Recent results in combinatorial number theory: Subset sum problems

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In the early sixties Erdős and Heilbronn formulated the following problem. Is it true, that if A is any set consisting of k distinct integers, then the number of different sums $a + a'$ $(a, a' \in A,\ a \neq a')$ is at least $2k - 3$? Simple as stated, this innocent looking relative of the well-known (and easy) Cauchy–Davenport Theorem had been a permanent challenge for three decades, until it was finally settled in the affirmative by Dias da Silva and Hamidoune a few years ago. Their algebraic proof has been later simplified by Alon, Nathanson, and Ruzsa.

The solution of this conjecture, including its generalization for more summands, opened up new horizons in the study of representation of integers as subset sums. The aim of this talk is to give a brief survey on this topic followed by a discussion of the below indicated new results.

Let $f(n, m)$ denote the maximum cardinality of a subset A of the set of the first n positive integers such that there is no $B \subset A$ the sum of whose elements is m. For every positive integer s, we determine the precise value of $f(n, m)$ in the range $c_1(s)n < m < c_2(s)n^2$ of integers m not divisible by s. This, in turn, confirms a conjecture of Alon in the case when the smallest positive integer that does not divide m is at most $O(\log n)$.

Earlier results by Alon, Lipkin, and Alon–Freiman utilized the analytic circle method. Our proof is completely elementary, apart from the application of the above mentioned generalization of the Erdős–Heilbronn conjecture.

Our first result, combined with other combinatorial ideas, allows us to confirm the following conjecture of Lev. Let $1 \leq a_1 < \ldots < a_l \leq n \leq 2l - const$ denote integers, and let $S = a_1 + \ldots + a_l$. Then every integer in the interval $[2n - 2l + 1, S - 2n + 2l - 1]$ is represented as the sum of some a_i's, and this interval cannot be extended. In particular, there exist $\varepsilon_1, \ldots, \varepsilon_l \in \{-1, +1\}$ such that $|\varepsilon_1 + \ldots + \varepsilon_l| \leq 1$ and $|\varepsilon_1 a_1 + \ldots + \varepsilon_l a_l| \leq 1$.