One-variable and Multi-variable Integral Calculus over the Levi-Civita Field and Applications

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The Levi-Civita Fields $\mathcal{R}$ and $\mathcal{C}$

- **Let** $\mathcal{R} = \{f : \mathbb{Q} \to \mathbb{R} | \textbf{supp}(f) \textbf{ is left-finite}\}$**.**

- **For** $x \in \mathcal{R}$**, define**

$$\lambda(x) = \begin{cases} \textbf{min(supp(}x\textbf{))} & \textbf{if } x \neq 0 \\ \infty & \textbf{if } x = 0 \end{cases}.$$

- **Arithmetic on** $\mathcal{R}$**: Let** $x, y \in \mathcal{R}$**. We define** $x + y$ **and** $x \cdot y$ **as follows. For** $q \in \mathbb{Q}$**, let**

$$\begin{aligned} (x+y)[q] &= x[q] + y[q] \\ (x \cdot y)[q] &= \sum_{q_1+q_2=q} x[q_1] \cdot y[q_2]. \end{aligned}$$

  Then $x + y \in \mathcal{R}$ **and** $x \cdot y \in \mathcal{R}$**.**

Result: $(\mathcal{R}, +, \cdot)$ **is a field.**

<u>Definition</u>: $\mathcal{C} := \mathcal{R} + i\mathcal{R}$**. Then** $(\mathcal{C}, +, \cdot)$ **is also a field.**

Order in $\mathcal{R}$

- **Define the relation $\leq$ on $\mathcal{R} \times \mathcal{R}$ as follows:**

$x \leq y$ **if** $x = y$ **or** $(x \neq y$ **and** $(x-y)[\lambda(x-y)] < 0)$.

- $(\mathcal{R}, +, \cdot, \leq)$ **is an ordered field.**

- $\mathcal{R}$ **is real closed.**

$$\Downarrow$$

$\mathcal{C}$ **is algebraically closed.**

- **The map $E : \mathbb{R} \to \mathcal{R}$, given by**

$$E(r)[q] \ = \begin{cases} r & \textbf{if } q = 0 \\ 0 & \textbf{else} \end{cases},$$

is an order preserving embedding.

- **There are infinitely small and infinitely large elements in $\mathcal{R}$: The number d, given by**

$$d[q] = \begin{cases} 1 & \textbf{if } q = 1 \\ 0 & \textbf{else} \end{cases},$$

is infinitely small; while d^{-1} is infinitely large.

For $x \in \mathcal{R}$, define

$$|x| = \begin{cases} x & \textbf{if } x \geq 0 \\ -x & \textbf{if } x < 0 \end{cases};$$

$$|x|_u = \begin{cases} e^{-\lambda(x)} & \textbf{if } x \neq 0 \\ 0 & \textbf{if } x = 0 \end{cases}.$$

For $z = x + iy \in \mathcal{C}$, define

$$|z| = \sqrt{|x|^2 + |y|^2};$$

$$\begin{aligned} |z|_u &= \begin{cases} e^{-\lambda(z)} & \textbf{if } z \neq 0 \\ 0 & \textbf{if } z = 0 \end{cases} \\ &= \max\{|x|_u, |y|_u\} \textbf{ since } \lambda(z) = \min\{\lambda(x), \lambda(y)\}. \end{aligned}$$

Note that $|\cdot|$ and $|\cdot|_u$ induce the same topology τ_v on $\mathcal{R}$ (or $\mathcal{C}$). Moreover, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|$ in $\mathcal{R}$.

Topological Structure of $\mathcal{R}$

Properties of the Topology τ_v (induced by $|\cdot|$ or $|\cdot|_u$):

- $(\mathcal{R}, \tau_v)$ is a disconnected topological space.
- $(\mathcal{R}, \tau_v)$ is Hausdorff.
- There are no countable bases.
- The topology induced to $\mathbb{R}$ is the discrete topology.
- $(\mathcal{R}, \tau_v)$ is not locally compact.
- τ_v is zero-dimensional (i.e. it has a base consisting of clopen sets).
- τ_v is not a vector topology.
- For all $x \in \mathcal{R}$ (or $\mathcal{C}$): $x = \sum_{n=1}^{\infty} x[q_n] \cdot d^{q_n}$.

Weak Topology τ_w: Induced by the family of semi-norms $(\|\cdot\|_r)_{r\in\mathbb{Q}}$, where $\|\cdot\|_r : \mathcal{R} \to \mathbb{R}$ is given by

$$\|x\|_r = \sup\{|x[q]| : q \leq r\}.$$

It is a metric topology, induced by the metric

$$\Delta(x, y) = \sum_{k\in\mathbb{N}} 2^{-k} \frac{\|x-y\|_k}{1+\|x-y\|_k}.$$

Properties of the Weak Topology:

- $(\mathcal{R}, \tau_w)$ **is Hausdorff with countable bases.**
- τ_w **is a vector topology.**
- **The topology induced on $\mathbb{R}$ is the usual order topology on $\mathbb{R}$.**
- $(\mathcal{R}, \tau_w)$ **is not locally bounded (hence not locally compact).**

Further Useful Notations

For $x, y \in \mathcal{R}$**, we say**

- $x \sim y$ **if** $\lambda(x) = \lambda(y)$**.**

- $x \approx y$ **if** $x \sim y$ **and** $x[\lambda(x)] = y[\lambda(y)]$**.**

- $x =_r y$ **if** $x[q] = y[q]$ $\forall q \leq r$**.**

- **For nonnegative** x, y **in** $\mathcal{R}$**, say** $x \ll y$ **if** $x < y$ **and** $x \not\sim y$**; say** $x \gg y$ **if** $y \ll x$**.**

Examples: $100 \sim 1$**;** $3 + d \approx 3$**;** $d^q \ll 1$ **if** $q > 0$**.**

Uniqueness of $\mathcal{R}$ and $\mathcal{C}$

- **$\mathcal{R}$ is the smallest Cauchy-complete and real closed non-Archimedean field extension of $\mathbb{R}$.**
  - **It is small enough so that the $\mathcal{R}$-numbers can be implemented on a computer, thus allowing for computational applications.**

- **$\mathcal{C}$ is the smallest Cauchy-complete and algebraically closed non-Archimedean field extension of $\mathbb{C}$.**

Power Series and Analytic Functions

[Shamseddine-2011]. *Absolute and relative extrema, the mean value theorem and the inverse function theorem for analytic functions on a Levi-Civita field, Contemporary Mathematics, American Mathematical Society, Volume 551 (Advances in Non-Archimedean Analysis), pp. 257-268.*

