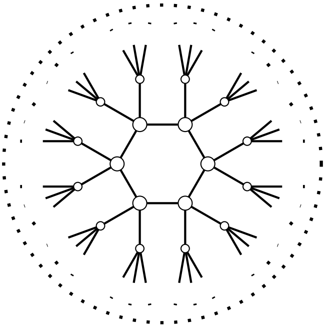
$$\mathrm{Conf}(1) \cong \mathrm{PGL}(2,\mathbb{R})$$

$$\partial T_p = P^1(\mathbb{Q}_p), \quad T_p = \mathrm{SL}(2,\mathbb{Q}_p)/K \quad (K = \mathrm{SL}(2,\mathbb{Z}_p)),$$

$$\mathrm{Conf}_p \cong \mathrm{PGL}(2,\mathbb{Q}_p)$$

¶Manin & Marcolli, *ATMP* 5 (2001)

One can even construct “black holes” in the same way, as quotients of this geometry by certain free (Schottky) subgroups:



The p-adic BTZ black hole (pictured for $p = 3$).

So it's off to the races:

One can now try to understand the simplest instances of holography: for instance, free bulk scalar fields propagating without backreaction, just as in Witten's paper.[✠]

All of the necessary facts—compatible group actions, a bulk Klein-Gordon equation with a well-posed Dirichlet problem at infinity—are probably well-known to the audience!

✠ Heydeman, Marcolli, IAS, and Stoica, `arXiv:1605.07639`; see also Gubser, Knaute, Parikh, Samberg, and Witaszczyk, *CMP* 352.3 (2017)

The Klein–Gordon equation:[ℵ]

$$\triangle\phi = \sum_{v'\sim v} \big(\phi(v') - \phi(v)\big) = m^2\phi.$$

Solutions analogous to $K(x)$:

$$\phi_\kappa(v) = p^{\kappa\langle v,x\rangle}.$$

Here $\langle v,x\rangle$ is the distance from v to the boundary point x, regularized to be zero at the (arbitrary) center vertex C.

[ℵ]For instance, Zabrodin, *CMP* 123.3 (1989).

The corresponding mass eigenvalue is

$$m_\kappa^2 = p^\kappa + p^{1-\kappa} - (p+1).$$

Thus, the BF bound is $m_\kappa^2 \geq -(\sqrt{p}-1)^2$.

Just as in the normal case, there are solutions in the bulk whose mass-squared is negative (but bounded from below)!

These solutions provide a bulk/boundary Green's function:

$$\phi(v) = \frac{p}{p+1} \int_{\mathbb{Q}_p} d\mu_0(x)\, \phi_0(x) p^{\langle v,x \rangle}.$$

Bulk fields of mass m_κ couple to boundary operators of conformal dimension κ:♠

$$\langle \mathcal{O}_\kappa(x) \mathcal{O}_\kappa(y) \rangle \sim \frac{1}{|x-y|_p^{2\kappa}}.$$

♠For p-adic CFT, see e.g. Melzer, *Int. J. Mod. Phys. A* 4.18 (1989)

Some intuitive features of ordinary AdS/CFT—for example, that the radial coordinate represents a scale in the boundary theory—are present even more sharply in the p-adic case.

For instance, if the boundary field ϕ_0 is a single mode (additive character of $\mathbb{Q}_p$), it stops contributing to the reconstruction of bulk physics abruptly, at a height determined by its wavelength!

Cool; what should one do next?

- We have some entries in the dictionary, but we don't have a precise pair of dual theories. (In the normal setting, the original construction of such pairs was motivated by string theory.)
- To that end, it would be good to understand more about possible boundary field theory models.
- Also, what about gravity? That's a key ingredient in the normal setting.

In the remainder of the talk, I'll talk about some small steps toward answering these questions.

As I tried to emphasize earlier, p-adic analogues of familiar field theory models (such as the $O(N)$ model) can often be defined straightforwardly.

One interesting feature: Computations in these theories sometimes exhibit universal answers, independent of which place the theory is defined at!

As an example, leading-order anomalous dimensions in $O(N)$[Ж] for the operators ϕ and ϕ^2:

$$\gamma_{\phi,\phi^2} = \mathrm{Res}_{\delta=0} g_{\phi,\phi^2}(\delta) + O(1/N^2)$$

where the functions g_{ϕ,ϕ^2} are given by

$$g_\phi(\delta) = \frac{1}{N} \frac{B(n-s,\delta-s)}{B(n-s,n-s)},$$

$$g_{\phi^2}(\delta) = -\frac{2}{N} \frac{B(n-s,\delta-s)}{B(n-s,n-s)} + \frac{1}{N} \frac{B(\delta,\delta)}{B(n-s,n-s)} \left(2 \frac{B(n-s,n-2s)}{B(n-s,n-s)} - 1 \right).$$

[Ж] Gubser, Jepsen, Parikh, & Trundy, `arXiv:1703.04202`

These results apply equally well in every case, assuming the special functions involved are defined uniformly!

