Spin Models, Association Schemes and $\Delta - Y$ Transformation

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In the world of knot and link invariants, we are interested in Spin Models and their classification. V. Jones introduced a construction of a link invariant based on the statistical mechanical concept of Spin Models. Jones studied only the symmetric case; Kawagoe, Munemasa and Watatani generalized it by removing the symmetry condition.

A Spin Model is defined on a directed graph G by assigning to each edge e a square matrix $w(e)$ whose rows and columns are indexed by a given finite set X. Let $c :\rightarrow X$ be an arbitrary coloring of the vertices of G with elements of X. Then with each edge e from v to v' is associated the $(c(v), c(v'))$ entry of $w(e)$. The product over all edges of this number is called the weight of the coloring c, and the sum of weights of all colorings is called the partition function.

The main idea of Jones is to represent every link by a plane graph with signed edges. Jones defines on this signed graph a Spin Model for which the matrix associated with any edge is choosen according to signs among two matrices. Then he gives a set of equations which, when satisfied by the two matrices, guarantee that the partition function (after an adequate normalization) is a link invariant.

F. Jaeger studied the relation of Spin Models and association schemes. Association schemes, a concept from Algebraic Combinatorics, are important for the study of the several areas of Combinatorics, for example, Distance–Regular Graphs, Codes, Design Theory, etc.

The question of the relation between Spin Models and association schemes (Bose-Mesner algebras or BM-algebras) was finally answered by K. Nomura: he gave a simple algebraic relation. The second invariance equation of the Spin Models generates a BM-algebra($N(W)$) and the third invariance equation tells us that the weight matrix of the Spin Models belongs to the BM-algebra of Nomura. The non-symmetric case is treated by Jaeger, Matsumoto, and Nomura. In particular every non-symmetric Spin Model generates a dual pair of BM–algebras. We are interested in knowing when there exists a Spin Model for a dual pair of BM–algebras.

On the other hand, in recent works Jaeger computed the partition function by using only local transformations on graphs. For this assume that all matrices assigned to edges belong to a given BM-algebra. This is always possible by using the the BM-algebra of Nomura. If a graph contains loops, pendant edges, edges in series or in parallel, one can easily compute the partition function on a reduced graph for which the assignment of matrices to edges has been modified in an appropriate way. In particular if a graph is series-parallel, the partition function can be computed by interating this process. Moreover, Jaeger extended the concept of series-parallel evaluation to all plane graphs by considering also the ΔY-transformations. The evaluation process which relies on Epifanov's Theorem and the fact that all matrices assigned to edges belong to a BM-algebra(*exactly triple regular*). Moreover, we give a simple extension to important classes of nonplanar graphs.