[Shamseddine-Berz-2007]. *Intermediate value theorem for analytic functions on a Levi-Civita field, Bulletin of the Belgian Mathematical Society-Simon Stevin, Volume 14, pp. 1001-1015.*

[Shamseddine-Berz-2005]. *Analytical properties of power series on Levi-Civita fields, Annales Mathmatiques Blaise Pascal, Volume 12 # 2, pp. 309-329.*

[Shamseddine-Berz-2001]. *Convergence on the Levi-Civita field and study of power series, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, 2001, ISBN 0-8247-0611-0, pp. 283-299.*

Theorem (Strong Convergence Criterion): Let (a_n) be a sequence in $\mathcal{R}$, and let

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n} \right) \textbf{ in } \mathbb{R}.$$

Let $x_0 \in \mathcal{R}$ be fixed and let $x \in \mathcal{R}$ be given. Then $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges strongly if $\lambda(x - x_0) > \lambda_0$ and is divergent if $\lambda(x - x_0) < \lambda_0$ or if $\lambda(x - x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

Theorem (Weak Convergence Criterion): **Let (a_n), λ_0 and x_0 be as above, and let $x \in \mathcal{R}$ be such that $\lambda(x-x_0) = \lambda_0$. For each $n \geq 0$, let $b_n = a_n d^{n\lambda_0}$. Suppose that the sequence (b_n) is regular and write $\cup_{n=0}^{\infty}$supp$(b_n) = \{q_1, q_2, \ldots\}$; with $q_{j_1} < q_{j_2}$ if $j_1 < j_2$. For each n, write $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$; let**

$$r = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|^{1/n} : j \geq 1\right\}}.$$

Then $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely weakly in $\mathcal{R}$ if $|(x-x_0)[\lambda_0]| < r$ and is weakly divergent in $\mathcal{R}$ if $|(x-x_0)[\lambda_0]| > r$.

Corollary (Power Series with Real Coefficients): **Let $\sum_{n=0}^{\infty} a_n X^n$, $a_n \in \mathbb{R}$, be a power series with classical radius of convergence equal to η. Let $x \in \mathcal{R}$, and let $A_n(x) = \sum_{i=0}^{n} a_i x^i \in \mathcal{R}$. Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series on $\mathcal{R}$.**

Transcendental Functions: **For any $x \in \mathcal{R}$, at most finite in absolute value, define**

$$\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}; \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}; \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}; \\
\cosh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}; \\
\sinh(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Definition: **Let** $a < b$ **in** $\mathcal{R}$ **be given and let** $f : [a, b] \to \mathcal{R}$**. Then** f **is analytic on** $[a, b]$ **means for all** $x \in [a, b]$ **there exists a positive** $\delta \sim b - a$ **in** $\mathcal{R}$**, and there exists a sequence** $(a_n(x))$ **in** $\mathcal{R}$ **such that**

$$f(y) = \sum_{n=0}^{\infty} a_n(x)(y - x)^n$$

for all $y \in (x - \delta, x + \delta) \cap [a, b]$**.**

Lemma: **Let** $f, g : [a, b] \to \mathcal{R}$ **be analytic on** $[a, b]$ **and let** $\alpha \in \mathcal{R}$ **be given. Then** $f + \alpha g$ **and** $f \cdot g$ **are analytic on** $[a, b]$**.**

Lemma: **Let** $f : [a, b] \to \mathcal{R}$ **be analytic on** $[a, b]$**, let** $g : [c, e] \to \mathcal{R}$ **be analytic on** $[c, e]$**, and let** $f([a, b]) \subset [c, e]$**. Then** $g \circ f$ **is analytic on** $[a, b]$**.**

Main Results on Analytic Functions: Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then

- f is bounded on $[a, b]$. In particular,
$$i(f) := \min\{\lambda(f(x)) : x \in [a, b]\}$$
exists and is called the index of f on $[a, b]$.

- **Intermediate Value Theorem:** f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$.

- **Differentiability of the Analytic Functions:** f is infinitely often differentiable on $[a, b]$, and for all $m \in \mathbb{N}$, we have that $f^{(m)}$ is analytic on $[a, b]$. Moreover, if f is given around $x_0 \in [a, b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$, then

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n \cdots (n - m + 1)\, a_n(x_0)(x - x_0)^{n-m}.$$

In particular, we have that

$$a_m(x_0) = \frac{f^{(m)}(x_0)}{m!} \text{ for all } m = 0, 1, 2, \ldots.$$

- **Extreme Value Theorem:** f **assumes a maximum and a minimum on** $[a, b]$**.**

- **The Mean Value Theorem:** **There exists** $c \in (a, b)$ **such that**
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- **Corollary:** **The following are true.**

(i) If $f'(x) \neq 0$ **for all** $x \in (a, b)$ **then either**

(a) $f'(x) > 0$ **for all** $x \in (a, b)$ **and** f **is strictly increasing on** $[a, b]$**, or**

(b) $f'(x) < 0$ **for all** $x \in (a, b)$ **and** f **is strictly decreasing on** $[a, b]$**.**

(ii) If $f'(x) = 0$ **for all** $x \in (a, b)$ **then** f **is constant on** $[a, b]$**.**

Measure Theory and Integration

[Shamseddine-Flynn-2016]. *Measure theory and Lebesgue-like integration in two and three dimensions over the Levi-Civita field, Contemporary Mathematics, American Mathematical Society, Volume 665 (Advances in non-Archimedean Analysis), pp. 289- 325.*

[Shamseddine-2013]. *New results on integration on the Levi-Civita field, Indagationes Mathematicae, Volume 24 # 1, pp. 199-211.*

[Shamseddine-Berz-2003]. *Measure theory and integration on the Levi-Civita field, Contemporary Mathematics, American Mathematical Society, Volume 319 (Ultrametric Functional Analysis), pp. 369-387.*

Measure Theory and Integration on $\mathcal{R}$

Definition (Measurable Set): **We say that $A \subset \mathcal{R}$ is measurable if $\forall\ \epsilon > 0$ in $\mathcal{R}$, $\exists$ sequences of mutually disjoint intervals (I_n) and (J_n) such that**

$\bigcup_{n=1}^{\infty} I_n \subset A \subset \bigcup_{n=1}^{\infty} J_n$,

$\sum_{n=1}^{\infty} l(I_n)$ **and** $\sum_{n=1}^{\infty} l(J_n)$ **converge in $\mathcal{R}$, and**

$$\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon.$$

Definition (The Measure of a Measurable Set): **Suppose** $A \subset \mathcal{R}$ **is a measurable set. Then for every** $k \in \mathbb{N}$ **there are two sequences of mutually disjoint intervals** $(I_n^k)_{n=1}^{\infty}$ **and** $(J_n^k)_{n=1}^{\infty}$ **such that** $\bigcup_{n=1}^{\infty} I_n^k \subset A \subset \bigcup_{n=1}^{\infty} J_n^k$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ **and** $\sum_{n=1}^{\infty} l\left(J_n^k\right)$ **both converge, and** $\sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) < d^k$.

