p-ADIC ANALYSIS: LECTURES NOTES FOR A MINI-COURSE IN THE L. SANTALÓ RESEARCH SUMMER SCHOOL 2019. PALACIO DE LA MAGDALENA, SANTANDER, SPAIN, JUNE 24-28, 2019.

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Contents

1. Introduction

These notes aim to provide a fast introduction to p-adic analysis assuming basic knowledge in algebra and analysis. We cannot provide detailed proofs, for an in-depth discussion, the reader may consult [1], [23], [24], see also [12], [14], [21], [22]. We focus on basic aspects of analysis involving complex-valued functions.

In the last thirty years p-adic analysis have received great attention due to its connections with physics, biology, cryptography, and several mathematical theories, see e.g. [1], [2], [17], [18], [15], [16], [24], [25], and the references therein. As a consequence of all this, nowadays, p-adic analysis is having a tremendous expansion. Let us mention a couple examples. First, the developing of the theory of p-adic pseudodifferential equations, which is a theory connected with several fields, see e.g. [1], [18], [15], [16], [24], [25] and the reference therein. Second, the deep connection local zeta functions and string amplitudes, see e.g. [5], [6], and the references therein.

The L. Santaló Research School 2019 aims to provide an introduction to the area of local zeta functions. In the Archimedean case, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the study of local zeta functions was initiated by Gel'fand and Shilov [11]. The meromorphic continuation of the local zeta functions was established, independently, by Atiyah [3] and Bernstein [4], see also [10, Theorem 5.5.1 and Corollary 5.5.1]. The main motivation was that the meromorphic continuation of Archimedean local zeta functions implies the existence of fundamental solutions (i.e. Green functions) for differential operators with constant coefficients. It is important to mention here, that in the p-adic framework, the existence of fundamental solutions for pseudodifferential operators is also a consequence of the fact that the Igusa local zeta functions admit a meromorphic continuation, see [18, Chapter 10] and [25, Chapter 5]. In the 70s, Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero [10]. For an elementary introduction to the basic aspects of local zeta functions the reader may consult [20].

2. p-Adic numbers: essential facts

2.1. Basic facts.

In this section we summarize the basic aspects of the field of p-adic numbers, for an in-depth discussion the reader may consult [1, 12, 13, 14, 21, 22, 23] and [24].

Definition 1. *Let F be a field. A norm (or an absolute value) on F is a real-valued function, $|\cdot|$, satisfying*

(i) $|x| = 0 \Leftrightarrow x = 0$;
(ii) $|xy| = |x||y|$;
(iii) $|x+y| \leq |x| + |y|$ *(triangle inequality), for any* $x, y \in F$.

Definition 2. *A norm $|\cdot|$ is called non-Archimedean (or ultrametric), if it satisfies*

$$|x+y| \leq \max\{|x|, |y|\}. \tag{2.1}$$

Notice that (2.1) implies the triangle inequality.

Example 1. *The trivial norm is defined as*

$$|x|_{trivial} = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

From now on we will work only with non-trivial norms.

Definition 3. *Let p be a fixed prime number, and let x be a nonzero rational number. Then, $x = p^k \frac{a}{b}$, with $p \nmid ab$, and $k \in \mathbb{Z}$. The p-adic absolute value (or p-adic norm) of x is defined as*

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Exercise 1. *The function $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. In addition, show that $|x+y|_p = \max\{|x|_p, |y|_p\}$ when $|x|_p \neq |y|_p$.*

We set $\mathbb{R}_+ := \{x \in \mathbb{R} : \ x \geq 0\}$. We denote by $\mathbb{N}$ the set of non-negative integers.

Definition 4. *Let X be a non-empty set. A distance, or metric, on X is a function $d : X \times X \to \mathbb{R}_+$ satisfying the following properties:*

(i) *$d(x,y) = 0$ if and only if $x = y$;*
(ii) *$d(x,y) = d(y,x)$;*
(iii) *$d(x,y) \leq d(x,z) + d(z,y)$ for any x, y, z in X.*

The pair (X,d) is called a metric space.

Example 2. *Let F be a field endowed with a norm $|\cdot|$. The distance $d(x,y) := |x-y|$, for x, y in F, is called the induced distance by $|\cdot|$. The pair (F,d) is a metric space.*

Definition 5. *Let (X,d) be a metric space. The metric d is called non-Archimedean if*

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} \text{ for any } x,\ y,\ z \in X.$$

Example 3. *Take $X = \mathbb{Q}$, and d the distance induce by the p-adic norm $|\cdot|_p$, for a fixed prime p. Then d is a non-Archimedean.*

Definition 6. *Let (X,d) be a metric space. A sequence $\{a_i\}_{i\in\mathbb{N}}$ in X is called a Cauchy sequence, if for any $\epsilon > 0$ there exists N such that $d(a_m, a_n) < \epsilon$ whenever both $m > N$, $n > N$.*

Definition 7. *Two metrics d_1 and d_2 on a set X are called equivalent if a sequence is Cauchy with respect to d_1 if and only if it is Cauchy with respect to d_2. We say that two norms are equivalent if they induce equivalent metrics.*

Exercise 2. *Let α be a fixed positive real number. For $x \in \mathbb{Q}$, we define $\|x\| = |x|_\infty^\alpha$, where $|\cdot|_\infty$ denotes the standard absolute value. Show that $\|\cdot\|$ is a norm if and only if $\alpha \leq 1$, and that in that case it is equivalent to the norm $|\cdot|_\infty$.*

Theorem 2.1 (Ostrowski, [14]). *Any non trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ or to the standard absolute value $|\cdot|_\infty$.*

Remark 1. *(i) Let F be a field endowed with a norm $|\cdot|$. We introduce a topology on F by giving a basis of open sets consisting of the open balls $B_r(a)$ with center a and radius $r>0$:*

$$B_r(a)=\{x\in F: |x-a|<r\}.$$

(ii) A sequence of points $\{x_i\}_{i\in\mathbb{N}}\subset F$ is called Cauchy if

$$|x_m-x_n|\to 0, \qquad m,n\to\infty.$$

(iii) A field F with a non trivial absolute value $|\cdot|$ is said to be complete if any Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$ has a limit point $x^*\in F$, i.e. if $|x_n-x^*|\to 0$, $n\to\infty$. This is equivalent to the fact that (F,d), with $d(x,y)=|x-y|$, is a complete metric space.

(iv) Let (X,d), (Y,D) be two metric spaces. A bijection $\rho: X\to Y$ satisfying

$$D(\rho(x),\rho(x'))=d(x,x'),$$

is called an isometry.

The following fact is well-known, see e.g. [19].

Theorem 2.2. *Let (M,d) be a metric space. There exists a complete metric space $(\widetilde{M},\widetilde{d})$, such that M is isometric to a dense subset of $\widetilde{M}$. This space $\widetilde{M}$ is unique up to isometries, that is, if $\widetilde{M}_0$ is a complete metric space having M as a dense subspace, then $\widetilde{M}_0$ is isometric to $\widetilde{M}$.*

Exercise 3. *Let $(F,|\cdot|)$ be a valued field, where $|\cdot|$ is a non-Archimedean absolute value. Assume that F is complete with respect to $|\cdot|$. Then, the series $\sum_{k\geq 0}a_k$, $a_k\in F$ converges if an only if $\lim_{k\to\infty}|a_k|=0$.*

2.2. **The field of p-adic numbers.**

Lemma 2.1. *Consider the set*

$$\mathbb{Q}_p:=\left\{x=p^\gamma\sum_{i=0}^{\infty}x_ip^i:\ \gamma\in\mathbb{Z},\ x_i\in\{0,1,\ldots,p-1\},\ x_0\neq 0\right\}\cup\{0\}$$

endowed with the p-adic norm $|\cdot|_p$. Then, the following assertions hold.

(i) *$(\mathbb{Q}_p,|\cdot|_p)$ is a complete metric space;*
(ii) *$\mathbb{Q}$ is dense in $\mathbb{Q}_p$;*
(iii) *$\mathbb{Q}_p$ is a field of characteristic zero;*
(iv) *the completion of $(\mathbb{Q},|\cdot|_p)$ is $(\mathbb{Q}_p,|\cdot|_p)$.*

Proof. (i) We first note that series $x=p^\gamma\sum_{i=0}^{\infty}x_ip^i$ converges in the p-adic norm. Let $\left\{x^{(m)}\right\}_{m\in\mathbb{N}}$ be a Cauchy series, with

$$x^{(m)}=p^{\gamma_m}\sum_{i=0}^{\infty}x_i^{(m)}p^i.$$

Then, given p^{-L} there exists $M \in \mathbb{N}$ such that

$$\left|x^{(m)} - x^{(n)}\right|_p < p^{-L} \text{ for } m \geq n > M \Leftrightarrow ord(x^{(m)} - x^{(n)}) > L \text{ for } m \geq n > M,$$

which implies the existence of $\gamma \in \mathbb{Z}$, $x_i \in \{0, 1, \ldots, p-1\}$ for $i = 1, \ldots, L$ such that

$$x^{(m)} = p^\gamma \sum_{i=0}^{L} x_i p^i \text{ for } m > M.$$

Since L can be taken arbitrarily large, there exists $x = p^\gamma \sum_{i=0}^{\infty} x_i p^i \in \mathbb{Q}_p$ such that

$$\left|x - x^{(m)}\right|_p < p^{-L} \text{ for } m > M.$$

Which implies that $x^{(m)} \to x$.

(ii) We set

$$\mathbb{Z}_p := \left\{ x \in \mathbb{Q}_p : x = p^\gamma \sum_{i=0}^{\infty} x_i p^i ; \gamma \in \mathbb{N},\ x_0 \neq 0 \right\}.$$

Then any $x \in \mathbb{Q}_p \smallsetminus \{0\}$ can be written as $x = p^\gamma \widetilde{x}$, with $\widetilde{x} \in \mathbb{Z}_p$ and $|\widetilde{x}|_p = 1$. Given p^{-L}, with $L \in \mathbb{N}$, we have to show the existence of $\frac{a}{b} \in \mathbb{Q}$ such that $\left|x - \frac{a}{b}\right|_p < p^{-L}$. We take $b^{-1} = p^\gamma$ and $a \in \mathbb{Z}$ satisfying $|a - \widetilde{x}|_p < p^{-L+\gamma}$.

(iii) We first show that $\mathbb{Z}_p$ is a ring. Take

$$x = p^\alpha \sum_{i=0}^{\infty} x_i p^i \text{ with } \alpha \in \mathbb{N},\ x_0 \neq 0, \quad y = p^\beta \sum_{i=0}^{\infty} y_i p^i \text{ with } \beta \in \mathbb{N},\ y_0 \neq 0.$$

And set $\gamma = \min\{\alpha, \beta\}$ and

$$z = p^\gamma \sum_{i=0}^{\infty} z_i p^i \text{ with } \alpha \in \mathbb{N},\ z_0 \neq 0.$$

We now define the digits z_is by the following formulae:

$$p^\gamma \sum_{i=0}^{L-1} z_i p^i \equiv p^\alpha \sum_{i=0}^{L-1} x_i p^i + p^\beta \sum_{i=0}^{L-1} y_i p^i \mod p^L \text{ for } L \in \mathbb{N} \smallsetminus \{0\}. \tag{2.2}$$

Here $A \equiv B \bmod p^L$ means p^L divides $A - B$. Now, we define $x + y = z$. Notice that (2.2) determines uniquely all the digits z_i's. For the product xy, we set

$$w = p^{\alpha+\beta} \sum_{i=0}^{\infty} w_i p^i \text{ with } w_0 \neq 0.$$

Then the digits w_is are uniquely determined by the following formulae:

$$p^{\alpha+\beta} \sum_{i=0}^{L-1} w_i p^i \equiv \left(p^\alpha \sum_{i=0}^{L-1} x_i p^i \right) \left(p^\beta \sum_{i=0}^{L-1} y_i p^i \right) \mod p^L \text{ for } L \in \mathbb{N} \smallsetminus \{0\}. \tag{2.3}$$

Now we define $xy = w$. It is not difficult to verify that $(\mathbb{Z}_p, +, \cdot)$ is a commutative ring. Furthermore, by using (2.3) one verifies that $xy = 0$ implies that $x = 0$ or $y = 0$. This means that $\mathbb{Z}_p$ is a domain. Finally, we notice that the field of fractions of $\mathbb{Z}_p$ is precisely $\mathbb{Q}_p$. In order to verify this assertion it is necessary to use Exercise 4.

(iv) It follows from (i)-(iii) by using Theorem 2.2. □

The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the distance induced by $|\cdot|_p$. By Lemma 2.1, any p-adic number $x \neq 0$ has a unique representation of the form

$$x = p^{\gamma} \sum_{i=0}^{\infty} x_i p^i,$$

where $\gamma = \gamma(x) \in \mathbb{Z},\ x_i \in \{0, 1, \ldots, p-1\},\ x_0 \neq 0$. The integer γ is called the *p-adic order of x,* and it will be denoted as $ord(x)$. By definition $ord(0) = +\infty$.

Example 4. *The formula $\frac{-1}{(p-1)} = \sum_{i=0}^{\infty} p^i$ holds true in $\mathbb{Q}_p$, i.e.*

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$

Indeed, set

$$\begin{aligned} z^{(n)} &:= (p-1) + (p-1)p + \cdots + (p-1)p^n \\ &= (p-1)\frac{p^{n+1}-1}{p-1} = p^{n+1} - 1. \end{aligned}$$

Then $\lim_{n\to\infty} z^{(n)} = \lim_{n\to\infty} p^{n+1} - 1 = 0 - 1 = -1$, *since* $|p^{n+1}|_p = p^{-n-1}$.

Remark 2. *The unit ball*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \{x \in \mathbb{Q}_p : x = \sum_{i=i_0}^{\infty} x_i p^i, i_0 \geq 0\},$$

is a domain of principal ideals. Any ideal of $\mathbb{Z}_p$ has the form

$$p^m \mathbb{Z}_p = \{x \in \mathbb{Z}_p : x = \sum_{i \geq m} x_i p^i\},\ m \in \mathbb{N}.$$

Indeed, let $I \subseteq \mathbb{Z}_p$ be an ideal. Set $m_0 = \min_{x\in I} ord(x) \in \mathbb{N}$, and let $x_0 \in I$ such that $ord(x_0) = m_0$. Then $I = x_0\mathbb{Z}_p$.

From a geometric point of view, the ideals of the form $p^m\mathbb{Z}_p,\ m \in \mathbb{Z}$, constitute a fundamental system of neighborhoods around the origin in $\mathbb{Q}_p$.

The *residue field* of $\mathbb{Q}_p$ is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ (the finite field with p elements). The *group of units* of $\mathbb{Z}_p$ is

$$\mathbb{Z}_p^{\times} = \{x \in \mathbb{Z}_p : |x|_p = 1\}.$$

Exercise 4. *$x = x_0 + x_1 p + \cdots \in \mathbb{Z}_p$ is a unit if and only if $x_0 \neq 0$. Moreover if $x \in \mathbb{Q}_p \setminus \{0\}$, then $x = p^m u$, $m \in \mathbb{Z}$, $u \in \mathbb{Z}_p^{\times}$.*

2.3. **Topology of $\mathbb{Q}_p$.** Define

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^r\},\ r \in \mathbb{Z},$$

as the *ball with center a and radius p^r*, and

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^r\},\ r \in \mathbb{Z}$$

as the *sphere with center a and radius p^r*.

The topology of $\mathbb{Q}_p$ is quite different from the usual topology of $\mathbb{R}$. First of all, since $|\cdot|_p : \mathbb{Q}_p \to \{p^m : m \in \mathbb{Z}\} \cup \{0\}$, the radii are always integer powers of p, for the sake of brevity we just use the power in the notation $B_r(a)$ and $S_r(a)$. On the other hand, since the powers of p and zero form a discrete set in $\mathbb{R}$, in the definition of $B_r(a)$ and $S_r(a)$ we can always use '$\leq$'. Indeed,

$$\{x \in \mathbb{Q}_p : |x - a|_p < p^r\} = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{r-1}\} = B_{r-1}(a) \subset B_r(a).$$

Remark 3. *Notice that $B_r(a) = a + p^{-r}\mathbb{Z}_p$ and $S_r(a) = a + p^{-r}\mathbb{Z}_p^\times$.*

We declare that the $B_r(a)$, $r \in \mathbb{Z}$, $a \in \mathbb{Q}_p$, are open subsets. These sets form a basis for the topology of $\mathbb{Q}_p$.

Proposition 2.1. *$S_r(a)$, $B_r(a)$ are open and closed sets in the topology of $\mathbb{Q}_p$.*

Proof. We first show that $S_r(a)$ is open. Notice that

$$S_r(a) = \bigsqcup_{i \in \{1,\dots,p-1\}} a + p^{-r}i + p^{-r+1}\mathbb{Z}_p = \bigsqcup_{i \in \{1,\dots,p-1\}} B_{(r-1)}(a + p^{-r}i),$$

and consequently $S_r(a)$ is an open set.