Let the *local zeta function* be defined following Tate's thesis as

$$\zeta_p(s) = \frac{1}{1-p^{-s}}, \quad \zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2).$$

Then gamma and beta functions are defined for $\mathbb{R}^n$ or $\mathbb{Q}_p^n$ by the relations

$$\Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(n-s)}, \quad B_p(t_1,t_2) = \frac{\Gamma_p(t_1)\Gamma_p(t_2)}{\Gamma_p(t_1+t_2)},$$

where p is a prime or ∞.

We also considered models of interacting fermions,[♣] inspired by recent work connecting SYK and other related models to AdS_2/CFT_1.

To define fermionic models on the p-adic line, one needs to:

- use Grassmann field variables;
- replace the symmetric quadratic form (propagator) in the kinetic term with an antisymmetric one.

[♣] Gubser, Heydeman, Jepsen, Parikh, IAS, Stoica, & Trundy, `arXiv:1707.01087`

In the real case, the propagator of a free fermion is

$$G(k) \sim \frac{1}{k} = \frac{\mathrm{sgn}(k)}{|k|}.$$

So, by analogy, one can antisymmetrize the propagator by using a quadratic multiplicative character of the field.

It's also possible to antisymmetrize on flavor indices.

This leads one to the following class of actions for theories of interacting fermions:

$$S_{\text{free}} = \int d\omega \, \frac{1}{2} \phi^{abc}(-\omega) \, |\omega|_p^s \, \mathrm{sgn}(\omega) \, \phi^{abc}(\omega)$$

$$S_{\text{int}} = \int dt \, \phi^{abc} \phi^{ab'c'} \phi^{a'bc'} \phi^{a'b'c}$$

Here, the field is either commuting or anticommuting; pairs of flavor indices are contracted either with δ or with a fixed antisymmetric matrix Ω; and the sign character may be either "even" or "odd," meaning that $\mathrm{sgn}(-1) = \pm 1$.

For the kinetic term to be nontrivial, we must have that $\sigma_\psi \sigma_\Omega = \mathrm{sgn}(-1)$. In fact, exactly one specific collection of these choices leads to consistent behavior in the IR for each X!

There's also a parameter s, controlling the order of derivative appearing in the kinetic term; to ensure that the field has positive scaling dimension and that the interaction term we write is relevant, we ask that

$$\frac{1}{2} < s \leq 1.$$

Just like the ordinary Klebanov–Tarnopolsky model (and other models of SYK type), this theory is dominated in the large-N limit by the "melon" diagrams.

The melon diagrams can be resummed into an exact Schwinger–Dyson equation, determining the two-point function in the interacting large-N theory.

In the limit of large N and weak coupling, with g^2N^3 fixed, this Schwinger–Dyson equation is

$$G = F + \sigma_\Omega(g^2N^3)\,G \star G^3 \star F.$$

G is the interacting, and F the free, two-point function.

Solve in the IR to obtain universal limiting behavior:

$$G(t) = b\frac{\operatorname{sgn}(t)}{|t|^{1/2}}, \quad |t| \gg (g^2N^3)^{1/(2-4s)}$$

where

$$\frac{1}{b^4g^2N^3} = -\sigma_\Omega\Gamma(\pi_{-1/2,\mathrm{sgn}})\Gamma(\pi_{1/2,\mathrm{sgn}}).$$

Scaling in the IR limit is completely independent of the spectral parameter of the UV theory!

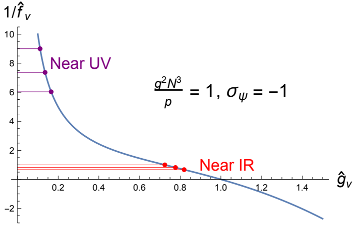
For fermionic theories with “direction-dependent” characters, one can do even better: it is possible to explicitly solve the Schwinger-Dyson equation for behavior at all scales, interpolating between the UV and the (universal) IR!

If $F(t) = f(|t|)\,\mathrm{sgn}(t)$ (and similarly for G), then

$$g = f + \sigma_\psi \frac{g^2 N^3}{p} |t|^2 g^4 f,$$

and this quartic can be explicitly solved for g when $\sigma_\psi = -1$.

Fermionic theories have a well-defined two-point function at all scales; the IR limit of theories with bosonic fields is problematic!



What about gravity?

Well, the only obvious “metric” data in the bulk space are the edge lengths. Unlike an actual metric, they don’t have any obvious gauge freedom (since they are geodesic lengths and hence physical), but they do couple naturally to matter through the Laplacian.

So one could try allowing them to fluctuate. Nonconstant configurations break the group action on the tree, but of course this is analogous to the normal situation.

When one does this,[ℷ] what kind of dynamics (action functional) should one choose?

A plausible action comes from a notion of *Ricci curvature* for graphs.[♡]

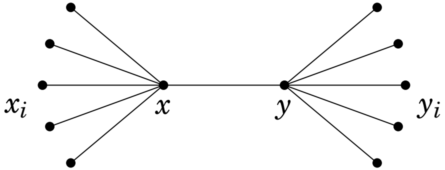
It's defined in a global fashion, by computing the rate of change in the average (Wasserstein) distance between two separated heat kernels at $t = 0$.