- $\left(\sum_{n=1}^{\infty} l\left(I_n^k\right)\right)_{k=1}^{\infty}$ **and** $\left(\sum_{n=1}^{\infty} l\left(J_n^k\right)\right)_{k=1}^{\infty}$ **are Cauchy sequences in** $\mathcal{R}$**. Therefore,** $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ **and** $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ **both exist in** $\mathcal{R}$**.**

- $\lim_{k\to\infty} \left(\sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right)\right) = 0.$

- **We define** $m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ **and we call this the measure of** A**.**

Consequences:

- **If $A \subset \mathcal{R}$ is measurable then**

$$\begin{aligned} m(A) &= \inf\Bigg\{\sum_{n=1}^{\infty} l\left(J_n\right) : J_n\text{'s } \textbf{are (mutually disjoint)} \\ &\qquad \textbf{intervals}, A \subset \bigcup_{n=1}^{\infty} J_n, \ \textbf{and} \\ &\qquad \sum_{n=1}^{\infty} l\left(J_n\right) \ \textbf{converges}\Bigg\} \\ &= \sup\Bigg\{\sum_{n=1}^{\infty} l\left(I_n\right) : I_n\text{'s } \textbf{are mutually disjoint} \\ &\qquad \textbf{intervals}, \bigcup_{n=1}^{\infty} I_n \subset A, \ \textbf{and} \\ &\qquad \sum_{n=1}^{\infty} l\left(I_n\right) \ \textbf{converges}\Bigg\}. \end{aligned}$$

- **If $A \subset \mathcal{R}$ is measurable then $m(A) \geq 0$.**
- **$I(a, b)$ is measurable and $m(I(a, b)) = b - a$.**
- **If $B \subset A \subset \mathcal{R}$ and if A and B are measurable, then $m(B) \leq m(A)$.**
- **If $A \subset \mathcal{R}$ is countable, then A is measurable and $m(A) = 0$.**

Proposition: **For each $k \in \mathbb{N}$, let $A_k \subset \mathcal{R}$ be measurable such that $(m(A_k))$ forms a null sequence. Then $\bigcup_{k=1}^{\infty} A_k$ is measurable and**

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m\left(A_k\right).$$

If the sets $(A_k)_{k=1}^{\infty}$ are mutually disjoint, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m\left(A_k\right).$$

Proposition: **Let $K \in \mathbb{N}$ be given and for each $k \in \{1, \ldots, K\}$, let A_k be measurable. Then $\bigcap_{k=1}^{K} A_k$ is measurable.**

Proposition: **Let $A, B \subset \mathcal{R}$ be measurable. Then**

$$m(A \cup B) = m(A) + m(B) - m(A \cap B).$$

Measurable Functions

Definition: **Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if $\forall\ \epsilon > 0$, $\exists$ a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ and f is analytic on I_n for all n; $\sum_{n=1}^{\infty} l\,(I_n)$ converges; and $m(A) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$.**

Proposition: **Let $f : I(a,b) \to \mathcal{R}$ be measurable. Then f is continuous almost everywhere on $I(a,b)$. Moreover, if f is differentiable on $I(a,b)$ and if $f' = 0$ everywhere, then f is constant on $I(a,b)$.**

Proposition: **Let $A, B \subset \mathcal{R}$ be measurable, let f be a measurable function on A and B. Then f is measurable on $A \cup B$ and $A \cap B$.**

Proposition: **Let $A \subset \mathcal{R}$ be measurable, let $f, g : A \to \mathcal{R}$ be measurable and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are measurable on A.**

Integration

Definition (Integral of an Analytic Function): **Let $f : I(a,b) \to \mathcal{R}$ be analytic on $I(a,b)$, and let F be an analytic anti-derivative of f on $I(a,b)$. Then the integral of f over $I(a,b)$ is the $\mathcal{R}$ number**

$$\int_{I(a,b)} f = \lim_{x \to b} F(x) - \lim_{x \to a} F(x).$$

$$\Downarrow$$

Definition (Integral of a Measurable Function): **Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then**

$$\int_A f = \lim_{\substack{\sum_{n=1}^{\infty} l(I_n) \to m(A) \\ \cup_{n=1}^{\infty} I_n \subset A \\ (I_n) \textbf{ are mutually disjoint} \\ f \textbf{ is analytic on } I_n}} \sum_{n=1}^{\infty} \int_{I_n} f.$$

Properties of the Integral:

- $\int_A \alpha = \alpha m(A)$.

- $f \leq 0$ **on** $A \Rightarrow \int_A f \leq 0$.

- $\int_A (f + \alpha g) = \int_A f + \alpha \int_A g$.

- $\int_{A \cup B} f = \int_A f + \int_B f - \int_{A \cap B} f$.

- $|f| \leq M$ **on** $A \Rightarrow \left| \int_A f \right| \leq Mm(A)$.

- (f_n) **converges uniformly to** f **on** A **and** f_n **is measurable on** A **for each** $n \Rightarrow f$ **is measurable on** A **and**

$$\lim_{n \to \infty} \int_A f_n = \int_A f.$$

Measure Theory and Integration on $\mathcal{R}^2$

Definition (Simple Region): Let $G \subset \mathcal{R}^2$. Then we say that G is a simple region if there exist constants $a, b \in \mathcal{R}, a \leq b$ and analytic functions $g_1, g_2 : I(a,b) \to \mathcal{R}, g_1 \leq g_2$ on $I(a,b)$ such that

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(g_1(y), g_2(y)), y \in I(a,b)\}.$$

Definition (Area of a Simple Region): Let $G \subset \mathcal{R}^2$ be a simple region given by

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(g_1(x), g_2(x)), x \in I(a,b)\}.$$

Then we define the area of G, denoted by $a(G)$, as:

$$a(G) = \int_{x \in I(a,b)} [g_2(x) - g_1(x)]$$

Lemma: **Let $H, G \subset \mathcal{R}$ be simple regions. Then $H \cap G$, $H \setminus G$ and $H \cup G$ can each be written as a finite union of mutually disjoint simple regions.**