In order to show that $S_r(a)$ is closed, we take a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of $S_r(a)$ converging to $\widetilde{x}_0 \in \mathbb{Q}_p$. We must show that $\widetilde{x}_0 \in S_r(a)$. Note that $x_n = a + p^{-r}u_n$, $u_n \in \mathbb{Z}_p^\times$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we have

$$|x_n - x_m|_p = p^r|u_n - u_m|_p \to 0, \qquad n, m \to \infty,$$

thus $\{u_n\}_{n\in\mathbb{N}}$ is also Cauchy, and since $\mathbb{Q}_p$ is complete $u_n \to \widetilde{u}_0$. Then $x_n \to a + p^{-r}\widetilde{u}_0$, so in order to conclude our proof we must verify that $\widetilde{u}_0 \in \mathbb{Z}_p^\times$. Because u_m is arbitrarily close to $\widetilde{u}_0$, their p-adic expansions must agree up to a big power of p, hence $\widetilde{u}_0 \in \mathbb{Z}_p^\times$.

A similar argument shows that $B_r(a)$ is closed. □

Lemma 2.2. *If $b \in B_r(a)$ then $B_r(b) = B_r(a)$, i.e. any point of the ball $B_r(a)$ is its center.*

Proof. Let $x \in B_r(b)$, then

$$|x - a|_p = |x - b + b - a|_p \leq \max\{|x - b|_p, |b - a|_p\} \leq p^r,$$

i.e. $B_r(b) \subseteq B_r(a)$. Since $a \in B_r(b)$ (i.e. $|b - a|_p = |a - b|_p \leq p^r$), we can repeat the previous argument to show that $B_r(a) \subseteq B_r(b)$. □

Exercise 5. *Show that any to balls in $\mathbb{Q}_p$ are either disjoint or one is contained in another.*

Exercise 6. *Show that the boundary of any ball is the empty set.*

Theorem 2.3. [1, Sec. 1.8] *A set $K \subset \mathbb{Q}_p$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p$.*

2.4. **The $n-$dimensional p-adic space.** We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq d} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.$$

We define $ord(x) = \min_{1\leq i\leq n}\{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a separable complete ultrametric space (here, separable means that $\mathbb{Q}_p^n$ contains a countable dense subset, which is $\mathbb{Q}^n$).

For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n \; : \; ||x-a||_p \leq p^r\}$ *the ball of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r^n$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x_i \in \mathbb{Q}_p \; : \; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 = \mathbb{Z}_p$. We also denote by $S_r^n(a) = \{x \in \mathbb{Q}_p^n \; : \; ||x-a||_p = p^r\}$ *the sphere of radius p^r with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S_r^n(0) := S_r^n$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $\left(\mathbb{Z}_p^\times\right)^n \subsetneq S_0^n$.

As a topological space $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. Two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other. As in the one dimensional case, a subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$. Since the balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$, one has that $\left(\mathbb{Q}_p^n, ||\cdot||_p\right)$ is a locally compact topological space.

3. Integration on $\mathbb{Q}_p^n$

3.1. **Measure theory: a basic dictionary.** The notion of measure of a set is a mathematical abstraction of the naive notions of length of a segment, the area of a plane figure, and the volume of a body.

Let X be a non-empty set. We want to introduce a notion of measure for a class of subsets of X. A suitable class is a σ-algebra of subsets of X. Denote by $\mathcal{P}(X)$ the power set of X, then a subset $\Sigma \subset \mathcal{P}(X)$ is called a σ-algebra, if it satisfies the following properties:
(i) $X \in \Sigma$;
(ii) Σ is closed under complementation: if $A \in \Sigma$ then $A^c := X \smallsetminus A \in \Sigma$;
(iii) Σ is closed under countable unions: if $A_i \in \Sigma$ for $i \in \mathbb{N}$, then $\cup_{i\in\mathbb{N}} A_i \in \Sigma$.

Notice that if follows from the above definition that $\varnothing \in \Sigma$, and that Σ is closed under countable intersections. The elements of Σ are called *measurable sets*, which means that we can assign a measure to these sets. The pair (X, Σ) is called a *measurable space*. Assume that (Y, Λ) is another measurable space and that $f : (X, \Sigma) \to (Y, \Lambda)$ is a function between measurable spaces. The function f is called a *measurable function* if the preimage of every measurable set is measurable.

Example 5. *Let X be a non-empty set. the following are some simple examples of σ-algebras.*

(i) $\Sigma = \{X, \varnothing\}$, *this is the trivial* σ*-algebra;*
(ii) $\Sigma = \mathcal{P}(X)$, *this is the discrete* σ*-algebra;*
(iii) $\Sigma = \{X, \varnothing, A, A^c\}$ *is the* σ*-algebra generated by the subset* A*.*

Example 6. *Let F be a family of subsets of X. Then there exists a unique smallest σ-algebra $\sigma(F)$ which contains any set in F. The σ-algebra $\sigma(F)$ is called the σ-algebra generated by F, it agrees with intersection of all the σ-algebras containing F. If (X, d) is a metric space, the σ-algebra generated by the open balls is called the Borel σ-algebra of X.*

Definition 8. *Let (X, Σ) be a measurable space. A function $\mu : \Sigma \to [0, +\infty]$ is called a measure if it satisfies the following properties:*
(i) $\mu(\varnothing) = 0$;
(ii) for any countable collection A_i, $i \in \mathbb{N}$, of pairwise disjoint sets in Σ,

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

Let μ be a measure on (X, Σ). The following are some basic properties of μ:
(i) monotonicity: if A_1 and A_2 are measurable sets with $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$;
(ii) subadditivity: for any countable collection A_i, $i \in \mathbb{N}$, of measurable sets in Σ,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mu(A_i);$$

(iii) continuity from below: if A_i, $i \in \mathbb{N}$, are measurable sets in Σ such that $A_i \subset A_{i+1}$ for all i, then the union of the sets A_i is measurable, and

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i);$$

(iv) continuity from above: if A_i, $i \in \mathbb{N}$, are measurable sets in Σ such that $A_{i+1} \subset A_i$ for all i, then the intersection of the sets A_i is measurable. In addition, if A_0 has finite measure, then

$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

3.2. **The Haar measure.**

Theorem 3.1. [9, Thm B. Sec. 58] *Let $(G, \cdot)$ be a locally compact topological, Abelian group. There exists a regular Borel measure μ_{Haar} (called a Haar measure of G), unique up to multiplication by a positive constant, such that $\mu_{Haar}(U) > 0$ for every non empty Borel open set U, and $\mu_{Haar}(x \cdot E) = \mu_{Haar}(E)$, for every Borel set E.*

Notation 1. *We will denote the Haar measure by dx, then $\mu_{Haar}(U) = \int_U dx$.*

Exercise 7. *Prove that $(\mathbb{Q}_p, +)$, respectively $(\mathbb{Q}_p^\times, \cdot)$, are locally compact topological groups. Since $\mathbb{Q}_p$, respectively $\mathbb{Q}_p^\times = \mathbb{Q}_p \backslash \{0\}$, are metric spaces, the continuity of the sum, respectively* *of the product, means that if* $x_n \to x$, *and* $y_n \to y$, *then* $x_n + y_n \to x + y$, *respectively* $x_n y_n \to xy$.

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, by Theorem 3.1 there exists a measure dx, which is invariant under translations, i.e. $d(x+a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique.

In the n-dimensional case, $(\mathbb{Q}_p^n, +)$ is also locally compact topological group. We denote by $d^n x$ the Haar measure normalized by the condition $\int_{\mathbb{Z}_p^n} d^n x = 1$. This measure agrees with the product measure $dx_1 \cdots dx_n$, and it also satisfies that $d^n(x+a) = d^n x$, for $a \in \mathbb{Q}_p^n$.

The open compact balls of $\mathbb{Q}_p^n$, e.g. $a + p^m \mathbb{Z}_p^n$, generate the Borel σ-algebra of $\mathbb{Q}_p^n$. The measure $d^n x$ assigns to each open compact subset U a nonnegative real number $\int_U d^n x$, which satisfies

$$\int_{\cup_{k=1}^{\infty} U_k} d^n x = \sum_{k=1}^{\infty} \int_{U_k} d^n x, \tag{3.1}$$

for all compact open subsets U_k in $\mathbb{Q}_p^n$, which are pairwise disjoint, and verify $\cup_{k=1}^{\infty} U_k$ is still compact. In addition,

$$\int_{a+U} d^n x = \int_U d^n x.$$

Remark 4. *Let* $\mathcal{B}(\mathbb{Q}_p^n)$ *be the Borel* σ*-algebra on* $\mathbb{Q}_p^n$. *Let* $d^n x$ *be the normalized Haar measure on* $\left(\mathbb{Q}_p^n, \mathcal{B}(\mathbb{Q}_p^n)\right)$. *The Haar measure of a Borel set* A *is denoted as* $\mu_{Haar}^{(n)}(A)$. *The fact that* $d^n x$ *is a regular measure means that for any measurable subset* A *of* $\mathbb{Q}_p^n$ *is holds that*

$$\begin{aligned} \mu_{Haar}^{(n)}(A) &= \sup \left\{ \mu_{Haar}^{(n)}(F) : F \subset A,\ F \text{ compact and measurable} \right\} \\ &= \inf \left\{ \mu_{Haar}^{(n)}(G) : G \supset A,\ G \text{ open and measurable} \right\}. \end{aligned}$$

3.3. **Integration of locally constant functions.** A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be *locally constant* if for every $x \in \mathbb{Q}_p^n$ there exists an open compact subset U, containing x, and such that $f(x) = f(u)$ for all $u \in U$.

Exercise 8. *Every locally constant function is continuous.*

Any locally constant function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ can be expressed as a linear combination of characteristic functions of the form

$$\varphi(x) = \sum_{n=1}^{\infty} c_k 1_{U_k}(x), \tag{3.2}$$

where $c_k \in \mathbb{C}$,

$$1_{U_k}(x) = \begin{cases} 1 & \text{if} \quad x \in U_k \\ \\ 0 & \text{if} \quad x \notin U_k, \end{cases}$$

and $U_k \subseteq \mathbb{Q}_p^n$ is an open compact for every k. Indeed, there exists a covering $\{U_i\}_{i\in\mathcal{N}}$ of $\mathbb{Q}_p^n$ such that each U_i is open compact and $\varphi\mid_{U_i}$ is a constant function. Since $\left(\mathbb{Q}_p^n, \|\cdot\|_p\right)$ is a separable metric space any open cover has a countable subcover, consequently we may take $\mathcal{N}=\mathbb{N}$.

Let $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ be a locally constant function as in (3.2). Assume that $A = \bigsqcup_{i=1}^{k} U_i$, the symbol $\bigsqcup$ means disjoint union, i.e. the sets U_i are pairwise disjoint, with U_i open compact. Then, we define

$$\int_A \varphi(x)\, d^n x = c_1 \int_{U_1} d^n x + \cdots + c_n \int_{U_k} d^n x. \tag{3.3}$$

We recall that given a function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ the *support of* φ is the set

$$Supp(\varphi) = \overline{\{x \in \mathbb{Q}_p^n \ : \ \varphi(x) \neq 0\}}.$$

A locally constant function with compact support is called a *test function* or a *Bruhat-Schwartz function*. These functions form a $\mathbb{C}$-vector space denoted as $\mathcal{D} := \mathcal{D}\left(\mathbb{Q}_p^n\right)$. From (3.1) and (3.3) one has that the mapping

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \int_{\mathbb{Q}_p^n} \varphi \, d^n x, \end{array} \tag{3.4}$$

is a well-defined linear functional.

3.4. **Integration of continuous functions with compact support.** We now extend the integration to a larger class of functions. Let U be a open compact subset of $\mathbb{Q}_p^n$. We denote by $C(U,\mathbb{C})$ the space of all the complex-valued continuous functions supported on U, endowed with the supremum norm, i.e. for $\varphi \in C(U,\mathbb{C})$, we set

$$\|\varphi\| = \sup_{x\in U} |\varphi(x)|.$$

We denote by $C_0(\mathbb{Q}_p^n,\mathbb{C})$ the space of all the complex-valued continuous functions vanishing at infinity, endowed also with the supremum norm. The function φ vanishes at infinity, if given $\varepsilon > 0$, there exists a compact subset K such that $|\varphi(x)| < \varepsilon$, if $x \notin K$.

It is known that $\mathcal{D}$ is dense in $C_0(\mathbb{Q}_p^n,\mathbb{C})$, see e.g. [23, Prop. 1.3]. We identify $C(U,\mathbb{C})$ with a subspace of $C_0(\mathbb{Q}_p^n,\mathbb{C})$, therefore $\mathcal{D}$ is dense in $C(U,\mathbb{C})$.

We fix an open compact subset U and consider the functional (3.4), since

$$\left| \int_U \varphi \, d^n x \right| \leq \sup_{x\in U} |\varphi(x)| \int_U d^n x,$$

then functional (3.4) has a unique extension to $C(U,\mathbb{C})$.

This means that if $f \in C(U,\mathbb{C})$ and $\{f_m\}_{m\in\mathbb{N}}$ is any sequence in $\mathcal{D}$ approaching f in the supremum norm, then

$$\int_{\mathbb{Q}_p^n} f(x) \ d^n x = \lim_{m\to\infty} \int_{\mathbb{Q}_p^n} f_m(x) \ d^n x. \tag{3.5}$$

More generally, if $f_m \to f$, with f, $f_m \in C(U,\mathbb{C})$ for $m \in \mathbb{N}$, then (3.5) holds.

3.4.1. *Some remarks on uniform convergence.* We recall the notion of uniform convergence. Let E be a non-empty set. Let $f_n : E \to \mathbb{C}$, $n \in \mathbb{N}$ be a sequence of complex-valued functions. We say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on E with limit $f : E \to \mathbb{C}$, if for every $\epsilon > 0$, there exists a natural number N such that for all $n > N$ and any $x \in E$, $|f_n(x) - f(x)| < \epsilon$, which is equivalent to say for every $\epsilon > 0$, there exists a natural number N such that for all $n > N$, $\sup_{x\in E} |f_n(x) - f(x)| < \epsilon$.

The Weierstrass M-test is a very useful criterion for determining the uniform convergence of sequences. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions $f_n : E \to \mathbb{C}$ and let M_n be a sequence of positive real numbers such that $|f_n(x)| < M_n$ for all $x \in E$ and $n \in \mathbb{N}$. If $\sum_n M_n$ converges, then $\sum_n f_n$ converges uniformly on E.

3.4.2. *Some remarks on convergent power series.* Let us denote by $\mathbb{C}[[z_1, \ldots, z_n]]$, the ring of formal power series with coefficients in $\mathbb{C}$. An element of this ring has the form

$$\sum_i c_i z^i = \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} c_{i_1,\ldots,i_n} z_1^{i_1} \ldots z_n^{i_n}, \text{ where } i = (i_1, \ldots, i_n) \in \mathbb{N}^n, \text{ and the } c_{i_1,\ldots,i_n}\text{s are in } \mathbb{C}.$$

A formal series $\sum_i c_i z^i$ is said to be convergent if there exists a positive real number R such that $\sum_i c_i a^i$ converges for any $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying $\max_i |a_i| < R$. The convergent series form a subring of $\mathbb{C}[[x_1, \ldots, x_n]]$, which will be denoted as $\mathbb{C}\langle\langle x_1, \ldots, x_n\rangle\rangle$.

