A New Proof of Three Terminal $\Delta - Y$ Reducibility for Planar Graphs

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A graph is star–triangle reducible if it can be reduced to a vertex by a sequence of series–parallel reductions and star–triangle transformations. Terminals are distinguished vertices which cannot be deleted by reductions and transformations.

One of the more important sets of reductions is the series–parallel reductions. These apply to graphs which may include loops (edges whose two end–vertices are identical), and parallel edges (two edges with the same pair of end–vertices). The four reductions are:

- $R0$ *Loop reduction:* Delete a loop.
- $R1$ *Degree–one reduction:* Delete a degree–one vertex and its incident edge.
- $R2$ *Series Reduction:* Delete a degree–two vertex y and its two incident edges xy and yz, and add in a new edge xz.
- $R3$ *Parallel reduction:* Delete one of a pair of parallel edges.

Each of these reductions decreases the number of edges in a graph. A connected graph is series–parallel reducible if it can be reduced to a single vertex by a sequence of these operations. Two other transformations of graph are important. A star (Y) is a vertex of degree three. A triangle (Δ) is a cycle of length three. The transformations are:

- $Y\Delta$ *Wye–delta transformation:* Delete a vertex w and its three incident edges wx, wy, wz, and add in a triangle xyz.
- ΔY Delta–wye transformation: Delete the edges of a triangle xyz, and add in a new vertex w and new edges wx, wy, and wz.

Two graphs are star–triangle–equivalent if one can be changed into the other by a sequence of $Y\Delta$– and ΔY–transformations. A connected graph is star–triangle reducible if it can be reduced to a single vertex by a sequence of series–parallel reductions and $Y\Delta$– or ΔY–transformations. Not every graph is star–triangle–reducible: for example, a simple graph of minimal degree four and girth four admits no reductions or transformations.

No characterization of star–triangle–reducible graphs is known, but some partial results are interesting. For example, Akers and Lehman conjectured that every planar graph is star–triangle–reducible. This was first proved by Epifanov in 1966. Simpler proofs are given by Feo and Provan, and Truemper. A minor of G is a graph formed by a sequence of edge deletions, edge contractions, and deletion of isolated vertices. Gitler proved that the class of graphs with no $K(5)$ minor are reducible and that the class of graphs with no $K(3,3)$ minor are reducible. Archedeacon, Colbourn, Gitler and Provan caracterized projective planar reducible graphs.

A variation on star–triangle–reducibility is to forbid reductions on some distinguished vertices. Specifically, let $T(V(G))$ be a set of terminals. A terminal cannot be deleted in a degree–one or series reduction, nor can it be deleted in a $Y\Delta$–transformation. If a graph with terminals can be reduced to eliminate all non–terminal vertices, then we say it is (terminal) star–triangle–reducible.

Akers also conjectured that any 3–terminal planar graph is star–triangle–reducible. This conjecture was proved by Gitler. We give in this talk a new proof of this result. On the other hand, there are examples of infinite families of 4–terminal planar non–star–triangle–reducible graphs.

The complexity of reducing a planar graph with n vertices is still open. The fastest known algorithms have complexity $O(n2)$ and they seem difficult to improve.

The study of star–triangle transformations has played an important role to solve problems like the classification of spin models using association schemes and their associated invariants by Jaeger, the characterization of graph embeddings in 3–space with certain knotting properties by Seymour, Robertson and Thomas. Also we can list the work of Robertson and Vitray which characterizes the minimal embeddings of projective planar graphs, the characterization of 3–polytopes by Granbaum and its relevance in the sums of squares problem by Calvillo, Gitler and Martnez. It is also important to notice its directs application to practical problems like reliability, flows and finding minimal paths, calculating the potential of electrical circuits, counting perfect matchings and spanning trees for reducible graphs.