Definition (Measurable Set): **Let $A \subset \mathcal{R}^2$. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$ there exist two sequences of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ such that**

$$\bigcup_{n=1}^{\infty} G_n \subset A \subset \bigcup_{n=1}^{\infty} H_n,$$

$\sum_{n=1}^{\infty} a(G_n)$ **and** $\sum_{n=1}^{\infty} a(H_n)$ **both converge, and**

$$\sum_{n=1}^{\infty} a(H_n) - \sum_{n=1}^{\infty} a(G_n) < \epsilon.$$

Definition (The Measure of a Measurable Set): Suppose $A \subset \mathcal{R}^2$ is a measurable set. Then for every $k \in \mathbb{N}$ there are two sequences of mutually disjoint simple regions $(G_n^k)_{n=1}^\infty$ and $(H_n^k)_{n=1}^\infty$ such that $\bigcup_{n=1}^\infty G_n^k \subset A \subset \bigcup_{n=1}^\infty H_n^k$, $\sum_{n=1}^\infty G_n^k$ and $\sum_{n=1}^\infty H_n^k$ both converge, and $\sum_{n=1}^\infty a(H_n^k) - \sum_{n=1}^\infty a(G_n^k) < d^k$.

- $\left(\sum_{n=1}^\infty a(G_n^k)\right)_{k=1}^\infty$ and $\left(\sum_{n=1}^\infty a(H_n^k)\right)_{k=1}^\infty$ are Cauchy sequences in $\mathcal{R}$. Therefore, $\lim_{k\to\infty} \sum_{n=1}^\infty a(G_n^k)$ and $\lim_{k\to\infty} \sum_{n=1}^\infty a(H_n^k)$ both exist in $\mathcal{R}$.

- $\lim_{k\to\infty} \left(\sum_{n=1}^\infty a(H_n^k) - \sum_{n=1}^\infty a(G_n^k)\right) = 0$.

- We define $m(A) = \lim_{k\to\infty} \sum_{n=1}^\infty a(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^\infty a(H_n^k)$ and we call this the measure of A.

Consequences:

- **If** $A \subset \mathcal{R}^2$ **is measurable then**

$$\begin{aligned} m(A) &= \inf\left\{\sum_{n=1}^{\infty} a(H_n) : H_n\text{'s are mutually disjoint}\right. \\ &\qquad \text{simple regions}, A \subset \bigcup_{n=1}^{\infty} H_n, \text{ and} \\ &\qquad \left.\sum_{n=1}^{\infty} a(H_n) \text{ converges}\right\} \\ &= \sup\left\{\sum_{n=1}^{\infty} a(G_n) : G_n\text{'s are mutually disjoint}\right. \\ &\qquad \text{simple regions}, \bigcup_{n=1}^{\infty} G_n \subset A, \text{ and} \\ &\qquad \left.\sum_{n=1}^{\infty} a(G_n) \text{ converges}\right\}. \end{aligned}$$

- **If** $A \subset \mathcal{R}^2$ **is measurable then** $m(A) \geq 0$**.**

- **If** $B \subset A \subset \mathcal{R}^2$ **and if** A **and** B **are measurable, then** $m(B) \leq m(A)$**.**

- **If** $A \subset \mathcal{R}^2$ **is countable, then** A **is measurable and** $m(A) = 0$**.**

Proposition: **For each $k \in \mathbb{N}$, let $A_k \subset \mathcal{R}^2$ be measurable such that $(m(A_k))$ forms a null sequence. Then $\bigcup_{k=1}^{\infty} A_k$ is measurable and**

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m\left(A_k\right).$$

If the sets $(A_k)_{k=1}^{\infty}$ are mutually disjoint, then equality holds.

Proposition: **Let $K \in \mathbb{N}$ be given and for each $k \in \{1, \ldots, K\}$, let $A_k \subset \mathcal{R}^2$ be measurable. Then $\bigcap_{k=1}^{K} A_k$ is measurable.**

Definition: **Let $A \subset \mathcal{R}^2$ be a simple region.**

If $A = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$, we define $\lambda_x(A) = \lambda(b - a)$ and $\lambda_y(A) = i(h_2(x) - h_1(x))$ on $I(a, b)$.

If $A = \{(x, y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a, b)\}$, we define $\lambda_y(A) = \lambda(b - a)$ and $\lambda_x(A) = i(h_2(y) - h_1(y))$ on $I(a, b)$.

If $\lambda_x(A) = \lambda_y(A) = 0$ then we say that A is finite.

Measurable Functions

Definition: **Let $A \subset \mathcal{R}^2$ be a simple region. Then we say that $f : A \to \mathcal{R}^2$ is analytic on A if for every $(x_0, y_0) \in A$, there is a simple region A_0 around (x_0, y_0) satisfying $\lambda_x(A_0) = \lambda_x(A)$, $\lambda_y(A_0) = \lambda_y(A)$ and a sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if $(x_0 + s, y_0 + t) \in A \cap A_0$, then**

$$f(x_0 + s, y_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} s^i t^j.$$

Proposition: **Let $A \subset \mathcal{R}^2$ be a simple region and let $f : A \to \mathcal{R}$ be an analytic function on A. Let $a < b$ in $\mathcal{R}$ and let $g : I(a, b) \to \mathcal{R}$ be analytic on $I(a, b)$ such that for every $x \in I(a, b)$, $(x, g(x)) \in A$. Then $F(x) := f(x, g(x))$ is an analytic function on $I(a, b)$.**

Proposition: **Let $A \subset \mathcal{R}^2$ be simple and let $f : A \to \mathcal{R}$ be analytic. Then f is bounded on A.**

Definition (Measurable Function): Let $A \subset \mathcal{R}^2$ be measurable and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if $\forall\ \epsilon > 0$, there exists a sequence of mutually disjoint simple regions (G_n) such that $G_n \subset A$ and f is analytic on G_n for all n; $\sum_{n=1}^{\infty} a\,(G_n)$ converges; and $m(A) - \sum_{n=1}^{\infty} a(G_n) \leq \epsilon$.

Proposition: Let $A \subset \mathcal{R}^2$ be a measurable set and let $f : A \to \mathcal{R}$ be measurable on A. Then f is given locally by a power series almost everywhere on A. Moreover, if $\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} f(x, y) = 0$ everywhere on A then $f(x, y)$ is constant on A.

Proposition: Let $A, B \subset \mathcal{R}^2$ be measurable, let f be a measurable function on A and B. Then f is measurable on $A \cup B$ and $A \cap B$.