If for $\sum_i c_i z^i$ there exists $\sum_i c_i^{(0)} x^i \in \mathbb{R}\langle\langle x_1, \ldots, x_n\rangle\rangle$ (a real convergent series) such that $|c_i| \le c_i^{(0)}$ for all $i \in \mathbb{N}^n$, we say that $\sum_i c_i^{(0)} x^i$ is a *dominant series* for $\sum_i c_i x^i$ and write

$$\sum c_i x^i << \sum c_i^{(0)} x^i.$$

Exercise 9. *A formal power series is convergent if and only if it has a dominant series.*

Example 7. *We set $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $|k| := k_1 + \ldots + k_n$. Let $f(z) = \sum_k c_k z^k$ be a complex convergent power series on $\max_i |z_i| < R$. This series has a dominant series, and by the Weierstrass M-test, the sequence $\sum_{|k|\le M} c_k z^k$ converges uniformly to $f(z)$ on $\max_i |z_i| < R$.*

We construct a 'radial function' on $|x_i|_p \le p^L < R$ for $i = 1, \ldots, n$, i.e. on the ball B_L^n, by taking

$$f(|x_1|_p, \ldots, |x_n|_p) = \sum_k c_k |x|_p^k := \sum_{(k_1,\ldots,k_n)} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n |x_i|_p^{k_i},$$

for $x = (x_1, \ldots, x_n) \in B_L^n$. Notice that

$$\sup_{x\in B_L^n} \left| f(|x_1|_p, \ldots, |x_n|_p) - \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n |x_i|_p^{k_i} \right|$$
$$\le \sup_{\max_i|z_i|<R} \left| f(z_1, \ldots, z_n) - \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^n z_i^{k_i} \right|,$$

and consequently $\sum_{|k|\leq M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^{n} |x_i|_p^{k_i}$ *converges uniformly to* $f(|x_1|_p,\ldots,|x_n|_p)$ *on* B_L^n. *Then*

$$\int_{B_L^n} f(|x_1|_p,\ldots,|x_n|_p)d^n x = \lim_{M\to\infty} \sum_{|k|\leq M} c_{(k_1,\ldots,k_n)} \int_{B_L^n} \prod_{i=1}^{n} |x_i|_p^{k_i}\, d^n x$$

$$= \lim_{M\to\infty} \sum_{|k|\leq M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^{n} \int_{B_L} |x_i|_p^{k_i}\, dx_i. \tag{3.6}$$

3.5. **The change of variables formula in dimension one.** Let us start by establishing the formula:

$$d\,(ax) = |a|_p\, dx,\ a \in \mathbb{Q}_p^{\times}, \tag{3.7}$$

which means the following:

$$\int_{aU} dx = |a|_p \int_U dx,$$

for every Borel set $U \subseteq \mathbb{Q}_p$, for instance an open compact subset. Indeed, consider

$$\begin{array}{rccc} T_a: & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p \\ & & & \\ & x & \longmapsto & ax, \end{array}$$

with $a \in \mathbb{Q}_p^{\times}$. T_a is a topological and algebraic isomorphism. Then $U \mapsto \int_{aU} dx$ is a Haar measure for $(\mathbb{Q}_p,+)$, and by the uniqueness of such measure, there exists a positive constant $C(a)$ such that $\int_{aU} dx = C(a) \int_U dx$. To compute $C(a)$ we can pick any open compact set, for instance $U = \mathbb{Z}_p$, and then we must show

$$\textstyle\int_{a\mathbb{Z}_p} dx = C(a) = |a|_p\,.$$

Let us consider first the case $a \in \mathbb{Z}_p$, i.e. $a = p^l u$, $l \in \mathbb{N}$, $u \in \mathbb{Z}_p^{\times}$. Fix a system of representatives $\{b\}$ of $\mathbb{Z}_p/p^l\mathbb{Z}_p$ in $\mathbb{Z}_p$, then

$$\mathbb{Z}_p = \bigsqcup_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} b + p^l\mathbb{Z}_p,$$

and

$$\begin{aligned} 1 = \int_{\mathbb{Z}_p} dx &= \sum_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{b+p^l\mathbb{Z}_p} dx = \sum_{b\in\mathbb{Z}_p/p^l\mathbb{Z}_p} \int_{p^l\mathbb{Z}_p} dx \\ &= \#\left(\mathbb{Z}_p/p^l\mathbb{Z}_p\right) \int_{p^l\mathbb{Z}_p} dx, \end{aligned}$$

i.e.

$$p^{-l} = |a|_p = \int_{p^l\mathbb{Z}_p} dx = \int_{a\mathbb{Z}_p} dx.$$

The case $a \notin \mathbb{Z}_p$ is treated in a similar way.

Now, if we take $f : U \to \mathbb{C}$, where U is a Borel set, then

$$\int\limits_{U} f(x)\, dx = |a|_p \int\limits_{a^{-1}U - a^{-1}b} f(ay+b)\ dy, \text{ for any } a \in \mathbb{Q}_p^{\times}, b \in \mathbb{Q}_p.$$

The formula follows by changing variables as $x = ay + b$. Then we get $dx = d(ay+b) = d(ay) = |a|_p\, dy$, because the Haar measure is invariant under translations and formula (3.7).

Example 8. *For any $r \in \mathbb{Z}$,*

$$\int\limits_{B_r} dx = \int\limits_{p^{-r}\mathbb{Z}_p} dx = p^r \int\limits_{\mathbb{Z}_p} dy = p^r.$$

Example 9. *For any $r \in \mathbb{Z}$,*

$$\int\limits_{S_r} dx = \int\limits_{B_r} dx - \int\limits_{B_{r-1}} dx = p^r - p^{r-1} = p^r\left(1 - p^{-1}\right).$$

Example 10. *Take $U = \mathbb{Z}_p \smallsetminus \{0\}$. We show that*

$$\int\limits_{U} dx = \int\limits_{\mathbb{Z}_p} dx = 1.$$

Notice that U is not compact, since the sequence $\{p^n\}_{n\in\mathbb{N}} \subseteq U$ converges to $0 \notin U$. Now, by using

$$\mathbb{Z}_p \smallsetminus \{0\} = \bigsqcup_{j=0}^{\infty} \left\{x \in \mathbb{Z}_p : |x|_p = p^{-j}\right\},$$

we have

$$\begin{aligned}\int\limits_{\mathbb{Z}_p\smallsetminus\{0\}} dx &= \sum_{j=0}^{\infty} \int\limits_{p^j\mathbb{Z}_p^{\times}} dx && \textit{(by changing variables as } x = p^j y,\ dx = p^{-j}dy\textit{)}\\ &= \left(\sum_{j=0}^{\infty} p^{-j}\right) \int\limits_{\mathbb{Z}_p^{\times}} dy = \frac{1-p^{-1}}{1-p^{-1}} = 1.\end{aligned}$$

This calculation shows that $\mathbb{Z}_p \smallsetminus \{0\}$ has Haar measure 1 and that $\{0\}$ has Haar measure 0.

Example 11. *Set*

$$Z(s) := \int\limits_{\mathbb{Z}_p\smallsetminus\{0\}} |x|_p^s\ dx,\ s \in \mathbb{C} \textit{ with } \operatorname{Re}(s) > -1.$$

We prove that $Z(s)$ has a meromorphic continuation to the whole complex plane as a rational function of p^{-s}.

Indeed,

$$\begin{aligned}Z(s) &= \int\limits_{\mathbb{Z}_p\smallsetminus\{0\}} |x|_p^s\ dx = \sum_{j=0}^{\infty} \int\limits_{|x|_p = p^{-j}} |x|_p^s\ dx = \sum_{j=0}^{\infty} p^{-js} \int\limits_{|x|_p=p^{-j}} dx\\ &= \left(1-p^{-1}\right) \sum_{j=0}^{\infty} p^{-j(s+1)} \ \textit{(here we need the hypothesis } \operatorname{Re}(s) > -1\textit{)}\\ &= \frac{(1-p^{-1})}{1-p^{-1-s}}, \quad \textit{for } \operatorname{Re}(s) > -1.\end{aligned}$$

We now note that the right hand-side is defined for any complex number $\operatorname{Re}(s) \neq -1$*, therefore, it gives a meromorphic continuation of* $Z(s)$ *to the half-plane* $\operatorname{Re}(s) < -1$*. Thus, we have shown that* $Z(s)$ *has a meromorphic continuation to the whole* $\mathbb{C}$ *with a simple pole at* $\operatorname{Re}(s) = -1$*.*

Example 12. *We compute*

$$Z(s, x^2-1) = \int_{\mathbb{Z}_p} \left|x^2-1\right|_p^s \, dx, \quad \textit{for} \quad \operatorname{Re}(s) > -1,\, p \neq 2.$$

Let us take $\{0, 1, \ldots, p-1\} \subset \mathbb{Z} \subset \mathbb{Z}_p$ *as a system of representatives of* $\mathbb{F}_p \simeq \mathbb{Z}_p/p\mathbb{Z}_p$*. Then*

$$\mathbb{Z}_p = \bigsqcup_{j=0}^{p-1} (j + p\mathbb{Z}_p),$$

and

$$\begin{aligned} Z(s, x^2-1) &= \sum_{j=0}^{p-1} \int_{j+p\mathbb{Z}_p} \left|(x-1)\,(x+1)\right|_p^s dx \\ &= p^{-1} \sum_{j=0}^{p-1} \int_{\mathbb{Z}_p} \left|(j-1+py)\,(j+1+py)\right|_p^s dy, \quad (x = j + py). \end{aligned}$$

Let us consider first the integrals in which $j \mp 1 + py \in \mathbb{Z}_p^\times$*, i.e. the reduction mod* p *of* $j \mp 1$ *is a nonzero element of* $\mathbb{F}_p$*, in this case*

$$\int_{\mathbb{Z}_p} \left|(j-1+py)\,(j+1+py)\right|_p^s dy = 1,$$

and since $p \neq 2$ *there are exactly* $p-2$ *of those* j*'s, then*

$$\begin{aligned} Z(s, x^2-1) &= (p-2)\,p^{-1} + p^{-1} \int_{\mathbb{Z}_p} \left|py\,(2+py)\right|_p^s dy + p^{-1} \int_{\mathbb{Z}_p} \left|(-2+py)\,py\right|_p^s dy \\ &= (p-2)\,p^{-1} + 2p^{-1-s} \int_{\mathbb{Z}_p} |y|_p^s \, dy = (p-2)\,p^{-1} + 2p^{-1-s} \frac{1-p^{-1}}{1-p^{-1-s}}. \end{aligned}$$

Exercise 10. *Take* $q(x) = \prod_{i=1}^{r} (x-\alpha_i)^{e_i} \in \mathbb{Z}_p[x]$*,* $\alpha_i \in \mathbb{Z}_p$*,* $e_i \in \mathbb{N} \smallsetminus \{0\}$*. Assume that* $\alpha_i \not\equiv \alpha_j \mod p$*. By using the methods presented in examples 11 and 12 compute the integral*

$$Z(s, q(x)) = \int_{\mathbb{Z}_p} |q(x)|_p^s \, dx.$$

3.6. **Improper Integrals.** Our next task is the integration of functions that do not have compact support. A function $f : \mathbb{Q}_p^n \to \mathbb{C}$ is said to be *locally integrable*, $f \in L^1_{loc}$, if

$$\int_K f(x)\, d^n x$$

exists for every compact subset K.

Definition 9 (Improper Integral)**.** *A function $f \in L^1_{loc}$ is said to be integrable in $\mathbb{Q}_p^n$, if*

$$\lim_{N\to+\infty} \int\limits_{B_N^n} f(x)\, d^n x = \lim_{N\to+\infty} \sum_{j=-\infty}^{N} \int\limits_{S_j^n} f(x)\, d^n x$$

exists. If the limit exists, it is denoted as $\int_{\mathbb{Q}_p^n} f(x)\, d^n x$, and we say that the improper integral exists.

Note that in this case,

$$\int\limits_{\mathbb{Q}_p^n} f(x)\, d^n x = \sum_{j=-\infty}^{+\infty} \int\limits_{S_j^n} f(x)\, d^n x.$$

Example 13. *The function $|x|_p$ is locally integrable but not integrable.*

Example 14. *Let $f : \mathbb{Q}_p \to \mathbb{C}$ be a radial function i.e. $f(x) = f(|x|_p)$. If $\sum\limits_{j=-\infty}^{+\infty} f(p^j)\, p^j < +\infty$. Then*

$$\int\limits_{\mathbb{Q}_p} f\big(|x|_p\big) dx = \sum_{j=-\infty}^{+\infty} \int\limits_{|x|_p = p^j} f\big(|x|_p\big)\, dx = \left(1 - p^{-1}\right) \sum_{j=-\infty}^{+\infty} f\left(p^j\right) p^j.$$

Exercise 11. *By using $\sum\limits_{r=0}^{+\infty} r p^{-r} = \frac{p}{(p-1)^2}$, show that*

$$\int\limits_{\mathbb{Z}_p} \ln\big(|x|_p\big)\, dx = -\frac{\ln p}{p-1}.$$

3.7. **Further remarks on integrals of continuous functions with compact support.**

Example 15 (Continuation of Example 7)**.** *With the notation given in Example 7 and using formula (3.6) and Example 11, we have*

$$\int\limits_{B_L^n} f(|x_1|_p, \ldots, |x_n|_p) d^n x = \lim_{M\to\infty} \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \int\limits_{B_L^n} \prod_{i=1}^{n} |x_i|_p^{k_i}\, d^n x$$

$$= \lim_{M\to\infty} \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^{n} \int\limits_{B_L^n} |x_i|_p^{k_i}\, dx_i = \lim_{M\to\infty} \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^{n} p^{L+Lk_i} \int\limits_{\mathbb{Z}_p} |y_i|_p^{k_i}\, dx_i$$

$$= \lim_{M\to\infty} \sum_{|k|\le M} c_{(k_1,\ldots,k_n)} \prod_{i=1}^{n} p^{L+Lk_i} \frac{(1-p^{-1})}{1-p^{-1-k_i}} = \left(1-p^{-1}\right)^n \sum_{k} \frac{p^{n(L+Lk_i)} c_{(k_1,\ldots,k_n)}}{\prod\limits_{i=1}^{n} \left(1-p^{-1-k_i}\right)}.$$

Example 16. *Let $f(x)=e^{-|x|_p}\Omega\left(|x|_p\right)$, where $\Omega\left(|x|_p\right)$ denotes the characteristic function of $\mathbb{Z}_p$. We compute first $\int f(x)dx$ by using Example 14:*

$$\begin{aligned}\int_{\mathbb{Q}_p} e^{-|x|_p}\Omega\left(|x|_p\right)dx &= \int_{\mathbb{Z}_p} e^{-|x|_p}dx=\sum_{j=0}^{\infty}\int_{p^j\mathbb{Z}_p^\times} e^{-|x|_p}dx\\ &= \sum_{j=0}^{\infty} e^{-p^{-j}}\int_{p^j\mathbb{Z}_p^\times}dx=\sum_{j=0}^{+\infty}p^{-j}\left(1-p^{-1}\right)e^{-p^{-j}}.\end{aligned}$$

We now compute the integral by using Example 15:

$$\int_{\mathbb{Z}_p} e^{-|x|_p}dx=\lim_{M\to\infty}\sum_{k=0}^{M}\frac{(-1)^k}{k!}\int_{\mathbb{Z}_p}|x|_p^k\,dx=\left(1-p^{-1}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\left(1-p^{-1-k}\right)}.$$

Consequently,

$$\sum_{j=0}^{\infty}p^{-j}\left(1-p^{-1}\right)e^{-p^{-j}}=\left(1-p^{-1}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\left(1-p^{-1-k}\right)}.$$

We invite the reader to verify this identity directly .

4. Change of variables formula

A function $h:U\to\mathbb{Q}_p$ is said to be *analytic on an open subset* $U\subseteq\mathbb{Q}_p^n$, if for every $b=(b_1,\dots,b_n)\in U$ there exists an open subset $\widetilde{U}\subset U$, with $b\in\widetilde{U}$, and a convergent power series $\sum_{i\in\mathbb{N}^n}a_i\left(x-b\right)^i$ for $x=(x_1,\dots,x_n)\in\widetilde{U}$, such that $h\left(x\right)=\sum_{i\in\mathbb{N}^n}a_i\left(x-b\right)^i$ for $x\in\widetilde{U}$, with $i=(i_1,\dots,i_n)$ and $\left(x-b\right)^i=\prod_{j=1}^n\left(x_j-b_j\right)^{i_j}$. In this case, $\frac{\partial}{\partial x_l}h\left(x\right)=\sum_{i\in\mathbb{N}^n}a_i\frac{\partial}{\partial x_l}\left(x-b\right)^i$ is a convergent power series.

Let U,V open subsets in $\mathbb{Q}_p^n$. A mapping $H:U\to V$, $H=(H_1,\dots,H_n)$ is called *analytic* if each H_i is analytic. The mapping H is said to be *bi-analytic* if H and H^{-1} are analytic.

Theorem 4.1. *Let $K_0,K_1\subset\mathbb{Q}_p^n$ be open compact subsets, and let $H=(H_1,\dots,H_n):K_1\to K_0$ be a bi-analytic map such that*

$$\det\left[\frac{\partial H_i}{\partial y_j}(z)\right]\neq 0,\ \textit{for any}\ \ z\in K_1.$$

If f is a continuous function on K_0, then

$$\int_{K_0} f\left(x\right)\ d^nx=\int_{K_1} f\left(\sigma(y)\right)\left|\det\left[\frac{\partial H_i}{\partial y_j}(y)\right]\right|_p\ d^ny,\qquad (x=H(y)).$$

For the proof of this theorem the reader may consult [10, Prop. 7.4.1] or [8, Section 10.1.2].