[ℷ] Gubser, Heydeman, Jepsen, Marcolli, Parikh, IAS, Stoica, & Trundy, JHEP 06 (2017) 157, `arXiv:1612.09580`

[♡] Lin, Lu, & Yau, *Tohoku Math. J. 2nd ser.* 63.4 (2011); see also Ollivier

On a tree-like graph, it reduces to the following function on edges of the graph:

$$\kappa_{xy} = \frac{b_{xy}}{d_x}\left(b_{xy} - \sum b_{xx_i}\right) + \frac{b_{xy}}{d_y}\left(b_{xy} - \sum b_{yy_i}\right)$$
$$\equiv \kappa_{x\to y} + \kappa_{y\to x}.$$



Here b denotes an inverse edge length, and $d_x = \sum_{x'\sim x} b^2_{xx'}$.

As one might expect, the Bruhat–Tits tree has constant negative curvature:

$$\kappa_{xy} = -2\frac{q-1}{q+1}.$$

So it seems reasonable to use κ to write an analogue of the Einstein–Hilbert term.

The resulting action for (e.g.) a scalar minimally coupled to gravity:

$$S = \sum_e \left((\kappa_e - 2\Lambda) + \frac{b_e^2}{2} |\delta_e \phi|^2 \right) + \sum_v \frac{1}{2} m^2 \phi_v^2.$$

Just like in ordinary gravity, we need to include a suitable boundary term (Gibbons–Hawking–York).

Let $\Gamma \subset T_p$ be a finite, connected subgraph, such that all vertices of Γ have valence either $p+1$ or 1. $\partial\Gamma$ is then the set of univalent vertices.

Then, one can truncate the action to Γ:

$$S_\Gamma = S_{\mathrm{EH}} + S_{\mathrm{bdy}} = \sum_{e\in\Gamma} (\kappa_e - 2\Lambda) + \sum_{x\in\partial\Gamma} \ell_x,$$

where

$$\ell_x = K + \sum_{\substack{y\sim x \\ y\notin\Gamma}} \kappa_{x\to y}.$$

What does the action evaluate to on-shell?

There are two undetermined constants, Λ and K, in it; one constraint ensures that S remains finite as $\Gamma \to T_p$:

$$K = \frac{2q}{q-1}\Lambda + q.$$

When this constraint is satisfied, the on-shell action is

$$S_{\text{cl}} = (2-2g)\left(1 + \frac{q+1}{q-1}\Lambda\right).$$

Here g is the genus (the result holds for the tree of any Mumford curve/higher-genus black hole). So the on-shell action is *topological!*

This suggests that this model is, perhaps, more like dilaton gravity in two dimensions than honest Einstein gravity.

Further evidence for that interpretation is provided by computing correlation functions of the operator T dual to edge-length fluctuations.

The two-point functions are as one would expect for a massless bulk field. But the three-point function vanishes identically, up to possible contact terms!

Here are some of those results more explicitly: the two point function of T is

$$\langle T(z_1)T(z_2)\rangle = \frac{p^n}{4}\frac{\zeta(2n)}{\zeta(n)^2}\frac{1}{|z_{12}|^{2n}}.$$

For an operator $\mathscr{O}$ of dimension Δ,

$$\langle T(z_1)\mathscr{O}(z_2)\mathscr{O}(z_3)\rangle = \frac{-\zeta(n)\zeta(2\Delta)}{\zeta(2\Delta-n)\zeta(-\Delta)\zeta(\Delta-n)}\frac{1}{|z_{12}|^n|z_{13}|^n|z_{23}|^{2\Delta-n}}.$$

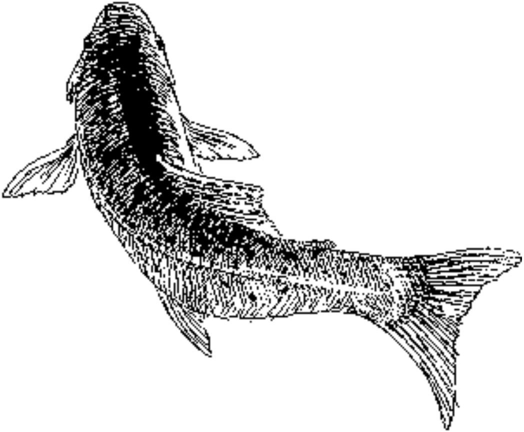
And finally,

$$\langle T(z_1)T(z_2)T(z_3)\rangle = 0!$$

A few words of outlook:

- The path-integral of the bulk theory defines a tensor network of sorts (albeit with infinite-dimensional local Hilbert spaces). Does it admit truncations? Does it have error-correcting properties?
- Connections to statistical mechanics models—for example, the Ising model on infinite trees?
- Can one rigorously compute the entanglement entropy of the vacuum state of the p-adic free boson? (I did this, but I am not convinced I trust the answer.)
- Bulk gauge fields/boundary conserved currents?
- Higher-spin gravity duals to the free $O(N)$ model?[‖]
- The list could easily go on. . .

[‖]A first step is in Gubser and Parikh, `arXiv:1704.01149`.



Thanks!

Drawing courtesy of Robert Savannah, U.S. Fish and Wildlife Service