Proposition: Let $A \subset \mathcal{R}^2$ be measurable, let $f, g : A \to \mathcal{R}$ be measurable and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are measurable on A.

Integration on $\mathcal{R}^2$

Definition: **Let $G \subset \mathcal{R}^2$ be a simple region given by**

$$G = \{(x, y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a, b)\}$$

and let $f : G \to \mathcal{R}$ be analytic on G. We define the integral of f over G as follows:

$$\iint_{(x,y)\in G} f(x, y) = \int_{x\in I(a,b)} \left[\int_{y\in I(h_1(x),h_2(x))} f(x, y) \right].$$

Proposition: **Let $G \subset \mathcal{R}^2$ be a simple region, let $f, g : G \to \mathcal{R}$ be analytic on G, and let $\alpha \in \mathcal{R}$ be given. Then**

- $\displaystyle\iint_{(x,y)\in G} \alpha = \alpha a(G)$;
- $\displaystyle\iint_{(x,y)\in G} (f + \alpha g)(x, y) = \iint_{(x,y)\in G} f(x, y) + \alpha \iint_{(x,y)\in G} g(x, y)$;
- **if $f \le g$ on G then** $\displaystyle\iint_{(x,y)\in G} f(x, y) \le \iint_{(x,y)\in G} g(x, y)$;
- **if $|f| \le M$ on G then** $\displaystyle\left| \iint_{(x,y)\in G} f(x, y) \right| \le Ma(G)$.

Integral of a Measurable Function: Let $A \subset \mathcal{R}^2$ be a measurable set, let $f : A \to \mathcal{R}$ be measurable on A, and let M be a bound for $|f|$ on A. For every $k \in \mathbb{N}$ there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, f is analytic on G_n^k, $\bigcup\limits_{n=1}^{\infty} G_n^k \subset A$, $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges, and $m(A) - \sum\limits_{n=1}^{\infty} a(G_n^k) \leq d^k$.

Since $\sum\limits_{n=1}^{\infty} a(G_n^k)$ converges we have that $\lim\limits_{n\to\infty} a(G_n^k) = 0$. Since $|\iint\limits_{(x,y)\in G_n^k} f(x,y)| \leq M a(G_n^k)$, it follows that $\lim\limits_{n\to\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y) = 0$. Therefore, for every $k \in \mathbb{N}$, $\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y)$ converges.

We show that $(\sum\limits_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n^k} f(x,y))_{k=1}^{\infty}$ is a Cauchy sequence and hence it converges. We define

$$\iint\limits_{(x,y)\in A} f(x,y) = \lim_{\substack{\sum\limits_{n=1}^{\infty} a(G_n) \to m(A), \\ \bigcup\limits_{n=1}^{\infty} G_n \subset A,\, G_n\text{'s mutually disjoint}, \\ f \text{ is analytic on } G_n \text{ for every n}}} \sum_{n=1}^{\infty} \iint\limits_{(x,y)\in G_n} f(x,y)$$

Theorem (Properties of the Double Integral): Let $A, B \subset \mathcal{R}^2$ be measurable sets, let $f, g : A, B \to \mathcal{R}$ be measurable functions on A, B, and let $\alpha \in \mathcal{R}$ be given. Then

- $\displaystyle\iint_{(x,y)\in A} \alpha = \alpha m(A)$;
- $\displaystyle\iint_{(x,y)\in A} (f+\alpha g)(x,y) = \iint_{(x,y)\in A} f(x,y) + \alpha \iint_{(x,y)\in A} g(x,y)$;
- **if** $f \leq g$ **on** A **then** $\displaystyle\iint_{(x,y)\in A} f(x,y) \leq \iint_{(x,y)\in A} g(x,y)$;
- **if** $|f| \leq M$ **on** A **then** $\displaystyle|\iint_{(x,y)\in A} f(x,y)| \leq Mm(A)$;
- $\displaystyle\iint_{(x,y)\in A\cup B} f(x,y) = \iint_{(x,y)\in A} f(x,y) + \iint_{(x,y)\in B} f(x,y) -$
  $$\iint_{(x,y)\in A\cap B} f(x,y);$$
- (f_n) **converges uniformly to** f **on** A **and** f_n **is measurable on** A **for each** $n \Rightarrow \displaystyle\lim_{n\to\infty} \iint_{(x,y)\in A} f_n(x,y)$

  exists and

  $$\lim_{n\to\infty} \iint_{(x,y)\in A} f_n(x,y) = \iint_{(x,y)\in A} f(x,y).$$

Measure Theory and Integration on $\mathcal{R}^3$

Definition (Simple Region): Let $S \subset \mathcal{R}^3$. Then we say that S is simple if there exists a simple region $A \subset \mathcal{R}^2$ and two analytic functions $h_1, h_2 : A \to \mathcal{R}$ such that $h_1 \leq h_2$ everywhere on A and

$$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}$$

or

$$S = \{(x, y, z) \in \mathcal{R}^3 : y \in I(h_1(x, z), h_2(x, z)), (x, z) \in A\}$$

or

$$S = \{(x, y, z) \in \mathcal{R}^3 : x \in I(h_1(y, z), h_2(y, z)), (y, z) \in A\}.$$

Definition (Volume of a Simple Region): Let

$$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}$$

be a simple region in $\mathcal{R}^3$, with A, h_1, and h_2 as above. Then we denote the volume of S with $v(S)$ and define it as

$$v(S) = \iint_{(x,y)\in A} \left[h_2(x, y) - h_1(x, y)\right].$$

A similar definition can be used in the other two cases.

Definition: Let
$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\}$
be a simple region in $\mathcal{R}^3$. We define $\lambda_x(S) = \lambda_x(A)$, $\lambda_y(S) = \lambda_y(A)$ and $\lambda_z(S) = i(h_2(x, y) - h_1(x, y))$ on A. We do similarly in the other two cases. Then we say that that S is a finite region if $\lambda_x(S) = \lambda_y(S) = \lambda_z(S) = 0$.