Example 17. *Set $U:=\mathbb{Q}_p\smallsetminus\left\{\frac{-d}{c}\right\}$ and $V:=\mathbb{Q}_p\smallsetminus\left\{\frac{a}{c}\right\}$, where $a,\ b,\ c,\ d\in\mathbb{Q}_p$, with $c\neq 0$. Consider the function*

$$\begin{array}{ccc} U & \to & V\\ \\ x & \to & y=\frac{ax+b}{cx+d},\end{array}$$

this is an analytic function in U, with inverse $x=\frac{dy-b}{-cy+a}$, which is analytic in V. Assume that $ad-bc\neq 0$, and take $\varphi:\mathbb{Q}_p\to\mathbb{C}$ a Bruhat-Schwartz function with support contained in V, then

$$\int\limits_{V}\varphi\left(y\right)dy=\int\limits_{U}\varphi\left(\frac{ax+b}{cx+d}\right)\frac{\left|ad-bc\right|_p}{\left|cx+d\right|_p^2}dx.$$

Example 18. *Compute $\int_{B_r^n}d^nx$, where $B_r^n=\left\{x\in\mathbb{Q}_p^n:\left\|x\right\|_p\leq p^r\right\}$. We first recall that*

$$B_r^n=p^{-r}\mathbb{Z}_p^n=\underbrace{\begin{array}{c}p^{-r}\mathbb{Z}_p\times\ldots\times p^{-r}\mathbb{Z}_p\\ n-copies\end{array}}.$$

By changing variables as $x_i=p^{-r}y_i$ for $i=1,\ldots,n$, we have $d^nx=p^{nr}d^ny$, and

$$\int_{B_r^n}d^nx=p^{nr}\int_{\mathbb{Z}_p^n}d^nx=p^{nr}.$$

Exercise 12. *Show that*

$$\int_{S_r^n}d^nx=p^{nr}\left(1-p^{-1}\right).$$

Exercise 13. *To generalize Example 11 to $\mathbb{Q}_p^n$, i.e. to show that the integral*

$$Z(s)=\int_{\mathbb{Z}_p^n\smallsetminus\{0\}}\left\|x\right\|_p^s d^nx,\ \text{for}\ \operatorname{Re}(s)>0,$$

admits an analytic continuation to the whole complex plane as a rational function of p^{-s}.

Exercise 14. *Let α be a real number. To show that*

$$I\left(\alpha\right)=\int_{\mathbb{Z}_p^n\smallsetminus\{0\}}\frac{1}{\left\|x\right\|_p^{\alpha}}d^nx<\infty\ \text{if}\ \alpha<n.$$

Exercise 15. *Let β be a real number. To show that*

$$J\left(\beta\right)=\int_{\mathbb{Q}_p^n\smallsetminus\mathbb{Z}_p^n}\frac{1}{\left\|x\right\|_p^{\beta}}d^nx<\infty\ \text{if}\ \beta>n.$$

Example 19. *Take $N\geq 4$ and complex variables s_{1j} and $s_{(N-1)j}$ for $2\leq j\leq N-2$ and s_{ij} for $2\leq i<j\leq N-2$. Put $\boldsymbol{s}:=(s_{ij})\in\mathbb{C}^{\boldsymbol{d}}$, where $\boldsymbol{d}=\frac{N(N-3)}{2}$ denotes the total number of indices ij. The Koba-Nielsen local zeta functions is defined as follows:*

$$Z^{(N)}\left(\boldsymbol{s}\right)=\int\limits_{\mathbb{Q}_p^{N-3}}\prod_{i=2}^{N-2}\left|x_j\right|_p^{s_{1j}}\left|1-x_j\right|_p^{s_{(N-1)j}}\prod_{2\leq i<j\leq N-2}\left|x_i-x_j\right|_p^{s_{ij}}\prod_{i=2}^{N-2}dx_i,$$

where $\prod_{i=2}^{N-2}dx_i$ is the normalized Haar measure on $\mathbb{Q}_p^{N-3}$. If $N=4$, we have

$$Z^{(4)}\left(\boldsymbol{s}\right)=\int\limits_{\mathbb{Q}_p}\left|x\right|_p^{s_{12}}\left|1-x\right|_p^{s_{32}}\ dx. \tag{4.1}$$

Notice that in integral (4.1) does not contain a test function, and consequently its convergence is not direct. In order to regularize it, we proceed as follows:

$$\begin{aligned} Z^{(4)}(s_{12},s_{32}) &= \int\limits_{\mathbb{Z}_p} |x|_p^{s_{12}} |1-x|_p^{s_{32}} \ dx + \int\limits_{\mathbb{Q}_p \smallsetminus \mathbb{Z}_p} |x|_p^{s_{12}} |1-x|_p^{s_{32}} \ dx \\ &=: Z_0^{(4)}(s_{12},s_{32}) + Z_1^{(4)}(s_{12},s_{32}). \end{aligned}$$

To study integral $Z_0^{(4)}(s_{12},s_{32})$, we use that $\mathbb{Z}_p = \bigsqcup_{j=0}^{p-1} j + p\mathbb{Z}_p$, to get

$$Z_0^{(4)}(s_{12},s_{32}) = \sum_{j=0}^{p-1} \int\limits_{j+p\mathbb{Z}_p} |x|_p^{s_{12}} |1-x|_p^{s_{32}} \ dx =: \sum_{j=0}^{p-1} Z_{0,j}^{(4)}(s_{12},s_{32}).$$

Now, for $j \neq 0, 1$,

$$Z_{0,j}^{(4)}(s_{12},s_{32}) = p^{-1}.$$

In the case $j = 0$,

$$Z_{0,0}^{(4)}(s_{12},s_{32}) = \int\limits_{p\mathbb{Z}_p} |x|_p^{s_{12}} \ dx = p^{-1-s_{12}} \left(\frac{1-p^{-1}}{1-p^{-1-s_{12}}} \right).$$

In the case $j = 1$,

$$\begin{aligned} Z_{0,1}^{(4)}(s_{12},s_{32}) &= \int\limits_{1+p\mathbb{Z}_p} |1-x|_p^{s_{32}} dx = p^{-1-s_{32}} \int\limits_{\mathbb{Z}_p} |y|_p^{s_{32}} dy \\ &= p^{-1-s_{32}} \left(\frac{1-p^{-1}}{1-p^{-1-s_{32}}} \right). \end{aligned}$$

Consequently,

$$Z_0^{(4)}(s_{12},s_{32}) = (p-2)\,p^{-1} + p^{-1-s_{32}} \left(\frac{1-p^{-1}}{1-p^{-1-s_{32}}} \right) + p^{-1-s_{32}} \left(\frac{1-p^{-1}}{1-p^{-1-s_{32}}} \right).$$

Notice that $Z_0^{(4)}(s_{12},s_{32})$ is holomorphic, and consequently the underlying integrals converge, in

$$\operatorname{Re}(s_{12}) > -1 \ \textit{and} \ \operatorname{Re}(s_{32}) > -1. \tag{4.2}$$

To study $Z_1^{(4)}(s_{12},s_{32})$, we first notice that

$$Z_1^{(4)}(s_{12},s_{32}) = \int\limits_{\mathbb{Q}_p \smallsetminus \mathbb{Z}_p} |x|_p^{s_{12}} |1-x|_p^{s_{32}} dx = \int\limits_{\mathbb{Q}_p \smallsetminus \mathbb{Z}_p} |x|_p^{s_{12}+s_{32}} dx.$$

Now, we use the change of variables:

$$x = \frac{1}{y}, \ dx = \frac{dy}{|y|_p^2},$$

then

$$\begin{aligned} Z_1^{(4)}(s_{12},s_{32}) &= \int\limits_{\mathbb{Q}_p\smallsetminus\mathbb{Z}_p} |x|_p^{s_{12}+s_{32}}\,dx = \int\limits_{p\mathbb{Z}_p} |y|_p^{-2-s_{12}-s_{32}}\,dy \\ &= p^{-1+2+s_{12}+s_{32}} \int\limits_{\mathbb{Z}_p} |y|_p^{-2-s_{12}-s_{32}}\,dy = p^{-1+2+s_{12}+s_{32}}\left(\frac{1-p^{-1}}{1-p^{s_{12}+s_{32}+1}}\right). \end{aligned}$$

Thus $Z_1^{(4)}(s_{12},s_{32})$ *is holomorphic in*

$$\mathrm{Re}(s_{12})+\mathrm{Re}(s_{32})+1<0. \tag{4.3}$$

Finally, it is not difficult to see that conditions (4.2)-(4.3) define an open set in $\mathbb{C}$.

The calculation presented in this example still requires some additional work, since it involves non compact subsets. The 'missing' part can be obtained easily by applying the dominated convergence theorem.

5. Additive characters

Given a nonzero p-adic number

$$x = x_{-m}p^{-m} + x_{-m+1}p^{-m+1} + \ldots + x_{-1}p^{-1} + x_0 + xp + \ldots \text{ with } x_{-m}\neq 0 \text{ and } m>0,$$

we define its *fractional part as*

$$\{x\}_p = x_{-m}p^{-m} + x_{-m+1}p^{-m+1} + \ldots + x_{-1}p^{-1} \in \mathbb{Q}.$$

If $x\in\mathbb{Z}_p$, we set $\{x\}_p := 0$. Now the function

$$\chi_p(x) := \exp 2\pi i \{x\}_p$$

is called the *standard additive character* of $\mathbb{Q}_p$ (more precisely of $(\mathbb{Q}_p,+)$). Notice that

$$\chi_p : (\mathbb{Q}_p,+) \to (S,\cdot)$$

is a continuous homomorphism from $(\mathbb{Q}_p,+)$ into the unit complex circle considered as a multiplicative group, i.e. χ_p satisfies the following:

(i) $\left|\chi_p(x)\right| = 1$ for $x\in\mathbb{Q}_p$;
(ii) $\chi_p(x+y) = \chi_p(x)\,\chi_p(y)$ for $x,\, y\in\mathbb{Q}_p$;
(iii) $\overline{\chi_p(x)} = \frac{1}{\chi_p(x)} = \chi_p(-x)$ for $x\in\mathbb{Q}_p$, where the bar means complex conjugate;
(iv) $\chi_p(x)\not\equiv 1$ for $x\in\mathbb{Q}_p\smallsetminus\mathbb{Z}_p$.

Example 20. *Let r be an integer. To show that*

$$\int_{B_r}\chi_p(\xi x)\,dx = \begin{cases} p^r & \text{if } \quad |\xi|_p \le p^{-r} \\ \\ 0 & \text{if } \ |\xi|_p \ge p^{-r+1}. \end{cases}$$

If $|\xi|_p\le p^{-r}$, *then* $|\xi x|_p \le 1$ *which means that* $\xi x\in\mathbb{Z}_p$ *and thus* $\chi_p(\xi x)\equiv 1$,

$$\int_{B_r}\chi_p(\xi x)\,dx = \int_{B_r} dx = p^r.$$

If $|\xi|_p \geq p^{-r+1}$, *there exists* $x_0 \in S_r$, *i.e.* $|x_0|_p = p^r$, *such that* $|x_0\xi|_p \geq p$, *i.e.* $x_0\xi \in \mathbb{Q}_p \smallsetminus \mathbb{Z}_p$ *and thus we may assumee that* $\chi_p(x_0\xi) \not\equiv 1$. *We now change variables as* $x = y + x_0$, *notice that* y *runs through* $p^{-r}\mathbb{Z}_p$, *to get*

$$\chi_p(\xi x_0) \int\limits_{|y|_p \leq p^r} \chi_p(\xi y)\, dy = \int\limits_{|x|_p \leq p^r} \chi_p(\xi x)\, dx,$$

which implies the announced formula.

Exercise 16. *Let* r *be an integer. For* $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, *we set* $x \cdot \xi := \sum_i x_i \xi_i$. *To show that*

$$\int_{B_r^n} \chi_p(x \cdot \xi)\, d^n x = \begin{cases} p^{rn} & \text{if} \quad \|\xi\|_p \leq p^{-nr} \\ \\ 0 & \text{if} \quad \|\xi\|_p \geq p^{-nr+1}. \end{cases}$$

Hint: remenber that $x = p^{ord(x)}\widetilde{x}$, *with* $\|\widetilde{x}\|_p = 1$ *and* $\xi = p^{ord(\xi)}\widetilde{\xi}$, *with* $\left\|\widetilde{\xi}\right\|_p = 1$. *Thus* $x \cdot \xi = p^{ord(x)+ord(\xi)}\widetilde{x} \cdot \widetilde{\xi}$.

Exercise 17. *Let* r *be an integer. To show that*

$$\int_{S_r} \chi_p(\xi x)\, dx = \begin{cases} p^r(1 - p^{-1}) & \text{if} \quad |\xi|_p \leq p^{-r} \\ \\ -p^{r-1} & \text{if} \quad |\xi|_p = p^{-r+1} \\ \\ 0 & \text{if} \quad |\xi|_p \geq p^{-r+2}. \end{cases}$$

Hint: use Example 20 and

$$\int_{S_r} \chi_p(\xi x)\, dx = \int_{B_r} \chi_p(\xi x)\, dx - \int_{B_{r-1}} \chi_p(\xi x)\, dx.$$

Exercise 18. *Extend the formula given in Example 14 to* n*-dimensional case, i.e. for radial functions* $f : \mathbb{Q}_p^n \to \mathbb{C}$ *i.e.* $f(x) = f(\|x\|_p)$.

Exercise 19. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a function such that* $\sum\limits_{r=0}^{\infty} |f(p^{-r})|\, p^{-r} < +\infty$. *Then*

$$\int\limits_{\mathbb{Q}_p} f\left(|x|_p\right) \chi_p(\xi x)\, dx = \frac{(1 - p^{-1})}{|\xi|_p} \sum_{r=0}^{\infty} f\left(\frac{p^{-r}}{|\xi|_p}\right) p^{-r} - \frac{1}{|\xi|_p} f\left(\frac{p}{|\xi|_p}\right) \quad \textit{for } \xi \neq 0,$$

in the sense of improper integrals.

By using this exercise with $f \equiv 1$, we get that

$$\int\limits_{\mathbb{Q}_p} \chi_p(\xi x)\, dx = \begin{cases} \infty & \text{if} \quad \xi = 0 \\ \\ 0 & \text{if} \quad \xi \neq 0 \end{cases} = \delta(\xi), \text{ the Dirac distribution.}$$

Which is the Fourier transform of the constant function 1 is the Dirac distribution (or Dirac delta function).

6. Fourier Analysis on $\mathbb{Q}_p^n$

6.1. **Some function spaces.** For $1 \leq \rho < \infty$, we denote by $L^\rho := L^\rho\left(\mathbb{Q}_p^n, d^n x\right)$, the $\mathbb{C}$-vector space of all the complex-valued and Borel measurable functions f satisfying

$$\|f\|_\rho := \left\{ \int\limits_{\mathbb{Q}_p^n} |f(x)|^\rho \, d^n x \right\}^{\frac{1}{\rho}} < \infty.$$

For $\rho = \infty$, $f \in L^\infty := L^\infty\left(\mathbb{Q}_p^n, d^n x\right)$, if

$$\|f\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{Q}_p^n} |f(x)| < \infty. \tag{6.1}$$

The condition appearing on the right-hand side in (6.1) means that function f is bounded almost everywhere, i.e. this condition may be false in a set of measure zero. L^ρ is a Banach space if we identify functions f and g satisfying $f(x) = g(x)$ almost everywhere. This means that $\|\cdot\|_\rho$ is norm, that that L^ρ is a complete metric space for the distance induced by $\|\cdot\|_\rho$.

We denote by $C_0 := C_0(\mathbb{Q}_p^n, \mathbb{C})$ the $\mathbb{C}$-vector space of continuous functions on $\mathbb{Q}_p^n$ that vanish at infinity endowed with the L^∞-norm. The condition 'vanish at infinity' means that for any $\epsilon > 0$ there exists a compact subset $K \subset \mathbb{Q}_p^n$ such that

$$|f(x)| < \epsilon \text{ for } x \in \mathbb{Q}_p^n \smallsetminus K.$$

Remark 5. *Lebesgue's dominated convergence theorem. Let $f_m : \mathbb{Q}_p^n \to \mathbb{C}$, $m \in \mathbb{N}$, be a sequence of complex-valued Borel measurable functions. Suppose that the sequence converges pointwise to a function f and that there exists an integrable function g such that $|f_m(x)| \leq g(x)$ for any $x \in \mathbb{Q}_p^n$ and all m, then f is integrable and*

$$\lim_{m \to \infty} \int_{\mathbb{Q}_p^n} f_m(x) \, d^n x = \int_{\mathbb{Q}_p^n} \lim_{m \to \infty} f_m(x) \, d^n x = \int_{\mathbb{Q}_p^n} f(x) d^n x.$$

6.2. **The Fourier transform.** Given $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n$, we define

$$x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n.$$

If $f \in L^1$ its Fourier transform is the function $\widehat{f}$ defined by

$$\widehat{f}(\xi) = \int\limits_{\mathbb{Q}_p^n} f(x) \, \chi_p(x \cdot \xi) \, d^n x.$$

We also use the notation $\mathcal{F}_{x \to \xi}(f)$, $\mathcal{F}(f)$ to denote the Fourier transform of f.

