Intercalate matrices I, II and III

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(3 talks)

In these three talks we study several combinatorial aspects of integer sum of squares formulae problem which can be stated as given a complete bipartite graph $K_{r,s}$ determine the smallest integer $r *_\zeta s$ and a signing $\sigma : E(K_{r,s}) \to \{-1, 1\}$ of the edges such that it is possible to color the edges of $K_{r,s}$ with $r *_\zeta s$ colors so that

i) all edges incident to a vertex have different colors

ii) every 4–cycle is colored with 2 or 4 colors

iii) every bicolored 4–cycle has precisely an odd number of its edges negatively signed.

We show that given a coloring of $K_{r,s}$ that satisfies i) and ii) either it also satisfies iii) or else there exist certain combinatorial obstructions. In fact it is shown that signability (fulfillment of iii) is equivalent to bipartiteness of certain binary matroids called intercalate matroids and obstructions correspond to its odd cycles.

The matrices that are used to represent (in the obvious way) colorings of bipartite graphs satisfying i) and ii) are called intercalate matrices in the literature of the sum of squares problem. An important class of such matrices are the ones that arise as summation tables of dyadic groups. Such matrices are called dyadic matrices. We will give some characterizations of them.

We will show also that for the dyadic case some criteria for signability can be derived from the study of a combinatorial optimization problem over a particular independence system. From this theory a characterization of Hurwitz–Radon matrices is obtained.

Finally we provide a geometrical setting for the study of intercalate matroids. We show the relationship of odd cycles of intercalate matroids to oriented matroid theory through zonotopes and to multimatroids through topological graph theory. It is emphasized the role of $\Delta - Y$ transformations and triality (a chain of duality transformations) in the construction of odd cycles of intercalate matroids. In particular we show how the smallest obstructions for signability of intercalate matrices are related to the embedings of graphs of the Petersen Family in several surfaces.