Definition (Analytic Function in $\mathcal{R}^3$): Suppose $S \subset \mathcal{R}^3$ is a simple region and let $f : S \to \mathcal{R}$. Then we say that f is analytic on S if for every $(x_0, y_0, z_0) \in S$ there exists a simple region $A \subset \mathcal{R}^3$ containing (x_0, y_0, z_0) and a sequence $(a_{ijk})_{i,j,k=0}^{\infty}$ in $\mathcal{R}$ such that $\lambda_x(A) = \lambda_x(S)$, $\lambda_y(A) = \lambda_y(S)$, $\lambda_z(A) = \lambda_z(S)$, and if $(x_0 + r, y_0 + s, z_0 + t) \in S \cap A$ then

$$f(x_0 + r, y_0 + s, z_0 + t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} r^i s^j t^k.$$

Proposition: Let $A \subset \mathcal{R}^3$ be a simple region, let $f, g : A \to \mathcal{R}$ be analytic on A, and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are analytic on A.

Proposition: Let $A \subset \mathcal{R}^3$ be a simple region and let $f : A \to \mathcal{R}$ be analytic on A. Let $B \subset \mathcal{R}^2$ be a simple region and let $g : B \to \mathcal{R}$ be an analytic function on B such that for every $(x, y) \in B$, $(x, y, g(x, y)) \in A$. Then $F(x, y) := f(x, y, g(x, y))$ is analytic on B.

Definition (Measurable Set): Let $S \subset \mathcal{R}^3$. Then we say that S is a measurable set if for every $\epsilon > 0$ there exist two sequences of mutually disjoint simple regions $(G_n)_{n=1}^{\infty}$ and $(H_n)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} G_n \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n$, $\sum\limits_{n=1}^{\infty} v(G_n)$ and $\sum\limits_{n=1}^{\infty} v(H_n)$ converge, and $\sum\limits_{n=1}^{\infty} v(H_n) - \sum\limits_{n=1}^{\infty} v(G_n) < \epsilon$.

Measure of a Measurable Set: Let $S \subset \mathcal{R}^3$ be a measurable set. For every $k \in \mathbb{N}$, there exist two sequences of mutually disjoint simple regions, $(G_n^k)_{n=1}^{\infty}$ and $(H_n^k)_{n=1}^{\infty}$, such that $\bigcup\limits_{n=1}^{\infty} G_n^k \subset S \subset \bigcup\limits_{n=1}^{\infty} H_n^k$, $\sum\limits_{n=1}^{\infty} v(G_n^k)$ and $\sum\limits_{n=1}^{\infty} v(H_n^k)$ converge, and $\sum\limits_{n=1}^{\infty} v(H_n^k) - \sum\limits_{n=1}^{\infty} v(G_n^k) < d^k$.

We show that $(\sum_{n=1}^{\infty} v(G_n^k))_{k=1}^{\infty}$ **and** $(\sum_{n=1}^{\infty} v(H_n^k))_{k=1}^{\infty}$ **are Cauchy sequences; and hence they converge. Moreover,**

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} v(G_n^k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} v(H_n^k).$$

We call this limit the measure of S **and we denote it by** $m(S)$**.**

$$\Downarrow$$

Similar properties to those in the one-dimensional and two-dimensional cases!

Definition (Measurable Function):

Let $S \subset \mathcal{R}^3$ **be measurable and let** $f : S \to \mathcal{R}$ **be bounded on** S**. Then we say that** f **is measurable on** S **if for every** $\epsilon > 0$ **in** $\mathcal{R}$**, there exists a sequence of mutually disjoint simple regions** $(G_n)_{n=1}^{\infty}$ **such that** $\bigcup_{n=1}^{\infty} G_n \subset S$**,** $\sum_{n=1}^{\infty} v(G_n)$ **converges,** $m(S) - \sum_{n=1}^{\infty} v(G_n) < \epsilon$**, and for every** $n \in \mathbb{N}$ f **is analytic on** G_n**.**

Integration in Three Dimensions

Definition (Integral of an Analytic Function over a Simple Region in $\mathcal{R}^3$): Let

$S = \{(x, y, z) \in \mathcal{R}^3 : z \in I(h_1(x, y), h_2(x, y)), (x, y) \in A\};$

and let $f : S \to \mathcal{R}$ be analytic on S. We define the integral of f over S as follows:

$$\iiint\limits_{(x,y,z)\in S} f(x, y, z) = \iint\limits_{(x,y)\in A} \left[\int\limits_{z\in I(h_1(x,y),h_2(x,y))} f(x, y, z) \right].$$

Consequences:

- **For any $\alpha \in \mathcal{R}$:** $\iiint\limits_{(x,y,z)\in S} \alpha = \alpha v(S)$.

- **If $|f(x)| \leq M$ for all $x \in S$ then**

$$\left| \iiint\limits_{(x,y,z)\in S} f(x, y, z) \right| \leq M v(S).$$

- **etc...**

Integral of a Measurable Function over a Measurable Set: Let $S \subset \mathcal{R}^3$ be a measurable set, let $f : S \to \mathcal{R}$ be measurable on S, and let M be a bound for f on S. For every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint simple regions $(G_n^k)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} G_n^k \subset S$, $\sum_{n=1}^{\infty} v(G_n^k)$ converges, $m(S) - \sum_{n=1}^{\infty} v(G_n^k) < d^k$, and for every $n \in \mathbb{N}$ f is analytic on G_n^k.

For every $k, n \in \mathbb{N}$:

$$\left| \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z) \right| \leq Mv(G_n^k).$$

It follows that

$$\lim_{n\to\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z) = 0$$

and so $\sum\limits_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z)$ converges.

We show that $\left(\sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z)\right)_{k=1}^{\infty}$ **is a Cauchy sequence and hence it converges.**

We define the limit to be the integral of f **over** S**:**

$$\iiint\limits_{(x,y,z)\in S} f(x,y,z) = \lim_{k\to\infty} \sum_{n=1}^{\infty} \iiint\limits_{(x,y,z)\in G_n^k} f(x,y,z).$$

$$\Downarrow$$

Similar properties of the triple integral as for double and single integrals!