Lemma 6.1. *(i) The mapping $f \to \widehat{f}$ is a bounded linear mapping from L^1 to L^∞ satisfying $\left\|\widehat{f}\right\|_\infty \leq \|f\|_1$.*

(ii) If $f \in L^1$, then $\widehat{f}$ is uniformly continuous.

Proof. (i)

$$\left|\widehat{f}\left(\xi\right)\right| = \left|\int\limits_{\mathbb{Q}_p^n} f\left(x\right)\chi_p\left(x\cdot\xi\right)d^nx\right| \leq \int\limits_{\mathbb{Q}_p^n}\left|f\left(x\right)\right|d^nx = \left\|f\right\|_1.$$

(ii) Notice that

$$\widehat{f}\left(\xi+h\right)-\widehat{f}\left(\xi\right) = \int\limits_{\mathbb{Q}_p^n} f\left(x\right)\chi_p\left(x\cdot\xi\right)\left\{\chi_p\left(x\cdot h\right)-1\right\}d^nx,$$

and since $\left|f\left(x\right)\chi_p\left(x\cdot\xi\right)\left\{\chi_p\left(x\cdot h\right)-1\right\}\right| \leq 2\left|f\left(x\right)\right| \in L^1$, by using the dominated convergence theorem,

$$\lim_{h\to 0}\left|\widehat{f}\left(\xi+h\right)-\widehat{f}\left(\xi\right)\right| \leq \int\limits_{\mathbb{Q}_p^n}\left|f\left(x\right)\right|\lim_{h\to 0}\left|\chi_p\left(x\cdot h\right)-1\right|d^nx = 0,$$

i.e. for any $\epsilon>0$, there is $\delta>0$, such that for any ξ', $\xi\in\mathbb{Q}_p^n$, with $\left\|\xi'-\xi\right\|_p = \left\|h\right\|_p < \delta$, it holds that $\left|\widehat{f}\left(\xi'\right)-\widehat{f}\left(\xi\right)\right| < \epsilon$. □

Remark 6. *The translation operator $\mathcal{T}_h$, $h\in\mathbb{Q}_p^n$, is defined by $\left(\mathcal{T}_hf\right)\left(x\right) = f\left(x-h\right)$. If $f\in L^1$, then $\widehat{\left(\mathcal{T}_hf\right)}\left(\xi\right) = \chi_p\left(\xi\cdot h\right)\widehat{f}\left(\xi\right)$, and $\mathcal{F}_{x\to\xi}(\chi_p\left(x\cdot h\right)f\left(x\right)) = \left(\mathcal{T}_{-h}\widehat{f}\right)\left(\xi\right) = \widehat{f}\left(\xi+h\right)$.*

We denote by $\Delta_k\left(x\right) := \Omega\left(p^{-k}\left\|x\right\|_p\right)$ the characteristic function of the ball $B_k^n = p^{-k}\mathbb{Z}_p^n$ for $k\in\mathbb{Z}$. A locally constant function with compact support (a test function) $\varphi:\mathbb{Q}_p^n\to\mathbb{C}$ is a linear combination of characteristic functions of balls, then

$$\varphi\left(x\right) = \sum_{i=1}^{m} c_i\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right).$$

Notice that $\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right)$ is the characteristic function of the ball $h_i + p^{-k_i}\mathbb{Z}_p^n$. We denote by $\mathcal{D}(\mathbb{Q}_p^n)$ the $\mathbb{C}$–vector space of test functions (the Bruhat-Schwartz).

We set $\delta_k\left(x\right) = p^{kn}\Omega\left(p^k\left\|x\right\|_p\right)$ for $k\in\mathbb{Z}$. Notice that δ_k satisfies

$$\int_{\mathbb{Q}_p^n}\delta_k\left(x\right)d^nx = 1 \text{ for any } k\in\mathbb{Z}_p.$$

Exercise 20. *To show that $\widehat{\Delta}_k\left(\xi\right) = \delta_k\left(\xi\right)$. Hence if $\varphi\left(x\right) = \sum_{i=1}^m c_i\mathcal{T}_{h_i}\Delta_{k_i}\left(x\right)$ is a test function, then $\widehat{\varphi}\left(\xi\right) = \sum_{i=1}^m c_i\chi_p\left(\xi\cdot h\right)\delta_k\left(\xi\right)$. Consequently, $\widehat{\varphi}\left(\xi\right)$ is also a test function.*

Remark 7. *Notice that $\lim_{k\to\infty}\Delta_k\left(x\right) = 1$, and that*

$$\lim_{k\to\infty}\delta_k\left(x\right) = \begin{cases}\infty & \text{if} \quad x=0\\ \\ 0 & \text{if} \quad x\neq 0.\end{cases}$$

Exercise 21. *Set $h_t\left(\xi\right) := \int_{\mathbb{Q}_p}\chi_p\left(\xi x\right)e^{-t|x|_p^\alpha}dx$, for $t>0$, $\alpha>0$. Show that $h_t\left(\xi\right)$ is a continuous function in x for $t>0$ fixed.*

Proposition 6.1 ([23, Chap. I, Proposition 1.3]). *$\mathcal{D}(\mathbb{Q}_p^n)$ is dense in C_0 as well as in L^ρ, $1 \leq \rho < \infty$.*

Proposition 6.2 (Riemann-Lebesgue Theorem). *If $f \in L^1$, then $\widehat{f}(\xi) \to 0$ as $\|\xi\|_p \to \infty$.*

Proof. For $g \in \mathcal{D}(\mathbb{Q}_p^n)$, $\widehat{g}$ has compact support. Fix $\epsilon > 0$, then there exists $g_\epsilon \in \mathcal{D}(\mathbb{Q}_p^n)$ such that $\|f - g_\epsilon\|_1 < \epsilon$, cf. Proposition 6.1. For $\xi \notin$supp $\widehat{g}_\epsilon$ we have

$$\left|\widehat{f}(\xi)\right| = \left|\widehat{f}(\xi) - \widehat{g}_\epsilon(\xi)\right| \leq \left\|\widehat{(f - g_\epsilon)}\right\|_\infty \leq \|f - g_\epsilon\|_1 < \epsilon.$$

□

Definition 10. *Given f, $g : \mathbb{Q}_p^n \to \mathbb{C}$ its convolution is the function*

$$\begin{aligned} h(x) &= f(x) * g(x) = \int_{\mathbb{Q}_p^n} f(x-z)g(z)d^nz \\ &= \int_{\mathbb{Q}_p^n} f(z)g(x-z)d^nz, \end{aligned}$$

when the defining integral exists.

Remark 8. *Young's inequality. Assume that $f \in L^\rho$, $g \in L^\sigma$ and $\frac{1}{\rho} + \frac{1}{\sigma} = \frac{1}{\gamma} + 1$ with $1 \leq \rho, \sigma, \gamma \leq \infty$. Then $\|f * g\|_\gamma \leq \|f\|_\rho \|g\|_\sigma$.*

The following proposition is left as an exercise to the reader.

Proposition 6.3. *If $f \in L^\rho$, $1 \leq \rho \leq \infty$, and $g \in L^1$, then $f * g \in L^\rho$ and $\|f * g\|_\rho \leq \|f\|_\rho \|g\|_1$.*

Remark 9. *Fubini's theorem. Let $f : \mathbb{Q}_p^{n+m} \to \mathbb{C}$ be a function such that the repeated integral*

$$\int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^m} f(x,y)d^my \right) d^nx$$

exists, then $f \in L^1\left(\mathbb{Q}_p^{n+m}, d^{n+m}x\right)$, and the following formulae hold:

$$\begin{aligned} \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^m} f(x,y)d^my \right) d^nx &= \int_{\mathbb{Q}_p^{n+m}} f(x,y)d^nxd^my \\ &= \int_{\mathbb{Q}_p^m} \left(\int_{\mathbb{Q}_p^n} f(x,y)d^nx \right) d^my. \end{aligned}$$

Lemma 6.2. *If f, $g \in L^1$, then $\widehat{f * g} = \widehat{f}\ \widehat{g}$.*

Proof. By Proposition 6.3 $f * g \in L^1$. The formula for the Fourier transform follows from Fubini's theorem. We invite the redear to verify this calculation. □

Lemma 6.3. *If f, $g \in L^1$, then*

$$\int_{\mathbb{Q}_p^n} \widehat{f}(y)g(y)\,d^ny = \int_{\mathbb{Q}_p^n} f(x)\widehat{g}(x)\,d^ny.$$

Proof. By Fubini's theorem and the definition of Fourier transform:

$$\int_{\mathbb{Q}_p^n} \widehat{f}(y)g(y)\,d^n y = \int_{\mathbb{Q}_p^n} \left\{ \int_{\mathbb{Q}_p^n} \chi_p(x\cdot y)\, f(x)\, d^n x \right\} g(y)\, d^n y$$

$$\int_{\mathbb{Q}_p^n} \left\{ \int_{\mathbb{Q}_p^n} \chi_p(x\cdot y)\, g(y)\, d^n y \right\} f(x)\, d^n x = \int_{\mathbb{Q}_p^n} f(x)\widehat{g}(x)\, d^n x.$$

□

6.3. **The Fourier transform on the space of test functions.** Let $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ be a locally constant function, this means that for each $x \in \mathbb{Q}_p^n$, there exists an integer $l = l(x)$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_l^n. \tag{6.2}$$

Since $B_l^n = \left\{ x \in \mathbb{Q}_p^n : \|x\|_p \le p^l \right\} = p^{-l}\mathbb{Z}_p^n$, condition (6.2) is equivalent to

$$\varphi\mid_{x+p^{-l}\mathbb{Z}_p^n} \equiv \varphi(x).$$

If φ is a test function, then supp φ is open compact, and consequently there exist a finite number of integers l_i and a finite number of points z_i in $\mathbb{Q}_p^n$ such that

$$\text{supp } \varphi = \bigsqcup_{i=1}^{r} z_i + p^{-l_i}\mathbb{Z}_p^n.$$

We set

$$k := \max_{1\le i\le r} -l_i.$$

Then $z_i + p^{-l_i}\mathbb{Z}_p^n \supset z_i + p^k\mathbb{Z}_p^n$ (i.e. $B_{-k}^n(z_i) \subset B_{l_i}^n(z_i)$) and

$$\varphi\mid_{x+p^{k}\mathbb{Z}_p^n} \equiv \varphi(x) \text{ for any } x \in \text{supp } \varphi.$$

This means that φ is constant on the cosets of $p^k\mathbb{Z}_p^n$ (i.e. on the cosets of $\mathbb{Q}_p^n/p^k\mathbb{Z}_p^n$). We now use the fact that supp φ is compact, which means that it is closed and bounded, then there exists an integer m such that

$$\text{supp } \varphi \subset p^m\mathbb{Z}_p^n.$$

Naturally, for any $x \in$supp φ, $x + p^k\mathbb{Z}_p^n \subset p^m\mathbb{Z}_p^n$, which implies that $k \ge m$.

In conclusion, $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ if and only there exist integers k, m, with $k \ge m$, such that φ is constant on the cosets of $p^k\mathbb{Z}_p^n$ (i.e. on the cosets of $p^m\mathbb{Z}_p^n/p^k\mathbb{Z}_p^n$) and is supported on $p^m\mathbb{Z}_p^n$. These functions form $\mathbb{C}$-vector space denoted as $\mathcal{D}_k^m(\mathbb{Q}_p^n) := \mathcal{D}_k^m$. We fix a set of representatives Is of $p^m\mathbb{Z}_p^n/p^k\mathbb{Z}_p^n =: G_{m,n}$, then the characteristic functions of the balls $I + p^k\mathbb{Z}_p^n$, $I \in G_{m,n}$ span $\mathcal{D}_k^m$, i.e.

$$\left\{ \Omega\left(p^k \|x - I\|_p \right) \right\}_{I\in G_{m,k}}$$

are a basis for $\mathcal{D}_k^m$. Notice that the dimension of $\mathcal{D}_k^m$ is $\#G_{m,k} = p^{(m-k)n}$.

Lemma 6.4. $\mathcal{F}(\mathcal{D}_k^m) \subset \mathcal{D}_{-m}^{-k}$.

Proof. Take $\varphi \in \mathcal{D}_k^m$, since $\varphi(x) = \sum_{I \in G_{m,k}} c_I \Omega\left(p^k \|x - I\|_p\right)$, and $\mathcal{F}$ is a linear operator, we may assume that $\varphi(x) = \Omega\left(p^k \|x - I\|_p\right)$. Then

$$\begin{aligned}\mathcal{F}_{x\to\xi}\left(\Omega\left(p^k \|x - I\|_p\right)\right) &= \int_{I + p^k \mathbb{Z}_p^n} \chi_p(\xi \cdot x)\, dx = p^{-nk}\chi_p(\xi \cdot I) \int_{\mathbb{Z}_p^n} \chi_p\left(p^k \xi \cdot y\right) dy \\ &= p^{-nk}\chi_p(\xi \cdot I)\, \Omega\left(p^{-k}\|\xi\|_p\right).\end{aligned}$$

Finally, we verify that if $\|\xi_0\|_p \le p^{-m}$, then

$$\chi_p(\xi \cdot I)\mid_{\xi_0 + p^m \mathbb{Z}_p^n} = \chi_p(\xi_0 \cdot I),$$

and consequently $p^{-nk}\chi_p(\xi \cdot I)\, \Omega\left(\left\|p^k \xi\right\|_p\right) \in \mathcal{D}_{-m}^{-k}$. □

Exercise 22. *Show the following assertion. The map*

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p^n) & \to & \mathcal{D}(\mathbb{Q}_p^n) \\ \varphi & \to & \widehat{\varphi}, \end{array} \tag{6.3}$$

where $\widehat{\varphi}(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\, \varphi(x)\, dx$, *is a well-defined linear operator, with inverse given by*

$$\varphi(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\, \widehat{\varphi}(\xi)\, d\xi.$$

In other words, the mapping (6.3) is a isomorphism of $\mathbb{C}$*-vector spaces on* $\mathcal{D}(\mathbb{Q}_p^n)$.

6.4. **The inverse Fourier transform.** One expects that the inverse Fourier transform be given by

$$f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\, \widehat{f}(\xi)\, d^n\xi = \int_{\mathbb{Q}_p^n} \overline{\chi_p(\xi \cdot x)}\, \widehat{f}(\xi)\, d^n\xi.$$

This formula does not always make sense since $\widehat{f}$ is not L^1 when $f \in L^1$.

Exercise 23. *Show that the Fourier transform of* $f(x) = \Omega\left(|x|_p\right) \ln\left(\frac{1}{|x|_p}\right)$ *is*

$$\widehat{f}(\xi) = \begin{cases} 0 & \text{if } |\xi|_p \le 1 \\ \left(\ln \frac{p}{1 - p^{-1}}\right) |\xi|_p^{-1} & \text{if } |\xi|_p > 1. \end{cases}$$

Definition 11. *If* g *is locally integrable and* $k \in \mathbb{Z}$, *we define*

$$A_k g = \int_{\mathbb{Q}_p^n} g(x)\, \Delta_k(x)\, d^n x = \int_{\|x\|_p \le p^k} g(x)\, d^n x.$$

Notice that if $g \in L^1$, then $A_k g \to \int_{\mathbb{Q}_p^n} g(x)\, d^n x$ as $k \to \infty$ (why?). Now, the limit $\lim_{k\to\infty} A_k g$ may exist even though $\int_{\mathbb{Q}_p^n} g(x)\, d^n x$ does not exist.

Exercise 24. *To show the following fact: if $f \in L^1$, $k \in \mathbb{Z}$, then*

$$\begin{aligned} A_k\left(\widehat{f}\chi_p(x\cdot)\right) &= \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\,\chi_p(x\cdot\xi)\,\Delta_k(\xi)\,d^n\xi \\ &= \int_{\mathbb{Q}_p^n} f(y)\,\delta_k(y-x)\,d^n y = p^{nk}\int_{\|x-y\|_p\leq p^{-k}} f(y)\,d^n y. \end{aligned}$$

Definition 12. *Let $f : \mathbb{Q}_p^n \to \mathbb{C}$ be locally integrable. A point $x \in \mathbb{Q}_p^n$ is called a regular point of f if*

$$f_k(x) := p^{nk}\int_{\|x-y\|_p\leq p^{-k}} f(y)\,d^n y \to f(x) \ \text{ as } k\to\infty.$$

Theorem 6.1 ([23, Theorem 1.14])**.** *Let f be a locally integrable function. There exists a zero measure subset $L = L(f)$ such that any $x \in \mathbb{Q}_p^n \smallsetminus L$ is a regular point of f.*

Exercise 25. *Assume that f is locally integrable and continuous at x. Show that*

$$p^{nk}\int_{\|x-y\|_p\leq p^{-k}} f(y)\,d^n y \to f(x) \ \text{ as } k\to\infty.$$

Corollary 6.1. *If $f \in L^1$, then*

$$A_k\left(\widehat{f}\ \overline{\chi_p(x\cdot)}\right) = \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\,\chi_p(-x\cdot\xi)\,\Delta_k(\xi)\,d^n\xi \to f(x)$$

almost everywhere. In particular, it converges at each point of continuity of f.

Proof. The corollary follows from Exercise 24, Theorem 6.1 and Exercise 25. ☐

Theorem 6.2. *If f and $\widehat{f}$ are both integrable then f is equal a.e. to a continuous function. With f modified (on a set of measure zero) to be continuous, we have*

$$f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi\cdot x)\,\widehat{f}(\xi)\,d^n\xi \quad \text{for all } x\in\mathbb{Q}_p^n. \tag{6.4}$$

Proof. If $\widehat{f}$ is integrable, then $A_k\left(\widehat{f}\ \chi_p(-\xi\cdot)\right)$ converges to a continuous function:

$$\int_{\mathbb{Q}_p^n} \chi_p(-\xi\cdot x)\,\widehat{f}(\xi)\,d^n\xi. \tag{6.5}$$

By Corollary 6.1, $f(x)$ agrees with (6.5) a.e. and consequently f is continuous almost everywhere. By modifying f on a set of measure zero we obtain formula (6.4). ☐

Exercise 26. *Define*

$$\mathcal{L}(\mathbb{Q}_p^n) := \left\{ f:\mathbb{Q}_p^n\to\mathbb{C}:\ f,\,\widehat{f}\in L^1 \text{ and } f,\,\widehat{f} \text{ are continuous.}\right\}$$

Then $\mathcal{L}(\mathbb{Q}_p^n) \xrightarrow{\mathcal{F}} \mathcal{L}(\mathbb{Q}_p^n)$ is an isomorphism of $\mathbb{C}$-vector spaces. In particular, the formula

$$f(0) = \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\,d^n\xi$$

holds.

Corollary 6.2. *If f, $g \in L^1$ and $\widehat{f} = \widehat{g}$ a.e., then $f(x) = g(x)$ a.e.*

Proof. Since $\widehat{(f-g)} = 0$, by Theorem 6.2, $(f-g)(x) = 0$ a.e. □

Remark 10. *Monotone convergence lemma (Levi's monotone convergence theorem). Let $h_k : \mathbb{Q}_p^n \to [0, \infty]$, $k \in \mathbb{N}$, be a sequence of non-negative Borel measurable functions satisfying*

$$0 \leq h_k(x) \leq h_{k+1}(x) \leq \infty \quad \text{for any } x \in \mathbb{Q}_p^n.$$

Assume that $h : \mathbb{Q}_p^n \to [0, \infty]$ is the pointwise limit of the sequence $\{h_k\}_{k \in \mathbb{N}}$. Then h is Borel measurable and

$$\lim_{k \to \infty} \int_{\mathbb{Q}_p^n} h_k(x)\, d^n x = \int_{\mathbb{Q}_p^n} h(x) d^n x.$$

Corollary 6.3. *If $f \in L^1$, $\widehat{f} \geq 0$ and f is continuous at zero, then $\widehat{f} \in L^1$ and $f(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) \widehat{f}(\xi)\, d^n\xi$ at each regular point of f. In particular, $f(0) = \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, d^n\xi$.*

Proof. We need only to show that $\widehat{f} \in L^1$. Since f and $\Delta_k \in L^1$, by Lemma 6.3 and Exercise 20, we have

$$\int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, \Delta_k(\xi)\, d^n\xi = \int_{\mathbb{Q}_p^n} f(\xi)\, \delta_k(\xi)\, d^n\xi.$$

By Theorem 6.1,

$$f(0) = \lim_{k \to \infty} \int_{\mathbb{Q}_p^n} f(\xi)\, \delta_k(\xi)\, d^n\xi = \lim_{k \to \infty} p^{nk} \int_{\|\xi\|_p \leq p^{-k}} \widehat{f}(\xi)\, d^n\xi,$$

i.e.

$$f(0) = \lim_{k \to \infty} \int_{\mathbb{Q}_p^n} \widehat{f}(\xi)\, \Delta_k(\xi)\, d^n\xi.$$

Finally, by using the fact that $\widehat{f} \geq 0$, and monotone convergence lemma, we have $\widehat{f} \in L^1$. □

7. The L^2-theory

Theorem 7.1. *If $f \in L^1 \cap L^2$, then $\left\| \widehat{f} \right\|_2 = \|f\|_2$.*

Proof. We set $g(x) := \overline{f}(-x)$, then $\widehat{g} = \overline{\widehat{f}}$. Since f, $g \in L^1$, $f * g \in L^1$ and

$$\widehat{f * g} = \widehat{f}\widehat{g} = \left|\widehat{f}\right|^2 \geq 0,$$

cf. Proposition 6.3 and Lemma 6.2.

Now, since f, $g \in L^2$, then $f * g$ is continuous, indeed, by using the Cauchy-Schwarz inequality,

$$\left|(f*g)(x+y)-(f*g)(x)\right| = \left|\int_{\mathbb{Q}_p^n} \left\{f(x+y-z)-f(x-z)\right\} g(z) d^n z\right|$$

$$\leq \sqrt{\int_{\mathbb{Q}_p^n} |g(z)|^2\, d^n z}\ \sqrt{\int_{\mathbb{Q}_p^n} |f(x+y-z)-f(x-z)|^2\, d^n z}$$

$$\|g\|_2 \sqrt{\int_{\mathbb{Q}_p^n} |f(u+y)-f(u)|^2\, d^n z} = \|g\|_2 \left\|f(\cdot+y)-f(\cdot)\right\|_2 \to 0 \text{ as } y \to 0,$$

by the dominated convergence theorem and the fact that

$$|f(u+y)-f(u)|^2 \leq 4\max\left\{|f(u+y)|^2, |f(u)|^2\right\} \leq 4\left\{|f(u+y)|^2+|f(u)|^2\right\}.$$

We now apply Corollary 6.3 to $f * g$, with $\widehat{f*g} \geq 0$, to get that $\widehat{f*g} = \left|\widehat{f}\right|^2 \in L^1$ and

$$(f*g)(0) = \int_{\mathbb{Q}_p^n} \left|\widehat{f}(\xi)\right|^2 d^n\xi = \int_{\mathbb{Q}_p^n} g(-z) f(z) d^n z = \int_{\mathbb{Q}_p^n} |f(z)|^2\, d^n z.$$

□

Remark 11. *Let $(Y, \|\cdot\|_Y)$ be a Banach space, this means that $(Y, \|\cdot\|_Y)$ is a normed complex space such that Y is a complete metric space for the distance induced by $\|\cdot\|_Y$. Let $(X, \|\cdot\|_X)$ be a complex normed space, and let $D(T)$ be a subspace of $(X, \|\cdot\|_X)$. Let $T : D(T) \to Y$ be a linear bounded operator, i.e. T satisfies $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for any α, $\beta \in \mathbb{C}$, and any x, $y \in D(T)$, and*

$$\|T\| := \sup_{x \in D(T)} \frac{\|T(x)\|_Y}{\|x\|_X} < \infty.$$

Then T has an extension $\widetilde{T} : \overline{D(T)} \to Y$, where $\overline{D(T)}$ denotes the closure of $D(T)$ in $(X, \|\cdot\|_X)$, with the same norm $\left\|\widetilde{T}\right\| = \|T\|$. If $\overline{D(T)} = X$, i.e. if $D(T)$ is dense in X, then $\widetilde{T}$ is unique.

Remark 12. *Let $T : X \to Y$ be a bounded (i.e. $\|T\| := \sup_{x \in X} \frac{\|T(x)\|_Y}{\|x\|_X} < \infty$) linear operator. Then T is continuous if and only if T is bounded.*

From this theorem, it follows that the mapping

$$\begin{array}{ccc} L^1 \cap L^2 & \to & L^2 \\ & & \\ f & \to & \widehat{f} \end{array}$$

is an L^2-isometry on $L^1 \cap L^2$, which is a dense subspace of L^2 (Why?). Thus, this mapping has an extension to an L^2-isometry from L^2 into L^2. We now extend the Fourier transform to L^2.

Definition 13. *For $f \in L^2$, let*

$$f_k := f \Delta_k, \quad \text{for } k \in \mathbb{N}, \tag{7.1}$$

and

$$\widehat{f}(\xi) := \lim_{k \to \infty} \widehat{f_k}(\xi) = \lim_{k \to \infty} \int_{\|x\|_p \leq p^k} \chi_p(\xi \cdot x) f(x) d^n x,$$

where the limit is taken in L^2.

Lemma 7.1. *If f, $g \in L^2$, then*

$$\int_{\mathbb{Q}_p^n} \widehat{f} \, g d^n y = \int_{\mathbb{Q}_p^n} f \, \widehat{g} d^n y.$$

Proof. We first notice that $f_k \underrightarrow{L^2} f$ and $g_k \underrightarrow{L^2} g$, and that f_k, $g_k \in L^1 \cap L^2$ for every k. Hence, by Lemma 6.3,

$$\int_{\mathbb{Q}_p^n} f_k \, \widehat{g_k} d^n x = \int_{\mathbb{Q}_p^n} \widehat{f_k} \, g_k d^n x.$$

Now, by using Theorem 7.1 and the Cauchy-Schwarz inequality,

$$\left| \int_{\mathbb{Q}_p^n} f_k \, \widehat{g_k} d^n x \right| \leq \|f_k\|_2 \|\widehat{g_k}\|_2 = \|f_k\|_2 \|g_k\|_2,$$

which means that the bilinear form

$$(f_k, g_k) \to \int_{\mathbb{Q}_p^n} f_k \, \widehat{g_k} d^n x$$

is bounded (and consequently continuous) in each variable in L^2, then

$$\int_{\mathbb{Q}_p^n} f_k \, \widehat{g_k} d^n x \underrightarrow{L^2} \int_{\mathbb{Q}_p^n} f \, \widehat{g} d^n x.$$

A similar result holds for the bilinear form $(f_k, g_k) \to \int_{\mathbb{Q}_p^n} \widehat{f_k} \, g_k d^n x$. □

Theorem 7.2. *The Fourier transform is unitary in L^2.*

Proof. We have to show that the Fourier transform is a bijective linear mapping that preserves the L^2-norm. We already know that $f \underrightarrow{\mathcal{F}} \widehat{f}$ is a linear L^2-isometry. It remains to show that that it is onto. By contradiction, we assume that $\mathcal{F}$ is not onto. Notice that $\mathcal{F}(L^2)$ is closed in L^2 because $\left\|\widehat{f}\right\|_2 = \|f\|_2$, if $\mathcal{F}(L^2) \neq L^2$, then by general theory of Hilbert spaces, $\mathcal{F}(L^2)$ has an orthogonal complement $\mathcal{F}(L^2)^\perp$ such that $L^2 = \mathcal{F}(L^2) \oplus \mathcal{F}(L^2)^\perp$, where for any $g \in \mathcal{F}(L^2)^\perp$ and any $\widehat{f} \in \mathcal{F}(L^2)$, $\left\langle \widehat{f}, g \right\rangle = 0$. Then, there exists $\overline{h} \in L^2$, $\left\|\overline{h}\right\|_2 \neq 0$ such that

$$\left\langle \widehat{f}, \overline{h} \right\rangle = \int_{\mathbb{Q}_p^n} \widehat{f} h d^n x = 0 \text{ for any } f \in L^2.$$

By using Lemma 7.1, $\widehat{h} = 0$, but $\left\|\overline{h}\right\|_2 = \|h\|_2 = \left\|\widehat{h}\right\|_2 = 0$, cf. Theorem 7.1, which contradicts $\left\|\overline{h}\right\|_2 \neq 0$. □

Exercise 27. *Show that* $\left\|\widehat{f}\right\|_2 = \|f\|_2$ *for* $f \in L^2$ *is equivalent to for any* f, $g \in L^2$,

$$\langle f, g\rangle = \left\langle \widehat{f}, \widehat{g}\right\rangle \text{ i.e. } \int_{\mathbb{Q}_p^n} f\ \overline{g} d^n x = \int_{\mathbb{Q}_p^n} \widehat{f}\ \overline{\widehat{g}} d^n x.$$

Exercise 28. *For* $f \in L^2$, *we set* $\widetilde{f}$ *for the reflection of* f *define as* $\widetilde{f}(x) = f(-x)$. *Show that if* $f \in L^2$, *then*

$$\mathcal{F}^{-1}(f) = \mathcal{F}\left(\widetilde{f}\right).$$

Theorem 7.3. *If* $f \in L^2$, *then*

$$\lim_{k\to\infty} \int_{\|x\|_p \le p^k} \chi_p(\xi \cdot x) f(x) d^n x = \widehat{f}(\xi) \quad \textit{almost everywhere.}$$

Proof. We use that Δ_k, f, $\widehat{f} \in L^2$, jointly with Lemma 7.1 and Exercise 27 to get

$$\begin{aligned}\lim_{k\to\infty} \int_{\|x\|_p \le p^k} \chi_p(\xi \cdot x) f(x) d^n x &= \lim_{k\to\infty} \int_{\mathbb{Q}_p^n} f(x) \overline{\chi_p(-\xi \cdot x)\Delta_k(x)} d^n x \\ &= \lim_{k\to\infty} \int_{\mathbb{Q}_p^n} \widehat{f}(\xi) \overline{\delta_k(\xi - x)} d^n x = p^{nk} \int_{\|\xi - x\|_p \le p^{-k}} \widehat{f}(\xi) d^n \xi.\end{aligned}$$

Now, since $\widehat{f} \in L^2$, then $\widehat{f} \in L^1_{loc}$, the result follows from Theorem 6.1. $\square$

Remark 13. *(i) The Fourier transform can be extended to* $L^1 + L^2$, *which means that if* $f = f_1 + f_2$, *with* $f_1 \in L^1$, $f_2 \in L^2$, *then* $\widehat{f} = \widehat{f_1} + \widehat{f_2}$, *where the Fourier transforms are defined in* L^1 *and* L^2 *respectively. The function* $\widehat{f}$ *is well-defined in* L^1_{loc}.

(ii) If $f \in L^1$, $g \in L^\rho$, $\rho \in [1, 2]$, *then* $\widehat{f * g} = \widehat{f}\ \widehat{g}$ *a.e., cf.* [23, Theorem 2.7].

8. $\mathcal{D}$ as a topological vector space

We define a topology on $\mathcal{D}$ as follows. We say that a sequence $\left\{\varphi_j\right\}_{j\in\mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

(C1) there are two fixed integers k and m such that each φ_j is constant on the cosets of $p^k\mathbb{Z}_p^n$ and is supported on $p^m\mathbb{Z}_p^n$, i.e. $\varphi_j \in \mathcal{D}_k^m$;

(C2) $\varphi_j \to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

We recall that $\mathcal{D}_k^m$ is $\mathbb{C}$-vector space of dimension $N_{m.k} := \#\left(p^m\mathbb{Z}_p^n / p^k\mathbb{Z}_p^n\right)$. Given $c = (c_1, \ldots, c_{N_{m.k}}) \in \mathbb{C}^{N_{m.k}}$, we set $\|c\|_{\mathbb{C}} = \max_i |c_i|$. Then $\left(\mathbb{C}^{N_{m.k}}, \|\cdot\|_{\mathbb{C}}\right)$ is a Banach space and

$$\left(\mathbb{C}^{N_{m.k}}, \|\cdot\|_{\mathbb{C}}\right) \simeq \mathcal{D}_k^m \text{ as topological spaces, (why?).}$$

A key fact is that uniform convergence in $\mathcal{D}_k^m$ agrees with the convergence in the supremum norm (L^∞-norm), which in turn agrees with the convergence in the $\|\cdot\|_{\mathbb{C}}$-norm.