The Delta Function on the Levi-Civita Field

Definition: **Let** $\delta : \mathcal{R} \to \mathcal{R}$ **be given by**

$$\delta(x) = \begin{cases} \frac{3}{4}d^{-3}(d^2 - x^2) & \textbf{if } |x| < d \\ \\ 0 & \textbf{if } |x| \geq d \end{cases}.$$

Proposition: **Let** $I \subset \mathcal{R}$ **be an interval. If** $(-d, d) \subset I$ **then**

$$\int_{x \in I} \delta(x) = 1.$$

Moreover, if $(-d, d) \cap I = \emptyset$ **then**

$$\int_{x \in I} \delta(x) = 0.$$

Proof: **If** $(-d, d) \subset I$ **then**

$$\int_{x \in I} \delta(x) = \int_{x \in (-d,d)} \frac{3}{4} d^{-3}(d^2 - x^2) = 1.$$

If $(-d, d) \cap I = \emptyset$ **then** $\delta(x) = 0$ **for all** $x \in I$**; hence**

$$\int_{x \in I} \delta(x) = \int_{x \in I} 0 = 0.$$

Proposition: **Let $I \subset \mathcal{R}$ be an interval containing $(-d, d)$. Then $\delta(x)$ has a measurable antiderivative on I that is equal to the Heaviside function on $I \cap \mathbb{R}$.**

Proof: **Let $H : I \to \mathcal{R}$ be given by**

$$H(x) = \begin{cases} 0 & \textbf{if } x \leq -d \\ \frac{3}{4}d^{-3}(d^2x - \frac{1}{3}x^3) + \frac{1}{2} & \textbf{if } -d < x < d \\ 1 & \textbf{if } x \geq d \end{cases}.$$

Then $H(x)$ is measurable and differentiable on I with $H'(x) = \delta(x)$. Moreover,

$$H(x)|_{\mathbb{R}} = \begin{cases} 0 & \textbf{if } x < 0 \\ 1/2 & \textbf{if } x = 0 \\ 1 & \textbf{if } x > 0 \end{cases}.$$

Proposition: **Let $a < b$ in $\mathcal{R}$ be such that $\lambda(b - a) < 1$; and let $f : [a, b] \to \mathcal{R}$ be an analytic function with $i(f) = 0$. Then for any $x_0 \in [a+d, b-d]$, we have that**

$$\int_{x \in [a,b]} f(x)\delta(x - x_0) =_0 f(x_0).$$

Proof: **Fix** $x_0 \in [a+d, b-d]$**. There exists** $\eta > 0$ **in** $\mathcal{R}$ **with** $\lambda(\eta) = \lambda(b-a)$ **such that, for any** $x \in [a, b]$ **satisfying** $|x - x_0| < \eta$**, we have that**

$$f(x) = f(x_0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Therefore,

$$\int_{x\in[a,b]} f(x)\delta(x - x_0) = \int_{x\in[x_0-d,x_0+d]} f(x)\delta(x - x_0)$$

$$= \int_{x\in[x_0-d,x_0+d]} f(x_0)\delta(x - x_0)$$

$$+ \int_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0)$$

$$= f(x_0) + \int_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \delta(x - x_0).$$

For any $x \in [x_0 - d, x_0 + d]$**,** $|x - x_0| \le d$**. Thus,**

$$
\begin{aligned}
&\left| \int_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x-x_0) \right| \\
&\leq \sum_{k=1}^{\infty} \frac{\left|f^{(k)}(x_0)\right|}{k!} d^k \int_{x\in[x_0-d,x_0+d]} \delta(x-x_0) \\
&= \sum_{k=1}^{\infty} \frac{\left|f^{(k)}(x_0)\right|}{k!} d^k.
\end{aligned}
$$

Since $i(f) = 0$ on $[a, b]$, it follows that for all $k \in \mathbb{N}$ $\lambda\left(f^{(k)}(x_0)(b-a)^k\right) \geq 0$ and hence $\lambda\left(f^{(k)}(x_0)d^k\right) > 0$. Thus,

$$
\lambda\left(\sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!} d^k\right) > 0.
$$

It follows that

$$
\lambda\left(\int_{x\in[x_0-d,x_0+d]} \sum_{k=1}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \delta(x-x_0)\right) > 0;
$$

and hence

$$
\int_{x\in[a,b]} f(x)\delta(x-x_0) =_0 f(x_0).
$$

Proposition: **Let** $a < b < c$ **in** $\mathcal{R}$ **be such that** $\lambda(b-a) < 1$ **and** $\lambda(c-b) < 1$**; let** $g : [a,b] \to \mathcal{R}$ **and** $h : [b,c] \to \mathcal{R}$ **be analytic functions satisfying** $g(b) = h(b)$ **and** $i(h) = i(g) = 0$**; and let the function** $f : [a,c] \to \mathcal{R}$ **be given by**

$$f(x) = \begin{cases} g(x) & \textbf{if } x \in [a,b) \\ h(x) & \textbf{if } x \in [b,c] \end{cases}.$$

Then for any $x_0 \in [a+d, c-d]$**, we have that**

$$\int\limits_{x \in [a,c]} f(x)\delta(x-x_0) =_0 f(x_0).$$

Definition (Delta Function in Two Dimensions):

Let $\delta_2 : \mathcal{R}^2 \to \mathcal{R}$ **be given by**

$$\delta_2(x, y) = \delta(x)\delta(y)$$

Proposition: Let $S \subset \mathcal{R}^2$ **be measurable. If** $(-d, d) \times (-d, d) \subset S$ **then**

$$\iint_S \delta_2(x, y) = 1.$$

If $(-d, d) \times (-d, d) \cap S = \emptyset$ **then**

$$\iint_S \delta_2(x, y) = 0.$$

Proposition: Let $S \subset \mathcal{R}^2$ **be a simple region with** $\lambda_x(S) < 1$ **and** $\lambda_y(S) < 1$**, let** $f : S \to \mathcal{R}$ **be an analytic function with index** $i(f) = 0$ **on** S**. Then, for any** $(x_0, y_0) \in S$ **that satisfies** $(x_0 - a, x_0 + a) \times (y_0 - a, y_0 + a) \subset S$ **for some positive** $a \gg d$ **in** $\mathcal{R}$**, we have that**

$$\iint\limits_{(x,y)\in S} f(x, y)\delta_2(x - x_0, y - y_0) =_0 f(x_0, y_0).$$

Definition (Delta Function in Three Dimensions):

Let $\delta_3 : \mathcal{R}^3 \to \mathcal{R}$ **be given by**

$$\delta_3(x, y, z) = \delta(x)\delta(y)\delta(z).$$

Proposition: Let $S \subset \mathcal{R}^3$ **be measurable. If** $(-d, d) \times (-d, d) \times (-d, d) \subset S$ **then**

$$\iiint_S \delta_3(x, y, z) = 1.$$

If $(-d, d) \times (-d, d) \times (-d, d) \cap S = \emptyset$ **then**

$$\iiint_S \delta_3(x, y, z) = 0.$$

Proposition: Let $S \subset \mathcal{R}^3$ **be a simple region with** $\lambda_x(S) < 1$**,** $\lambda_y(S) < 1$**,** $\lambda_z(S) < 1$**, and let** $f : S \to \mathcal{R}$ **be an analytic function on** S **with** $i(f) = 0$ **on** S**. Then, for any** $(x_0, y_0, z_0) \in S$ **that satisfies**

$$(x_0 - a, x_0 + a) \times (y_0 - a, y_0 + a) \times (z_0 - a, z_0 + a) \subset S$$

for some positive $a \gg d$ **in** $\mathcal{R}$**, we have that**

$$\iiint\limits_{(x,y,z)\in S} f(x, y, z)\delta_3(x - x_0, y - y_0, z - z_0) =_0 f(x_0, y_0, z_0).$$

Example (Damped Driven Harmonic Oscillator): Consider an underdamped, driven harmonic oscillator with mass m, viscous damping constant c, spring constant k, and driving force $f(t)$. Let $x(t)$ be the position of the oscillator at time t with $x(0) = 0$ and $\dot{x}(0) = 0$.