Exercise 29. *Show that* $\mathcal{D}$ *is a complete and separable topological vector space.*

Theorem 8.1. *The map*

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p^n) & \rightarrow & \mathcal{D}(\mathbb{Q}_p^n) \\ & & \\ \varphi & \rightarrow & \widehat{\varphi}, \end{array} \tag{8.1}$$

where $\widehat{\varphi}(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x)\,dx$, *is a homeomorphism of topological vector spaces , with inverse given by*

$$\varphi(x) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x)\widehat{\varphi}(\xi)\,d\xi.$$

Proof. We already know that (8.1) is an isomorphism of vector spaces. It remains to show that the continuity of $\mathcal{F}$ and $\mathcal{F}^{-1}$. Let $\varphi_j \underrightarrow{\mathcal{D}} \varphi$, i.e. $\varphi_j, \varphi \in \mathcal{D}_k^m$ for some integers m, k, and $\varphi_j \underrightarrow{\text{unif.}} \varphi$. Since $\widehat{\varphi_j}, \widehat{\varphi} \in \mathcal{D}_{-m}^{-k}$, in order to show that $\widehat{\varphi_j} \underrightarrow{\mathcal{D}} \widehat{\varphi}$, it is sufficient to establish that $\widehat{\varphi_j} \underrightarrow{\text{unif.}} \widehat{\varphi}$, i.e. $\left\|\widehat{\varphi_j} - \widehat{\varphi}\right\|_\infty \rightarrow 0$ as $j \rightarrow \infty$. By using that

$$\begin{aligned} \left\|\widehat{\varphi_j} - \widehat{\varphi}\right\|_\infty &\leq \int_{\mathbb{Q}_p^n} \left|\varphi_j - \varphi\right| d^n x \leq \left\|\varphi_j - \varphi\right\|_\infty \int_{p^m \mathbb{Z}_p^n} dx \\ &= p^{-mn} \left\|\varphi_j - \varphi\right\|_\infty \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

The continuity of the inverse Fourier transform is established by using the same argument. □

9. The space of distributions on $\mathbb{Q}_p^n$

The $\mathbb{C}$-vector space $\mathcal{D}'\left(\mathbb{Q}_p^n\right) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}$ is called the *Bruhat-Schwartz space of distributions.* We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'$ converges to T if

$$\lim_{j\rightarrow\infty} T_j(\varphi) = T(\varphi) \text{ for any } \varphi \in \mathcal{D}.$$

Exercise 30. *Define the map*

$$\begin{array}{ccc} \mathcal{D}' \times \mathcal{D} & \rightarrow & \mathbb{C} \\ & & \\ (T, \varphi) & \rightarrow & T(\varphi). \end{array}$$

Then (T, φ) *is a bilinear form which is continuous in* T *and* φ *separately. We call this map the pairing between* $\mathcal{D}'$ *and* $\mathcal{D}$. *From now on we will use* (T, φ) *instead of* $T(\varphi)$.

Exercise 31. *If* $f \in L^\rho$, $1 \leq \rho \leq \infty$, *then* f *induces a distribution. More precisely,*

$$(f, \varphi) = \int_{\mathbb{Q}_p^n} f\varphi d^n x$$

Remark 14. *If* T *is a distribution and* g *is a locally integrable function such that*

$$(T, \varphi) = \int_{\mathbb{Q}_p^n} g\varphi d^n x \text{ for all } \varphi \in \mathcal{D},$$

we identify T *with function* g. *In this case, some authors say that* T *is a regular distribution.*

Example 21. *(i) The distribution* $(\delta, \varphi) = \varphi(0)$ *is called the Dirac distribution.*
(ii) $(1, \varphi) = \int_{\mathbb{Q}_p^n} \varphi d^n x$.

Lemma 9.1. *Every linear functional on* $\mathcal{D}$ *is continuous, i.e.* $\mathcal{D}'$ *agrees with the algebraic dual of* $\mathcal{D}$.

Proof. Let $T : \mathcal{D} \to \mathbb{C}$ be a linear functional, and let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of test functions converging to 0. Then $\varphi_j \in \mathcal{D}_k^m$, for all j, consequently

$$\varphi_j(x) = \sum_{I \in G_{m,k}} \varphi_j(I)\, \Omega\left(p^k \|x - I\|_p\right),$$

and $\max_{I \in G_{m,k}} \left|\varphi_j(I)\right| \to 0$ as $j \to \infty$. Then

$$T\varphi_j(x) = \sum_{I \in G_{m,k}} \varphi_j(I)\, T\left(\Omega\left(p^k \|x - I\|_p\right)\right) = \sum_{I \in G_{m,k}} c_{k,I}\ \varphi_j(I),$$

and

$$\left|T\varphi_j(x)\right| \leq \sum_{I \in G_{m,k}} |c_{k,I}|\ \ \left|\varphi_j(I)\right| \leq \left\{\sum_{I \in G_{m,k}} |c_{k,I}|\right\} \max_{I \in G_{m,k}} \left|\varphi_j(I)\right| \to 0,$$

as $j \to \infty$. □

10. The Fourier transform on $\mathcal{D}'$

Definition 14. *For* $T \in \mathcal{D}'$ *its Fourier transform, denoted as* $\mathcal{F}(T)$ *or* $\widehat{T}$, *is the distribution defined as*

$$(\mathcal{F}(T), \varphi) = (T, \mathcal{F}(\varphi)) \text{ for all } \varphi \in \mathcal{D}.$$

Notice that since $\varphi \to \mathcal{F}(\varphi)$ *is a homeomorphism of* $\mathcal{D}$, $\mathcal{F}(T)$ *is well-defined.*

Example 22. $\widehat{\delta} = 1$. *Indeed,*

$$\left(\widehat{\delta}, \varphi\right) = (\delta, \widehat{\varphi}) = \widehat{\varphi}(0) = \int_{\mathbb{Q}_p^n} \varphi d^n x = (1, \varphi) \text{ for all } \varphi \in \mathcal{D}.$$

Definition 15. *The inverse Fourier transform of* $T \in \mathcal{D}'$, *denoted as* $\mathcal{F}^{-1}(T)$ *or* $\check{T}$, *is defined as*

$$\left(\mathcal{F}^{-1}(T), \varphi\right) = \left(T, \mathcal{F}^{-1}(\varphi)\right) \text{ for all } \varphi \in \mathcal{D}.$$

Remark 15. *Notice that* $\mathcal{F}^{-1}(T) \in \mathcal{D}'$ *and that* $\mathcal{F}^{-1}(\mathcal{F}(T)) = T$ *for any* $T \in \mathcal{D}'$.

Lemma 10.1. *The map* $T \to \mathcal{F}(T)$ *is a homeomorphism of* $\mathcal{D}'$ *onto* $\mathcal{D}'$.

Proof. The onto part follows from Remark 15. For the continuity, we proceed as follows. Let $T_j \underset{\to}{\mathcal{D}'} T$, i.e. $(T_j, \varphi) \underset{\to}{} (T, \varphi)$ for all $\varphi \in \mathcal{D}$. Then

$$\left(\widehat{T}_j, \varphi\right) = (T_j, \widehat{\varphi}) \to (T, \widehat{\varphi}) \text{ for all } \varphi \in \mathcal{D},$$

i.e.

$$\left(\widehat{T}_j, \varphi\right) \to \left(\widehat{T}, \varphi\right) \text{ for all } \varphi \in \mathcal{D}.$$

□

Definition 16. *Linear change of variables for distributions. For $A \in GL_n\left(\mathbb{Q}_p^n\right)$ and $b \in \mathbb{Q}_p^n$, we define*

$$(T\left(Ax+b\right),\varphi) = \frac{1}{|\det A|_p}\left(T,\varphi\left(A^{-1}\left(y-b\right)\right)\right).$$

Example 23. *Recall that $\mathcal{T}_b(\varphi\left(x\right)) = \varphi\left(x-b\right)$ for $b \in \mathbb{Q}_p^n$ and $\varphi \in \mathcal{D}$. Then, for $G \in \mathcal{D}'$,*

$$(\mathcal{T}_b G,\varphi) = (G,\mathcal{T}_{-b}\varphi).$$

Example 24. *The reflection operator, denoted as $\widetilde{\bullet}$, acting on $\varphi \in \mathcal{D}$ is defined as $\widetilde{\varphi}\left(x\right) = \varphi\left(-x\right)$. Then Then, for $G \in \mathcal{D}'$, $\left(\widetilde{G},\varphi\right) = (G,\widetilde{\varphi})$.*

Exercise 32. *Show that*

$$G \to \mathcal{F}\left(G\right),\ G \to \mathcal{F}^{-1}\left(G\right),\ G \to \mathcal{T}_b G,\ G \to \widetilde{G}$$

are homeomorphisms of $\mathcal{D}'$ onto $\mathcal{D}'$.

Definition 17. *For $G \in \mathcal{D}'$ and $\theta \in \mathcal{D}$, we define the distribution θG by*

$$(\theta G,\varphi) = (G,\theta\varphi) \text{ for all } \varphi \in \mathcal{D}.$$

Why is this definition correct?

Exercise 33. *For ψ, $\varphi \in \mathcal{D}$, $G \in \mathcal{D}'$, the maps*

$$G \to \varphi G, \quad \psi \to \psi\varphi$$

are continuous maps from $\mathcal{D}'$ into $\mathcal{D}'$ and from $\mathcal{D}$ into $\mathcal{D}$, respectively.

Exercise 34. *Show that $\mathcal{F}_{x\to\xi}\left(\chi_p\left(x\cdot\xi_0\right)\right) = \delta\left(\xi-\xi_0\right)$ in $\mathcal{D}'$.*

Example 25. *We want to compute $\mathcal{F}_{\xi\to x}^{-1}\left(\frac{1}{\|\xi\|_p^\alpha+m^2}\right)$, where $\alpha > 0$ and $m > 0$. Notice that if $\alpha > n$, then $\frac{1}{\|\xi\|_p^\alpha+m^2} \in L^1$ (Why?). In the general case, $\frac{1}{\|\xi\|_p^\alpha+m^2} \notin L^1$. If we use the notion of improper integral, we have*

$$\int_{\mathbb{Q}_p^n} \frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^\alpha+m^2}d^n\xi = \sum_{j=-\infty}^{+\infty}\ \int_{\|\xi\|_p=p^j} \frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^\alpha+m^2}d^n\xi.$$

The exact mathematical meaning of this formula is

$$\sum_{j=-\infty}^{N}\ \int_{\|\xi\|_p=p^j} \frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^\alpha+m^2}d^n\xi \overset{\mathcal{D}'}{\to} \mathcal{F}_{\xi\to x}^{-1}\left(\frac{1}{\|\xi\|_p^\alpha+m^2}\right) \text{ as } N \to \infty.$$

Indeed, for any test function φ, we have

$$\sum_{j=-\infty}^{N}\ \int_{\mathbb{Q}_p^n}\ \int_{\|\xi\|_p=p^j} \frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^\alpha+m^2}\varphi\left(x\right)d^n\xi d^n x = \int_{\mathbb{Q}_p^n}\ \int_{\|\xi\|_p\leq p^N} \frac{\chi\left(-\xi\cdot x\right)}{\|\xi\|_p^\alpha+m^2}\varphi\left(x\right)d^n\xi d^n x$$

$$= \int_{\|\xi\|_p\leq p^N} \frac{1}{\|\xi\|_p^\alpha+m^2}\left\{\int_{\mathbb{Q}_p^n}\chi\left(-\xi\cdot x\right)\varphi\left(x\right)d^n x\right\}d^n\xi = \int_{\|\xi\|_p\leq p^N} \frac{\mathcal{F}^{-1}(\varphi)}{\|\xi\|_p^\alpha+m^2}d^n\xi$$

$$= \int_{\mathbb{Q}_p^n} \frac{\mathcal{F}^{-1}(\varphi)}{\|\xi\|_p^\alpha+m^2}d^n\xi = \left(\frac{1}{\|\xi\|_p^\alpha+m^2},\mathcal{F}^{-1}(\varphi)\right) \text{ for } N \text{ sufficiently large.}$$

Definition 18. *We say that a locally constant function f belongs to $\mathcal{U}_{loc}$ if and only if f is constant on the cosets of $p^k\mathbb{Z}_p^n$ for some $k \in \mathbb{Z}$.*

Example 26. *(i) $\mathcal{D} \subset \mathcal{U}_{loc}$. (ii) Consider $f \in \mathcal{U}_{loc}$ such that $|f(y)| e^{-t\|y\|_p^\alpha} \in L^1$ for $t > 0$. Then*

$$h(x) = \int_{\mathbb{Q}_p^n} \{f(x-y) - f(x)\} e^{-t\|y\|_p^\alpha} d^n y \in \mathcal{U}_{loc}.$$

Proposition 10.1. *$f \in \mathcal{U}_{loc}$ if and only if $f \in \mathcal{D}'$ and there is $k \in \mathbb{Z}$ such that $\mathcal{T}_x f = f$ for any $x \in p^k\mathbb{Z}_p^n$.*

Proof. If $f \in \mathcal{U}_{loc}$ and $\mathcal{U}_{loc} \subset L^1_{loc}$, then f gives rise an element from $\mathcal{D}'$ satisfying $(\mathcal{T}_x f, \varphi) = (f, \varphi)$ for any $x \in p^k\mathbb{Z}_p^n$.

Conversely, assume that $f \in \mathcal{D}'$, and $\mathcal{T}_x f = f$, for all $x \in p^k\mathbb{Z}_p^n$. We set

$$h(y) = p^{kn}(f, \mathcal{T}_y \Delta_k).$$

Then $h(y) \in \mathcal{U}_{loc}$. Indeed, since for all $x \in p^k\mathbb{Z}_p^n$,

$$(\mathcal{T}_x h)(y) = p^{kn}(f, \mathcal{T}_{y-x}\Delta_k) = p^{kn}(\mathcal{T}_x f, \mathcal{T}_y \Delta_k) = p^{kn}(f, \mathcal{T}_y\Delta_k) = h(y).$$

Now we show that $f = h$ in $\mathcal{D}'$. It is sufficient to verify

$$(h, \varphi) = (f, \varphi) \text{ i.e. } p^{kn} \int_{\mathbb{Q}_p^n} (\mathcal{T}_{-y} f, \Delta_k)\varphi(y)\, d^n y = (f, \varphi),$$

for φ equals to the characteristic function of a ball of type $J + p^l\mathbb{Z}_p^n$. If $l \geq k$, we can replace φ by the characteristic function of $J + p^k\mathbb{Z}_p^n$. If $k > l$, we decompose $J + p^l\mathbb{Z}_p^n$ into cosets modulo $p^k\mathbb{Z}_p^n$, i.e.

$$J + p^l\mathbb{Z}_p^n = \bigsqcup_{J_i} J_i + p^k\mathbb{Z}_p^n,$$

where the J_is runs through a finite set of representatives. Now we can replace an replace φ by the characteristic function of $J_i + p^k\mathbb{Z}_p^n$. Finally,

$$\begin{aligned}
\left(h, 1_{J_i + p^k\mathbb{Z}_p^n}\right) &= p^{kn} \int_{J_i + p^k\mathbb{Z}_p^n} (\mathcal{T}_{-y} f, \Delta_k)\, d^n y = \int_{\mathbb{Z}_p^n} \left(\mathcal{T}_{-J_i - p^k u} f, \Delta_k\right) d^n y \\
&= \int_{\mathbb{Z}_p^n} \left(\mathcal{T}_{-p^k u} f, \mathcal{T}_{J_i}\Delta_k\right) d^n y = \int_{\mathbb{Z}_p^n} (f, \mathcal{T}_{J_i}\Delta_k)\, d^n y = (f, \mathcal{T}_{J_i}\Delta_k) \\
&= \left(f, 1_{J_i + p^k\mathbb{Z}_p^n}\right).
\end{aligned}$$

□

Definition 19. *For $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, $T * \varphi$ is defined by $\widehat{T * \varphi} = \widehat{\varphi}\, \widehat{T}$.*

Exercise 35. *Show that for $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, the map $T \to T * \varphi$ is a continuous map from $\mathcal{D}'$ into $\mathcal{D}'$. See Exercise 33.*

Exercise 36. *To show the following formulas for φ, $\psi \in \mathcal{D}$, $G \in \mathcal{D}'$ and $x \in \mathbb{Q}_p^n$:*
(i) $\widetilde{(\mathcal{T}_{-x}\varphi)} = \mathcal{T}_x\widetilde{\varphi}$;
(ii) $\widetilde{(\mathcal{T}_{-x}G)} = \mathcal{T}_x\widetilde{G}$;

(iii) $\widetilde{(\varphi * \psi)} = \widetilde{\varphi} * \widetilde{\psi}$;
(iv) $\widetilde{(\widetilde{\varphi} * \psi)} = \varphi * \widetilde{\psi}$.

Proposition 10.2 ([23, Theorem 3.15]). *The following three characterizations of $\varphi * G \in \mathcal{D}'$ for $\varphi \in \mathcal{D}$, $G \in \mathcal{D}'$ are equivalent:*
(i) $\widehat{\varphi * G} = \widehat{\varphi}\ \widehat{G}$;
(ii) $(\varphi * G, \psi) = (G, \widetilde{\varphi} * \psi)$ *for* $\psi \in \mathcal{D}$;
(iii) $\varphi * G$ *belongs to* $\mathcal{U}_{loc}$ *and agrees with function*

$$g(x) = (G, T_x \widetilde{\varphi}) = (G(y), \varphi(x - y)).$$

Exercise 37. *Show that the map* $G \rightarrow fG$, $f \in \mathcal{U}_{loc}$ *is a continuous map of* $\mathcal{D}'$ *into* $\mathcal{D}'$.

Definition 20. *A distribution* $G \in \mathcal{D}'$ *has compact support if there is a* $k \in \mathbb{Z}$ *such that* $\Delta_k G = G$.

We denote by $\mathcal{D}'_{comp}$ the $\mathbb{C}$-vector space of distributions with compact support.

Theorem 10.1. $f \in \mathcal{U}_{loc}$ *if and only if* $\widehat{f}$ *has compact support. In particular, if* $f \in \mathcal{D}'$ *then* $\mathcal{T}_x f = f$ *for all* $x \in p^k\mathbb{Z}_p^n$ *if and only if* $\Delta_k \widehat{f} = \widehat{f}$.

Proof. Assume that $f \in \mathcal{U}_{loc}$ and that $\mathcal{T}_x f = f$ (i.e. $f \mid_{x+p^k\mathbb{Z}_p^n} \equiv f(x)$). By using Proposition 10.2-(iii), we have

$$\begin{aligned}\mathcal{F}^{-1}\left(\Delta_k \widehat{f}\right) &= \mathcal{F}^{-1}(\Delta_k) * f = \mathcal{F}(\Delta_k) * f = \delta_k * f \\ = p^{nk} \int\limits_{\|x-y\|_p \le p^{-k}} f(y) d^n y &= p^{nk} \int\limits_{x+p^k\mathbb{Z}_p^n} f(y) d^n y = p^{nk} f(x) \int\limits_{x+p^k\mathbb{Z}_p^n} d^n y = f(x),\end{aligned}$$

i.e. $\Delta_k \widehat{f} = \widehat{f}$.

Now suppose that $\widehat{f} \in \mathcal{D}'$ satisfying $\Delta_k \widehat{f} = \widehat{f}$. Then for any $\varphi \in \mathcal{D}$ and for all $x \in p^k\mathbb{Z}_p^n$,

$$\begin{aligned}(\mathcal{T}_x f, \varphi) &= (f, \mathcal{T}_{-x}\varphi) = \left(\mathcal{F}(f), \mathcal{F}^{-1}(\mathcal{T}_{-x}\varphi)\right) \\ = \left(\mathcal{F}(f)(\xi), \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right) &= \left(\Delta_k(\xi) \widehat{f}(\xi), \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right) \\ = \left(\widehat{f}(\xi), \Delta_k(\xi) \chi_p(x \cdot \xi) \mathcal{F}^{-1}(\varphi)(\xi)\right) &= \left(\widehat{f}, \mathcal{F}^{-1}(\varphi)\right) = (f, \varphi).\end{aligned}$$

□

Remark 16. *if* $\widehat{f} \in \mathcal{D}'$ *satisfies* $\mathcal{T}_x \widehat{f} = \widehat{f}$ *for all* $x \in p^k\mathbb{Z}_p^n \Leftrightarrow f \in \mathcal{D}'$ *with* $\Delta_k f = f$. *Henc*

$$\begin{aligned}\left(\widehat{f}, \varphi\right) &= (f, \widehat{\varphi}) = (\Delta_k f, \widehat{\varphi}) = (f, \Delta_k \widehat{\varphi}) \\ &= \left(f(\xi), \int\limits_{\mathbb{Q}_p^n} \Delta_k(\xi) \chi_p(x \cdot \xi) \varphi(x) d^n x\right) \\ &= \int\limits_{\mathbb{Q}_p^n} \left(f(\xi), \Delta_k(\xi) \chi_p(x \cdot \xi)\right) \varphi(x) d^n x,\end{aligned}$$

i.e. $\widehat{f}(\xi) = \left(f(\xi), \Delta_k(\xi)\chi_p(x\cdot\xi)\right)$. *Here, it is necessary to use that if* $T \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$, $G \in \mathcal{D}'\left(\mathbb{Q}_p^m\right)$, *then* $T \times G \in \mathcal{D}'\left(\mathbb{Q}_p^{n+m}\right)$ *suc that*

$$\begin{aligned}(T(x) \times G(y), \phi(x,y)) &= (T(x), (G(y), \phi(x,y))) \\ &= (G(y), (T(x), \phi(x,y))), \text{ for } \phi(x,y) \in \mathcal{D}\left(\mathbb{Q}_p^{n+m}\right).\end{aligned}$$

Definition 21. *If* T, $W \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *and* W *has compact support, then* $T*W \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *is defined by*

$$\widehat{T*W} = \widehat{T}\,\widehat{W}.$$

Exercise 38. *If* $W \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *has compact support, then the map* $T \to T*W$ *is a continuous map from* $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *into* $\mathcal{D}'\left(\mathbb{Q}_p^n\right)$.

Hint: Notice that the following mappings are continuous:

$$\begin{array}{lcl}\mathcal{D}'\left(\mathbb{Q}_p^n\right) & \to & \mathcal{D}'\left(\mathbb{Q}_p^n\right) \\ \\ \widehat{T} & \to & \widehat{T}\,\widehat{W},\end{array}$$

and

$$\begin{array}{lll}\mathcal{D}'\left(\mathbb{Q}_p^n\right) & \xrightarrow{\text{mult. } \widehat{W}} & \mathcal{D}'\left(\mathbb{Q}_p^n\right) \\ \widehat{T} & \to & \widehat{T}\,\widehat{W} \\ \updownarrow \mathcal{F}^{-1} & & \updownarrow \mathcal{F}^{-1} \\ \mathcal{D}'\left(\mathbb{Q}_p^n\right) & \xrightarrow{*W} & \mathcal{D}'\left(\mathbb{Q}_p^n\right). \\ T & \to & T*W\end{array}$$

Theorem 10.2. *(i) Let* $T_1, \ldots, T_s \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *all but (at most) one being in* $\mathcal{U}_{loc}$. *Then* $T_1 \cdots T_s \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *is well-defined as a conmutative and associative product.*

(ii) Let $W_1, \ldots, W_s \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *all but (at most) one being in* $\mathcal{D}_{comp}$. *Then* $T_1 * \cdots * T_s \in \mathcal{D}'\left(\mathbb{Q}_p^n\right)$ *is well-defined as a conmutative and associative convolution product.*

11. Some additional examples

Example 27. *(i) Take* $\phi \in \mathcal{D}$, *then the associated distribution has compact support.*

(ii) $\delta \in \mathcal{D}'$ *has compact support (Why?).*

(iii) Let $T \in \mathcal{D}'$, *then* $\delta * T = T$. *Indeed,*

$$\widehat{\delta * T} = \widehat{\delta}\,\widehat{T} = 1\widehat{T} = \widehat{T}.$$

(iv) Let $T \in \mathcal{D}'$, *then* $\delta_k * T \in \mathcal{D}'$, *more precisely* $\delta_k * T \in \mathcal{U}_{loc}$. *Then*

$$\lim_{k\to\infty} \delta_k * T = T \text{ in } \mathcal{D}'.$$

Indeed,

$$\lim_{k\to\infty} \widehat{\delta_k * T} = \lim_{k\to\infty} \widehat{\delta_k}\widehat{T} = \lim_{k\to\infty} \Delta_k\widehat{T} = \widehat{T} \text{ in } \mathcal{D}'.$$

Example 28. *$\mathcal{D}$ is dense in $\mathcal{D}'$, every distribution is the weak limit of a sequence of test functions. Take $T \in \mathcal{D}'$, then $\delta_k * T \in \mathcal{U}_{loc}$, $\Delta_l(\delta_k * T) \in \mathcal{D}$ for any l, k. We now show that*

$$\lim_{l\to\infty}\left(\lim_{k\to\infty}(\Delta_l\widehat{(\delta_k * T)})\right) = \widehat{T} \text{ in } \mathcal{D}'.$$

Indeed, $\Delta_l\widehat{(\delta_k * T)} = \delta_l * \widehat{(\delta_k * T)} = \delta_l * \Delta_k\widehat{T}$. For a fixed δ_l, the map

$$\begin{array}{ccc} \mathcal{D}' & \rightarrow & \mathcal{D}' \\ & & \\ G & \rightarrow & \delta_l * G \end{array}$$

is continuous. Then $\lim_{k\to\infty}\delta_l * \Delta_k\widehat{T} = \delta_l * \widehat{T}$. And consequently,

$$\lim_{l\to\infty}\left(\lim_{k\to\infty}(\Delta_l\widehat{(\delta_k * T)})\right) = \lim_{l\to\infty}\delta_l * \widehat{T} = \widehat{T} \text{ in } \mathcal{D}'.$$

Example 29. *Let $q(x) \in \mathbb{Q}_p[x] \smallsetminus \mathbb{Q}_p$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p)$. The Igusa local zeta function attached to the pair (q,φ) is the distribution*

$$\begin{array}{cccc} Z_\varphi(s;q) & \mathcal{D}(\mathbb{Q}_p) & \rightarrow & \mathbb{C} \\ & & & \\ & \varphi & \rightarrow & \displaystyle\int_{\mathbb{Q}_p\smallsetminus q^{-1}(0)} \varphi(x)\,|q(x)|_p^s\,dx, \end{array}$$

for $\mathrm{Re}(s) > 0$. Here we use that for $a > 0$ and $s \in \mathbb{C}$, $a^s = e^{s\ln a}$. In this example, we show that $Z_\varphi(s;q)$ has a meromorphic continuatiom to the whole complex plane such that $Z_\varphi(s;q)$ is a rational function in p^{-s}. We set

$$q(x) = R(x)\prod_{i=1}^{l}(x-\alpha_i)^{e_i},$$

where the α_i are the roots of $q(x)$ belonging to the suppφ and $R(x) \neq 0$ for any $x \in$ suppφ. Then there exists a covering of suppφ such that:

(i) supp$\varphi = \bigsqcup_{i=1}^{M} B_t(\widetilde{x}_i)$;

(ii) each ball $B_t(\widetilde{x}_i)$ contains up most one root of $q(x)$, say α_i. In this case we change $B_t(\widetilde{x}_i)$ by $B_t(\alpha_i)$;

(iii) $|R(x)|_p\,|_{B_t(\widetilde{x}_i)} \equiv |R(\widetilde{x}_i)|_p$ for $i = 1,\ldots,M$.

The announced properties of the covering follow from the following considerations. Since $\mathbb{Q}_p$ is a metric space and $\alpha_i \neq \alpha_j$ if $i \neq j$, there exist $t' \in \mathbb{Z}$ such that

$$B_{t'}(\alpha_i) \cap B_{t'}(\alpha_j) = \emptyset \text{ if } i \neq j.$$

Let $\widetilde{x}_i \in$ suppφ, since $|R(x)|_p \neq 0$ for any $x \in$ suppφ, there exist a ball $B_{t''}(\widetilde{x}_i)$ such that

$$0 < \inf_{x\in B_{t''}(\widetilde{x}_i)}|R(x)|_p < \sup_{x\in B_{t''}(\widetilde{x}_i)}|R(x)|_p < |R(\widetilde{x}_i)|_p.$$

Then by the ultrametric inequality $|R(x)|_p\,|_{B_{t''}(\widetilde{x}_i)} = |R(\widetilde{x}_i)|_p$.

Now assuming that $\widetilde{x}_i = \alpha_i$ *for* $i = 1, \ldots, l$, *with* $M > l$, *we have*

$$\begin{aligned}
Z_{\varphi}(s;q) &= \sum_{i=1}^{M} \int_{B_t(\widetilde{x}_i)} |q(x)|_p^s \, dx \\
&= \sum_{i=1}^{l} \int_{B_t(\alpha_i)} |R_i(x)|_p^s \, |x-\alpha_i|_p^{e_i s} \, dx + \sum_{i=l+1}^{M} \int_{B_t(\widetilde{x}_i)} |q(x)|_p^s \, dx \\
&= \sum_{i=1}^{l} |R_i(\alpha_i)|_p^s \int_{B_t(\alpha_i)} |x-\alpha_i|_p^{e_i s} \, dx + p^{-t} \sum_{i=l+1}^{M} |q(\widetilde{x}_i)|_p^s \\
&= \sum_{i=1}^{l} |R_i(\alpha_i)|_p^s \, p^{-t-e_i s} \left(\frac{1-p^{-1}}{1-p^{-e_i s-1}} \right) + p^{-t} \sum_{i=l+1}^{M} |q(\widetilde{x}_i)|_p^s .
\end{aligned}$$

References

[1] Albeverio S., Khrennikov A., and Shelkovich V. M., *Theory of p-adic distributions: linear and non-linear models. London Mathematical Society, Lecture Note Series, 370*, Cambridge University Press, Cambridge, 2010.

[2] Anashin V., Khrennikov A. , *Applied Algebraic Dynamics,* De Gruyter Expositions in Mathematics, vol. 49, Walter de Gruyter GmbH & Co, Berlin-N.Y., 2009.

[3] Atiyah M. F., Resolution of Singularities and Division of Distributions, Comm. pure Appl. Math. (1970) 23, 145–150.

[4] Bernstein I. N., Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients. Functional Analysis and its Applications (1972) 5 (2), 1–16.

[5] Bocardo-Gaspar M., García-Compeán H., and Zúñiga-Galindo W. A., Regularization of p-adic String Amplitudes, and Multivariate Local Zeta Functions, Lett Math Phys (2019) 109: 1167. https://doi.org/10.1007/s11005-018-1137-1.

[6] Bocardo-Gaspar M., García-Compeán H., and Zúñiga-Galindo W. A., p-Adic string amplitudes in the limit p approaches to one. J. High Energ. Phys. (2018) 2018: 43. https://doi.org/10.1007/JHEP08(2018)043.

[7] Bocardo-Gaspar M., Veys Willem, Zúñiga-Galindo W. A.,Meromorphic Continuation of Koba-Nielsen String Amplitudes. Preprint 2019.

[8] Bourbaki, N., *Éléments de mathématique. Fasc. XXXVI. Variétés différentielles et analytiques. Fascicule de résultats (Paragraphes 8 à 15)*, Actualités Scientifiques et Industrielles, No. 1347 Hermann, Paris 1971.

[9] Halmos P., *Measure Theory*, D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[10] Igusa J.-I., A*n introduction to the theory of local zeta functions*, in AMS/IP Studies in Advanced Mathematics, 14, American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000

[11] Gel'fand I. M., Shilov G.E., *Generalized Functions vol 1*, Academic Press, New York and London, 1977.

[12] Gouvêa F., *p-adic numbers: An introduction. Universitext*, Springer-Verlag, Berlin, 1997.

[13] Katok S., *p-adic analysis compared with real.* Student Mathematical Library, 37. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2007.

[14] Koblitz N., *p-adic Numbers, p-adic Analysis and Zeta functions.* Second edition, Springer-Verlag, New York, 1984.

[15] Kochubei Anatoly N., *Pseudo-differential equations and stochastics over non-Archimedean fields*, Marcel Dekker, 2001.

[16] Kozyrev S. V., Methods and Applications of Ultrametric and p-Adic Analysis: From Wavelet Theory to Biophysics, *Proc. Steklov Inst. Math.* 274 (2011), Suppl. 1, 1-84.

[17] Khrennikov Andrei, *Non-archimedean analysis: quantum paradoxes, dynamical systems and biological models*, Mathematics and its Applications 427, Dordrecht: Kluwer Academic Publishers, 1997.

[18] Khrennikov Andrei, Kozyrev Sergei, Zúñiga-Galindo W. A. , *Ultrametric Equations and its Applications*, *Encyclopedia of Mathematics and its Applications (168)*, Cambridge University Press, 2018.

[19] Lang S., *Real and functional analysis*. Third edition. Graduate Texts in Mathematics, 142. Springer-Verlag, New York, 1993.

[20] Edwin León-Cardenal, W. A. Zúñiga-Galindo, An introduction to the Theory of local Zeta functions from scratch, Revista Integración, temas de matemáticas, (2019) 37, no.1, 45–76.

[21] Robert A., *A course in p-adic Analysis*, Springer-Verlag, 2000.

[22] Schikhof W. H., *Ultrametric calculus: An introduction to p-adic analysis*. Cambridge Studies in Advanced Mathematics, 4. Cambridge University Press, Cambridge, 2006.

[23] Taibleson M. H., Fourier analysis on local fields, Princeton University Press, 1975.

[24] Vladimirov V. S., Volovich I. V., and Zelenov E. I., *p-adic analysis and mathematical physics*. Series on Soviet and East European Mathematics 1, World Scientific, River Edge, NJ, 1994.

[25] Zúñiga-Galindo W. A. , *Pseudodifferential equations over non-Archimedean spaces*, *Lectures Notes in Mathematics 2174*, Springer, Cham, 2016.

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