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{f(t)}{m}.$$

Let $\gamma = \frac{c}{2\sqrt{mk}}$ and let $\omega_0 = \sqrt{\frac{k}{m}}$. Thus,

$$\ddot{x}(t) + 2\gamma\omega_0\dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}.$$

Consider the underdamped case: $\gamma^2\omega_0^2 - \omega_0^2 < 0$ (that is, $\gamma < 1$).

We first find a piecewise analytic solution to

$$\left(\frac{\partial^2}{\partial t^2} + 2\gamma\omega_0\frac{\partial}{\partial t} + \omega_0^2\right)G(t,t') = \delta(t-t').$$

We get:

$G(t,t') = e^{-\gamma\omega_0(t-t')}\left(A_1 \sin(\omega(t-t')) + B_1 \cos(\omega(t-t'))\right)$
if $t \le t' - d$**;**

$$G(t,t') = \begin{array}{l} e^{-\gamma\omega_0(t-t')}\left(A_2 \sin(\omega(t-t')) + B_2 \cos(\omega(t-t'))\right) \\ +\frac{3}{\omega_0^2}\left(\frac{d^2-(t-t')^2}{4} + \frac{\gamma(t-t')}{\omega_0} + \frac{1-4\gamma^2}{2\omega_0^2}\right) \end{array}$$
if $t' - d < t < t' + d$**; and**

$G(t,t') = e^{-\gamma\omega_0(t-t')}(A_3 \sin(\omega(t-t')) + B_3 \cos(\omega(t-t')))$
if $t \ge t' + d$**.**

We want the solution to satisfy the initial conditions $G(t'-d,t') = 0$ **and** $\frac{\partial}{\partial t}G(t,t')|_{t=t'-d} = 0$ **as well as continuity of** $G(t,t')$ **and** $\frac{\partial}{\partial t}G(t,t')$ **at** $t = t' - d$ **and** $t = t' + d$**.**

From the initial conditions we get

$$A_1 = B_1 = 0.$$

From the continuity of G and its derivative at $t = t' - d$ we then have

$$A_2 = \frac{3}{\omega_0^2} d^{-3} \exp(-\gamma\omega_0 d) \cdot \\ \left[\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2} \right) d \right) \frac{\cos \omega d}{\omega} \\ - \left(\frac{\gamma}{\omega_0} - \frac{1 - 4\gamma^2}{2\omega_0^2} \right) \sin \omega d \right]$$

$$B_2 = \frac{3}{\omega_0^2} d^{-3} \exp(-\gamma\omega_0 d) \cdot \\ \left[\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2} \right) d \right) \frac{\sin \omega d}{\omega} \\ + \left(\frac{\gamma}{\omega_0} - \frac{1 - 4\gamma^2}{2\omega_0^2} \right) \cos \omega d \right].$$

Finally, from the continuity of G and its derivative at $t = t' + d$ we get:

$$A_3 = \frac{3}{\omega_0^2} d^{-3} \exp(-\gamma\omega_0 d) \cdot \left[\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d \right) \frac{\cos\omega d}{\omega} - \left(\frac{\gamma}{\omega_0} - \frac{1-4\gamma^2}{2\omega_0^2} \right) \sin\omega d + \left(\frac{3\gamma}{2\omega_0} - \frac{2\gamma^3}{\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d \right) \frac{\cos\omega d}{\omega} + \left(\frac{\gamma}{\omega_0} - \frac{1-4\gamma^2}{2\omega_0^2} \right) \sin\omega d \right]$$

$$B_3 = \frac{3}{\omega_0^2} d^{-3} \exp(-\gamma\omega_0 d) \cdot \left[\left(\frac{2\gamma^3}{\omega_0} - \frac{3\gamma}{2\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d \right) \frac{\sin\omega d}{\omega} + \left(\frac{\gamma}{\omega_0} - \frac{1-4\gamma^2}{2\omega_0^2} \right) \cos\omega d - \left(\frac{3\gamma}{2\omega_0} - \frac{2\gamma^3}{\omega_0} + \left(\gamma^2 - \frac{1}{2}\right) d \right) \frac{\sin\omega d}{\omega} + \left(\frac{\gamma}{\omega_0} - \frac{1-4\gamma^2}{2\omega_0^2} \right) \cos\omega d \right].$$

Note that $A_3 =_0 \frac{1}{\omega}$ **and** $B_3 =_0 0$**; and hence**

$$G(t,t')|_{\mathbb{R}} =_0 \begin{cases} 0 & \textbf{if } t < t' \\ \frac{1}{\omega}\exp\left(-\gamma\omega_0(t-t')\right)\sin(\omega(t-t')) & \textbf{if } t \geq t' \end{cases}$$

which is the classical Green's function for this problem.

Now assume the driving force is given by

$$f(t) = \begin{cases} m\exp(-\gamma\omega_0 t) & \textbf{if } t \geq 0 \\ 0 & \textbf{if } t < 0 \end{cases}.$$

Then we obtain the real solution as:

$$x(t) =_0 \int_{t' \in [-d^{-1}, d^{-1}]} G(t,t')\frac{f(t')}{m}.$$

But $G(t,t') = 0$ **for** $t' > t + d$ **and** $f(t') = 0$ **for** $t' < 0$**; thus,**

$$x(t) =_0 \int\limits_{t' \in [0, t+d]} G(t,t') \exp(-\gamma\omega_0 t')$$

$$= \int\limits_{t' \in [0, t-d]} G(t,t') \exp(-\gamma\omega_0 t')$$

$$+ \int\limits_{t' \in [t-d, t+d]} G(t,t') \exp(-\gamma\omega_0 t')$$

$$=_0 e^{-\gamma\omega_0 t} \frac{\cos(\omega t) - 1}{\omega^2},$$

which agrees with the classical solution.